FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 37

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CONTENTS

1. APPLICATION OF COHOMOLOGY: HILBERT POLYNOMIALS AND FUNCTIONS, RIEMANN-ROCH, DEGREES, AND ARITHMETIC GENUS

We have now seen some powerful uses of Cech cohomology, to prove things about spaces of global sections, and to prove Serre vanishing. We will now see some classical constructions come out very quickly and cheaply.

In this section, we will work over a field k. Define $h^i(X, \mathcal{F}) := \dim_k H^i(X, \mathcal{F})$.

Suppose $\mathcal{F}$ is a coherent sheaf on a projective k-scheme X. Define the **Euler characteristic**

$$\chi(X, \mathcal{F}) := \sum_{i=0}^{\dim X} (-1)^i h^i(X, \mathcal{F}).$$

We will see repeatedly here and later that Euler characteristics behave better than individual cohomology groups. As one sign, notice that for fixed n, and $m \geq 0$,

$$h^0(\mathbb{P}^n_k, \mathcal{O}(m)) = \binom{n+m}{m} = \frac{(m+1)(m+2)\cdots(m+n)}{n!}.$$

Notice that the expression on the right is a polynomial in m of degree n. (For later reference, notice also that the leading coefficient is $m^n/n!$.) But it is not true that

$$h^0(\mathbb{P}^n_k, \mathcal{O}(m)) = \frac{(m+1)(m+2)\cdots(m+n)}{n!}$$

for *all* m — it breaks down for $m \leq -n-1$. Still, you can check that

$$\chi(\mathbb{P}^n_k, \mathcal{O}(m)) = \frac{(m+1)(m+2)\cdots(m+n)}{n!}.$$

So one lesson is this: if one cohomology group (usual the top or bottom) behaves well in a certain range, and then messes up, likely it is because (i) it is actually the Euler characteristic which is behaving well *always*, and (ii) the other cohomology groups vanish in that range.

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In fact, we will see that it is often hard to calculate cohomology groups (even h^0), but it can be easier calculating Euler characteristics. So one important way of getting a hold of cohomology groups is by computing the Euler characteristics, and then showing that all the *other* cohomology groups vanish. Hence the ubiquity and importance of *vanishing theorems*. (A vanishing theorem usually states that a certain cohomology group vanishes under certain conditions.) We will see this in action when discussing curves.

The following exercise shows another way in which Euler characteristic behaves well: it is *additive in exact sequences*.

1.A. EXERCISE. Show that if $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is an exact sequence of coherent sheaves on X, then $\chi(X, \mathcal{G}) = \chi(X, \mathcal{F}) + \chi(X, \mathcal{H})$. (Hint: consider the long exact sequence in cohomology.) More generally, if

$$0 \to \mathcal{F}_1 \to \cdots \to \mathcal{F}_n \to 0$$

is an exact sequence of sheaves, show that

$$\sum_{i=1}^{n} (-1)^i \chi(X, \mathcal{F}_i) = 0.$$

1.1. The Riemann-Roch Theorem for line bundles on a nonsingular projective curve. Suppose $\mathcal{L}$ is an invertible sheaf on a projective curve C over k. We tentatively define the **degree** of $\mathcal{L}$ as follows. Let s be a non-zero rational section on C. Let D be the divisor of zeros and poles of s:

$$D := \sum_{p \in C} \nu_p(s)[p]$$

Then define $\deg \mathcal{L} := \deg D = \sum \nu_p(s) \deg p$. Here $\deg p$ is the degree of the residue field of $\mathcal{O}_C$ at p, i.e. $\dim k\mathcal{O}_p = \deg p$. It isn't yet clear $\deg \mathcal{L}$ is well-defined: a priori it depends on the choice of s. Nonetheless you should prove the following.

1.B. EXERCISE: THE RIEMANN-ROCH THEOREM FOR LINE BUNDLES ON A NONSINGULAR PROJECTIVE CURVE. Show that

$$\chi(C, \mathcal{L}) = \deg \mathcal{L} + \chi(C, \mathcal{O}_C).$$

Here is a possible hint. Suppose $p \in C$ is a closed point of C, of degree d. Then twisting the closed exact sequence

$$0 \to \mathcal{O}_C(-p) \to \mathcal{O}_C \to \mathcal{O}_p \to 0$$

by $\mathcal{L}$ (as $\otimes \mathcal{L}$ is an exact functor) we obtain

$$0 \to \mathcal{L}(-p) \to \mathcal{L} \to \mathcal{O}_p \to 0$$

(where we are using a non-canonical isomorphism $\mathcal{L}|_p \cong \mathcal{O}_C|_p = \mathcal{O}_p$). Use the additivity of χ in exact sequences to show that the result is true for $\mathcal{L}$ if and only if it is true for $\mathcal{L}(-p)$. The result is also clearly true for $\mathcal{L} = \mathcal{O}$. Then argue by "induction" that it is true for all $\mathcal{L}$.

In particular, $\deg \mathcal{L}$ is well-defined!

1.C. EXERCISE. If $\mathcal{L}$ and $\mathcal{M}$ are two line bundles on a nonsingular projective curve C, show that $\deg \mathcal{L} \otimes \mathcal{M} = \deg \mathcal{L} + \deg \mathcal{M}$. (Hint: choose rational sections of $\mathcal{L}$ and $\mathcal{M}$.)

In fact we could have *defined* the degree of a line bundle $\mathcal{L}$ on a nonsingular projective curve C to be $\chi(C, \mathcal{L}) - \chi(C, \mathcal{O}_C)$. Then Riemann-Roch would be true by definition; but we would still want to relate this notion of degree to the classical notion of zeros and poles, which we would do by the argument in the previous paragraph. Otherwise, for example, Exercise 1.C isn't obvious from the cohomological definition.

Definition. Suppose C is a reduced projective curve (pure dimension 1, over a field k). If $\mathcal{L}$ is a line bundle on C, define $\deg \mathcal{L} = \chi(C, \mathcal{L}) - \chi(C, \mathcal{O}_C)$. If C is irreducible, and $\mathcal{F}$ is a coherent sheaf on C, define the **rank** of $\mathcal{F}$, denoted $\operatorname{rank} \mathcal{F}$, to be its rank at the generic point of C.

1.D. EASY EXERCISE. Show that the rank is additive in exact sequences: if $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is an exact sequence of coherent sheaves, show that $\operatorname{rank} \mathcal{F} - \operatorname{rank} \mathcal{G} + \operatorname{rank} \mathcal{H} = 0$.

Definition. Define

$$\deg \mathcal{F} = \chi(C, \mathcal{F}) - (\operatorname{rank} \mathcal{F})\chi(C, \mathcal{O}_C). \tag{1}$$

If $\mathcal{F}$ is a line bundle, we can drop the hypothesis of irreducibility in the definition.

This generalizes the notion of the degree of a line bundle on a nonsingular curve.

1.E. EASY EXERCISE. Show that degree is additive in exact sequences.

The statement (1) is often called Riemann-Roch for coherent sheaves (or vector bundles) on a projective curve.

If $\mathcal{F}$ is a coherent sheaf on X, define the **Hilbert function of** $\mathcal{F}$:

$$h_{\mathcal{F}}(n) := h^0(X, \mathcal{F}(n)).$$

The **Hilbert function of** X is the Hilbert function of the structure sheaf. The ancients were aware that the Hilbert function is "eventually polynomial", i.e. for large enough n, it agrees with some polynomial, called the **Hilbert polynomial** (and denoted $p_{\mathcal{F}}(n)$ or $p_X(n)$). This polynomial contains lots of interesting geometric information, as we will soon see. In modern language, we expect that this "eventual polynomiality" arises because the Euler characteristic should be a polynomial, and that for $n \gg 0$, the higher cohomology vanishes. This is indeed the case, as we now verify.

1.2. *Theorem. — If $\mathcal{F}$ is a coherent sheaf on a projective* k*-scheme* $X \hookrightarrow \mathbb{P}^n_k$*,* $\chi(X, \mathcal{F}(m))$ *is a polynomial of degree equal to* $\dim \operatorname{Supp} \mathcal{F}$*. Hence by Serre vanishing (Theorem 1.2(ii) in the class*

35/36 notes), for $m \gg 0$, $h^0(X, \mathcal{F}(m))$ is a polynomial of degree $\dim \operatorname{Supp} \mathcal{F}$. *In particular, for $m \gg 0$, $h^0(X, \mathcal{O}_X(m))$ is polynomial with degree* $= \dim X$.

Here $\mathcal{O}_X(m)$ is the restriction or pullback of $\mathcal{O}_{\mathbb{P}^n_k}(1)$. Both the degree of the 0 polynomial and the dimension of the empty set is defined to be -1. In particular, the only coherent sheaf Hilbert polynomial 0 is the zero-sheaf.

Proof. Define $p_{\mathcal{F}}(m) = \chi(X, \mathcal{F}(m))$. We will show that $p_{\mathcal{F}}(m)$ is a polynomial of the desired degree.

Step 1. Assume first that k is infinite. (This is one of those cases where even if you are concerned with potentially arithmetic questions over some non-algebraically closed field like $\mathbb{F}_p$, you are forced to consider the "geometric" situation where the base field is algebraically closed.)

$\mathcal{F}$ has a finite number of associated points. Then there is a hyperplane $x = 0$ ($x \in \Gamma(X, \mathcal{O}(1))$) missing this finite number of points. (This is where we use the algebraic closure, or more precisely, the infinitude of k.)

Then the map $\mathcal{F}(-1) \xrightarrow{\times x} \mathcal{F}$ is injective (on any affine open subset, $\mathcal{F}$ corresponds to a module, and x is not a zero-divisor on that module, as it doesn't vanish at any associated point of that module). Thus we have a short exact sequence

$$0 \to \mathcal{F}(-1) \to \mathcal{F} \to \mathcal{G} \to 0 \qquad (2)$$

where $\mathcal{G}$ is a coherent sheaf.

1.F. EXERCISE. Show that $\operatorname{Supp} \mathcal{G} = \operatorname{Supp} \mathcal{F} \cap V(x)$.

Hence $\dim \operatorname{Supp} \mathcal{G} = \dim \operatorname{Supp} \mathcal{F} - 1$ by Krull's Principal Ideal Theorem unless $\mathcal{F} = 0$ (in which case we already know the result, so assume this is not the case).

Twisting (2) by $\mathcal{O}(m)$ yields

$$0 \to \mathcal{F}(m-1) \to \mathcal{F}(m) \to \mathcal{G}(m) \to 0$$

Euler characteristics are additive in exact sequences, from which $p_{\mathcal{F}}(m) - p_{\mathcal{F}}(m-1) = p_{\mathcal{G}}(m)$. Now $p_{\mathcal{G}}(m)$ is a polynomial of degree $\dim \operatorname{Supp} \mathcal{F} - 1$.

The result follows from a basic fact about polynomials.

1.G. EXERCISE. Suppose f and g are functions on the integers, $f(m+1) - f(m) = g(m)$ for all m, and $g(m)$ is a polynomial of degree $d \geq 0$. Show that f is a polynomial of degree $d+1$.

Step 2: k finite.

1.H. EXERCISE. Complete the proof using Exercise 2.G from the notes from class 35/36 (on cohomology and change of base field), using $K = \overline{k}$.

$\square$

Definition. $p_{\mathcal{F}}(m)$ was defined in the above proof. If $X \subset \mathbb{P}^n$ is a projective k-scheme, define $p_X(m) := p_{\mathcal{O}_X}(m)$.

Example 1. $p_{\mathbb{P}^n}(m) = \binom{m+n}{n}$, where we interpret this as the polynomial $(m+1)\cdots(m+n)/n!$.

Example 2. Suppose H is a degree d hypersurface in $\mathbb{P}^n$. Then from the closed subscheme exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-d) \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_H \to 0,$$

we have

$$p_H(m) = p_{\mathbb{P}^n}(m) - p_{\mathbb{P}^n}(m-d) = \binom{m+n}{n} - \binom{m+n-d}{n}.$$

1.I. EXERCISE. Show that the twisted cubic (in $\mathbb{P}^3$) has Hilbert polynomial $3m+1$.

1.J. EXERCISE. Find the Hilbert polynomial for the dth Veronese embedding of $\mathbb{P}^n$ (i.e. the closed immersion of $\mathbb{P}^n$ in a bigger projective space by way of the line bundle $\mathcal{O}(d)$).

From the Hilbert polynomial, we can extract many invariants, of which two are particularly important. The first is the *degree*. Classically, the degree of a complex projective variety of dimension n was defined as follows. We slice the variety with n generally chosen hyperplane. Then the intersection will be a finite number of points. The degree is this number of points. Of course, this requires showing all sorts of things. Instead, we will define the *degree of a projective k-scheme of dimension* n to be leading coefficient of the Hilbert polynomial (the coefficient of m^n) times $n!$.

Using the examples above, we see that the degree of $\mathbb{P}^n$ in itself is 1. The degree of the twisted cubic is 3.

1.K. EXERCISE. Show that the degree is always an integer. Hint: by induction, show that any polynomial in m of degree k taking on only integral values must have coefficient of m^k an integral multiple of $1/k!$. Hint for this: if $f(x)$ takes on only integral values and is of degree k, then $f(x+1) - f(x)$ takes on only integral values and is of degree $k-1$.

1.L. EXERCISE. Show that the degree of a degree d hypersurface is d (preventing a notational crisis).

1.M. EXERCISE. Suppose a curve C is embedded in projective space via an invertible sheaf of degree d. In other words, this line bundle determines a closed immersion. Show that the degree of C under this embedding is d (preventing another notational crisis). (Hint: Riemann-Roch, Exercise 1.B.)

1.N. EXERCISE. Show that the degree of the dth Veronese embedding of $\mathbb{P}^n$ is d^n.

1.O. EXERCISE (BEZOUT'S THEOREM). Suppose X is a projective scheme of dimension at least 1, and H is a degree d hypersurface not containing any associated points of X. (For example, if X is a projective variety, then we are just requiring H not to contain any irreducible components of X.) Show that $\deg H \cap X = d \deg X$.

This is a very handy theorem! For example: if two projective plane curves of degree m and degree n share no irreducible components, then they intersect in mn points, counted with appropriate multiplicity. The notion of multiplicity of intersection is just the degree of the intersection as a k-scheme.

We trot out a useful example we have used before: let $k = \mathbb{Q}$, and consider the parabola $x = y^2$. We intersect it with the four lines, $x = 1$, $x = 0$, $x = -1$, and $x = 2$, and see that we get 2 each time (counted with the same convention as with the last time we saw this example).

If we intersect it with $y = 2$, we only get one point — but that's of course because this isn't a projective curve, and we really should be doing this intersection on $\mathbb{P}^2_k$ — and in this case, the conic meets the line in two points, one of which is "at ∞".

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1.P. EXERCISE. Show that the degree of the d-fold Veronese embedding of $\mathbb{P}^n$ is d^n in a different way (from Exercise 1.N) as follows. Let $\nu_d : \mathbb{P}^n \to \mathbb{P}^N$ be the Veronese embedding. To find the degree of the image, we intersect it with n hyperplanes in $\mathbb{P}^N$ (scheme-theoretically), and find the number of intersection points (counted with multiplicity). But the pullback of a hyperplane in $\mathbb{P}^N$ to $\mathbb{P}^n$ is a degree d hypersurface. Perform this intersection in $\mathbb{P}^n$, and use Bezout's theorem (Exercise 1.O).

There is another nice important of information residing in the Hilbert polynomial. Notice that $p_X(0) = \chi(X, \mathcal{O}_X)$, which is an *intrinsic* invariant of the scheme X, which does not depend on the projective embedding.

Imagine how amazing this must have seemed to the ancients: they defined the Hilbert function by counting how many "functions of various degrees" there are; then they noticed that when the degree gets large, it agrees with a polynomial; and then when they plugged 0 into the polynomial — extrapolating backwards, to where the Hilbert function and Hilbert polynomials didn't agree — they found a magic invariant!

Now we can finally see a nonsingular curve over an algebraically closed field that is provably not $\mathbb{P}^1$! Note that the Hilbert polynomial of $\mathbb{P}^1$ is $(m+1)/1 = m+1$, so $\chi(\mathcal{O}_{\mathbb{P}^1}) =$

1. Suppose C is a degree d curve in $\mathbb{P}^2$. Then the Hilbert polynomial of C is

$$p_{\mathbb{P}^2}(m) - p_{\mathbb{P}^2}(m-d) = (m+1)(m+2)/2 - (m-d+1)(m-d+2)/2.$$

Plugging in $m = 0$ gives us $-(d^2 - 3d)/2$. Thus when $d > 2$, we have a curve that cannot be isomorphic to $\mathbb{P}^1$! (I think I gave you an earlier exercise that there is a *nonsingular* degree d curve.)

Now from $0 \to \mathcal{O}_{\mathbb{P}^2}(-d) \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_C \to 0$, using $h^1(\mathcal{O}_{\mathbb{P}^2}(d)) = 0$, we have that $h^0(C, \mathcal{O}_C) = 1$. As $h^0 - h^1 = \chi$, we have

$$h^1(C, \mathcal{O}_C) = (d-1)(d-2)/2.$$

Motivated by geometry, we define the **arithmetic genus** of a scheme X as $1-\chi(X, \mathcal{O}_X)$. This is sometimes denoted $p_a(X)$. In the case of nonsingular complex curves, this corresponds to the topological genus. For irreducible reduced curves (or more generally, curves with $h^0(X, \mathcal{O}_X) \cong k$), $p_a(X) = h^1(X, \mathcal{O}_X)$. (In higher dimension, this is a less natural notion.)

We thus now have examples of curves of genus $0, 1, 3, 6, 10, \ldots$ (corresponding to degree 1 or $2, 3, 4, 5, \ldots$).

This begs some questions, such as: are there curves of other genera? (We'll see soon that the answer is yes.) Are there other genus 1 curves? (Not if k is algebraically closed, but yes otherwise.) Do we have all the curves of genus 3? (Almost all, but not quite.) Do we have all the curves of genus 6? (We're missing most of them.)

Caution: The Euler characteristic of the structure sheaf doesn't distinguish between isomorphism classes of nonsingular projective schemes over algebraically closed fields — for example, $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$ both have Euler characteristic 1, but are not isomorphic — $\operatorname{Pic} \mathbb{P}^2 \cong \mathbb{Z}$ while $\operatorname{Pic} \mathbb{P}^1 \times \mathbb{P}^1 \cong \mathbb{Z} \oplus \mathbb{Z}$.

Important Remark. We can restate the Riemann-Roch formula for curves (Exercise 1.B) as:

$$h^0(C, \mathcal{L}) - h^1(C, \mathcal{L}) = \deg \mathcal{L} - p_a + 1.$$

This is the most common formulation of the Riemann-Roch formula.

If C is a nonsingular irreducible projective complex curve, then the corresponding complex-analytic object, a compact *Riemann surface*, has a notion called the *genus* g, which is the number of holes. It turns out that $g = p_a$ in this case, and for this reason, we will often write g for p_a when discussing nonsingular (projective irreducible) curves, over any field.

1.3. *Complete intersections.* We define a **complete intersection** in $\mathbb{P}^n$ as follows. $\mathbb{P}^n$ is a complete intersection in itself. A closed subscheme $X_r \hookrightarrow \mathbb{P}^n$ of dimension r (with $r < n$) is a complete intersection if there is a complete intersection X_{r+1}, and X_r is an effective Cartier divisor in class $\mathcal{O}_{X_{r+1}}(d)$.

1.Q. EXERCISE. Show that if X is a complete intersection of dimension r in $\mathbb{P}^n$, then $H^i(X, \mathcal{O}_X(m)) = 0$ for all $0 < i < r$ and all m. Show that if $r > 0$, then $H^0(\mathbb{P}^n, \mathcal{O}(m)) \to H^0(X, \mathcal{O}(m))$ is surjective. (Hint: long exact sequences.)

Now X_r is the divisor of a section of $\mathcal{O}_{X_{r+1}}(m)$ for some m. But this section is the restriction of a section of $\mathcal{O}(m)$ on $\mathbb{P}^n$. Hence X_r is the scheme-theoretic intersection of X_{r+1} with a hypersurface. Thus inductively X_r is the scheme-theoretic intersection of $n-r$ hypersurfaces. (By Bezout's theorem, Exercise 1.O, $\deg X_r$ is the product of the degree of the defining hypersurfaces.)

1.R. EXERCISE (COMPLETE INTERSECTIONS ARE CONNECTED). Show that complete intersections of *positive* dimension are connected. (Hint: show $h^0(X, \mathcal{O}_X) = 1$.)

1.S. EXERCISE. Find the genus of the intersection of 2 quadrics in $\mathbb{P}^3$. (We get curves of more genera by generalizing this! At this point we need to worry about whether there are any nonsingular curves of this form. We can check this by hand, but later Bertini's Theorem will save us this trouble.)

1.T. EXERCISE. Show that the rational normal curve of degree d in $\mathbb{P}^d$ is *not* a complete intersection if $d > 2$. (Hint: If it *were* the complete intersection of $d-1$ hypersurfaces, what would the degree of the hypersurfaces be? Why could none of the degrees be 1?)

1.U. EXERCISE. Show that the union of 2 distinct planes in $\mathbb{P}^4$ is not a complete intersection. Hint: it is connected, but you can slice with another plane and get something not connected (see Exercise 1.R).

This is another important scheme in algebraic geometry that is an example of many sorts of behavior. We will see more of it later!

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