FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 32

RAVI VAKIL

CONTENTS

1. IMPORTANT EXAMPLE: INVERTIBLE SHEAVES AND MAPS TO PROJECTIVE SCHEMES

Theorem 1.1 will give one reason why line bundles are crucially important: they tell us about maps to projective space, and more generally, to quasiprojective A-schemes. Given that we have had a hard time naming any non-quasiprojective schemes, they tell us about maps to essentially all schemes that are interesting to us.

Before stating the theorem, we begin with some motivation. Recall that the data of a map to $\mathbb{A}^n$ corresponds to the choice of n functions, which could be called "coordinate functions". (The case $n = 1$ was an earlier exercise, and the general case is no harder.) Our goal is to give a similar characterization of maps to $\mathbb{P}^n$. We have already seen that a choice of $n+1$ functions on X with no common zeros yields a map to $\mathbb{P}^n$. However, this can't give *all* maps to $\mathbb{P}^n$: suppose $n > 0$ and consider the identity map $\mathbb{P}^n_k \to \mathbb{P}^n_k$. This map can't be described in terms of $n+1$ functions on X with no common zeros, as the only functions on $\mathbb{P}^n$ are constants, so they only maps $\mathbb{P}^n_k \to \mathbb{P}^n_k$ that can be described in terms of n functions with no common zeros are constant maps. The resolution of this problem is by considering not just functions — sections of the trivial invertible sheaf — but sections of any invertible sheaf.

1.1. *Important theorem. — Maps to $\mathbb{P}^n$ correspond to $n+1$ sections of a line bundle, not all vanishing at any point (i.e. generated by global sections), modulo global sections of $\mathcal{O}_X^*$.*

This is one of those important theorems in algebraic geometry that is easy to prove, but quite subtle in its effect on how one should think. It takes some time to properly digest.

The theorem describes all morphisms to projective space, and hence by the Yoneda philosophy, this can be taken as the *definition* of projective space: it defines projective space up to unique isomorphism.

Every time you see a map to projective space, you should immediately simultaneously keep in mind the invertible sheaf and sections.

Date: Friday, February 8, 2008.

Maps to projective schemes can be described similarly. For example, if $Y \hookrightarrow \mathbb{P}^2_k$ is the curve $x_2^2x_0 = x_1^3 - x_1x_0^2$, then maps from a scheme X to Y are given by an invertible sheaf on X along with three sections s_0, s_1, s_2, with no common zeros, satisfying $s_2^2s_0 - s_1^3 + s_1s_0^2 = 0$.

Here more precisely is the correspondence of Theorem 1.1. If you have $n+1$ sections, then away from the intersection of their zero-sets, we have a morphism. Conversely, if you have a map to projective space $f : X \to \mathbb{P}^n$, then we have $n+1$ sections of $\mathcal{O}_{\mathbb{P}^n}(1)$, corresponding to the hyperplane sections, $x_0, \dots, x_{n+1}$. then $f^*x_0, \dots, f^*x_{n+1}$ are sections of $f^*\mathcal{O}_{\mathbb{P}^n}(1)$, and they have no common zero.

So to prove this, we just need to show that these two constructions compose to give the identity in either direction.

Proof. Given $n+1$ sections $s_0, \dots, s_n$ of an invertible sheaf. We get trivializations on the open sets where each one vanishes. The transition functions are precisely s_i/s_j on $U_i \cap U_j$. We pull back $\mathcal{O}(1)$ by this map to projective space, This is trivial on the distinguished open sets. Furthermore, $f^*D(x_i) = D(s_i)$. Moreover, $s_i/s_j = f^*(x_i/x_j)$. Thus starting with the $n+1$ sections, taking the map to the projective space, and pulling back $\mathcal{O}(1)$ and taking the sections $x_0, \dots, x_n$, we recover the s_i's. That's one of the two directions.

Correspondingly, given a map $f : X \to \mathbb{P}^n$, let $s_i = f^*x_i$. The map $[s_0; \cdots; s_n]$ is precisely the map f. We see this as follows. The preimage of U_i is $D(s_i) = D(f^*x_i) = f^*D(x_i)$. So the right open sets go to the right open sets. And $D(s_i) \to D(x_i)$ is precisely by $s_j/s_i = f^*x_j/x_i$. $\square$

Here is some convenient language. A **linear system** on a k-scheme X is a k-vector space V (usually finite-dimensional), an invertible sheaf $\mathcal{L}$, and a linear map $\lambda : V \to \Gamma(X, \mathcal{L})$. Such a linear system is often called "V", with the rest of the data left implicit. If the map λ is an isomorphism, it is called a **complete linear system**, and is often written $|\mathcal{L}|$. Given a linear system, any point $x \in X$ on which all elements of the linear system V vanish, we say that x is a **base-point** of V. If V has no base-points, we say that it is **base-point-free**. The union of base-points is called the **base locus**. The base locus has a scheme-structure — the (scheme-theoretic) intersection of the vanishing loci of the elements of V (or equivalently, of a basis of V). In this incarnation, it is called the **base scheme** of the linear system.

A linear system is sometimes called a **linear series**. I'm not sure of the distinction between these two terms, so I'll not use this second terminology.

1.A. EXERCISE (AUTOMORPHISMS OF PROJECTIVE SPACE). Show that all the automorphisms of projective space $\mathbb{P}^n_k$ correspond to $(n+1) \times (n+1)$ invertible matrices over k, modulo scalars (also known as $PGL_{n+1}(k)$). (Hint: Suppose $f : \mathbb{P}^n_k \to \mathbb{P}^n_k$ is an automorphism. Show that $f^*\mathcal{O}(1) \cong \mathcal{O}(1)$. Show that $f^* : \Gamma(\mathbb{P}^n, \mathcal{O}(1)) \to \Gamma(\mathbb{P}^n, \mathcal{O}(1))$ is an isomorphism.)

Exercise 1.A will be useful later, especially for the case $n = 1$. In this case, these automorphisms are called **fractional linear transformations**.

(A question for experts: why did I not state that previous exercise over an arbitrary base ring A? Where does the argument go wrong in that case?)

Here are some more examples of these ideas in action.

Example 1. Consider the $n+1$ functions $x_0, \dots, x_n$ on $\mathbb{A}^{n+1}$ (otherwise known as $n+1$ sections of the trivial bundle). They have no common zeros on $\mathbb{A}^n - 0$. Hence they determine a morphism $\mathbb{A}^{n+1} - 0 \to \mathbb{P}^n$. (We've talked about this morphism before. But now we don't have to worry about gluing.)

Example 2: the Veronese morphism is $|\mathcal{O}_{\mathbb{P}^n}(d)|$. Consider the line bundle $\mathcal{O}_{\mathbb{P}^n}(m)$ on $\mathbb{P}^n$. We've checked that the number of sections of this line bundle are $\binom{n+m}{m}$, and they correspond to homogeneous degree m polynomials in the projective coordinates for $\mathbb{P}^n$. Also, they have no common zeros (as for example the subset of sections x_0^m, x_1^m, $\dots$, x_n^m have no common zeros). Thus the complete linear system is base-point-free, and determines a morphism $\mathbb{P}^n \to \mathbb{P}^{\binom{n+m}{m}-1}$. This is called the **Veronese morphism**. For example, if $n=2$ and $m=2$, we get a map $\mathbb{P}^2 \to \mathbb{P}^5$.

We have checked earlier that this is a closed immersion. How can you tell in general if something is a closed immersion, and not just a map? Here is one way.

1.B. EXERCISE. Suppose $\pi : X \to \mathbb{P}^n_A$ corresponds to an invertible sheaf $\mathcal{L}$ on X, and sections $s_0, \dots, s_n$. Show that π is a closed immersion if and only if

(i) each open set X_{s_i} is affine, and
(ii) for each i, the map of rings $A[y_0, \dots, y_n] \to \Gamma(X_{s_i}, \mathcal{O})$ given by $y_j \mapsto s_j/s_i$ is surjective.

Example 3: The rational normal curve. Recall that the image of the Veronese morphism when $n = 1$ is called a **rational normal curve of degree** m. Our map is $\mathbb{P}^1 \to \mathbb{P}^m$ given by $[x;y] \to [x^m; x^{m-1}y; \cdots; xy^{m-1}; y^m]$.

1.C. EXERCISE. If the image scheme-theoretically lies in a hyperplane of projective space, we say that it is **degenerate** (and otherwise, **non-degenerate**). Show that a base-point-free linear system V with invertible sheaf $\mathcal{L}$ is non-degenerate if and only if the map $V \to \Gamma(X, \mathcal{L})$ is an inclusion. Hence in particular a complete linear system is always non-degenerate.

1.D. EXERCISE. Suppose we are given a map $\pi : \mathbb{P}^1_k \to \mathbb{P}^n_k$ where the corresponding invertible sheaf on $\mathbb{P}^1_k$ is $\mathcal{O}(d)$. (We will later call this a *degree* d *map*.) Show that if $d < n$, then the image is degenerate. Show that if $d = n$ and the image is nondegenerate, then the image is isomorphic (via an automorphism of projective space, Exercise 1.A) to a rational normal curve.

Example 4: The Segre morphism in terms of a linear system. The Segre morphism can also be interpreted in this way. This is a useful excuse to define some notation. Suppose $\mathcal{F}$ is a quasicoherent sheaf on a Z-scheme X, and $\mathcal{G}$ is a quasicoherent sheaf on a Z-scheme Y. Let π_X, π_Y be the projections from $X \times_Z Y$ to X and Y respectively. Then $\mathcal{F} \boxtimes \mathcal{G}$ is defined to be $\pi_X^* \mathcal{F} \otimes \pi_Y^* \mathcal{G}$. In particular, $\mathcal{O}_{\mathbb{P}^m \times \mathbb{P}^n}(a, b)$ is defined to be $\mathcal{O}_{\mathbb{P}^m}(a) \boxtimes \mathcal{O}_{\mathbb{P}^n}(b)$ (over any base Z). The Segre morphism $\mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^{mn+m+n}$ corresponds to the complete linear system for the invertible sheaf $\mathcal{O}(1, 1)$.

When we first saw the Segre morphism, we saw (in different language) that this complete linear system is base-point-free. We also checked by hand that it is a closed immersion, essentially by Exercise 1.B.

1.E. FUN EXERCISE. Show that any map from projective space to a smaller projective space is constant (over a field). Hint: show that if $m < n$ then m non-empty hypersurfaces in $\mathbb{P}^n$ have non-empty intersection. For this, use the fact that any non-empty hypersurface in $\mathbb{P}^n_k$ has non-empty intersection with any subscheme of dimension at least 1.

1.F. EXERCISE. Show that a base-point-free linear system V on X corresponding to $\mathcal{L}$ induces a morphism to projective space $X \to \mathbb{P}V^* = \operatorname{Proj} \oplus_n \mathcal{L}^{\otimes n}$. The resulting morphism is often written $X \xrightarrow{|V|} \mathbb{P}^n$.

E-mail address: vakil@math.stanford.edu