FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 27

RAVI VAKIL

CONTENTS

1. QUASICOHERENT SHEAVES OF IDEALS, AND CLOSED SUBSCHEMES

The point of this section is that closed subschemes correspond precisely to quasicoherent sheaves of ideals.

Recall that if $i : X \hookrightarrow Y$ is a closed immersion, then we have a surjection of sheaves on Y: $\mathcal{O}_Y \twoheadrightarrow i_*\mathcal{O}_X$. (The i_* is often omitted, as we are considering the sheaf on X as being a sheaf on Y.) The sheaf $i_*\mathcal{O}_X$ is quasicoherent on Y; this is in some sense the definition of "closed subscheme". The kernel $\mathcal{I}_{X/Y}$ is a "sheaf of ideals" in Y: for each open subset of Y, the sections form an ideal in the ring of functions of Y. As quasicoherent sheaves on Y form an abelian category, $\mathcal{I}_{X/Y}$ is a *quasicoherent sheaf of ideals.*

Conversely, a quasicoherent sheaf of ideals $\mathcal{I} \hookrightarrow \mathcal{O}_Y$ defines a closed subscheme. This was stated in slightly different language in Exercise 1. Precisely, $\mathcal{I}$ is quasicoherent precisely if, for each distinguished open $\operatorname{Spec} A_f \hookrightarrow \operatorname{Spec} A$, $\mathcal{I}(\operatorname{Spec} A_f) = \mathcal{I}(\operatorname{Spec} A)_f$ (Definition B of quasicoherent sheaves), and this was one criterion for when ideals in affine open sets define a closed subscheme (Exercise 1). (An example of a non-quasicoherent sheaf of ideals was given in an earlier Exercise.)

We call

$$0 \to \mathcal{I}_{X/Y} \to \mathcal{O}_Y \to i_*\mathcal{O}_X \to 0 \tag{1}$$

the **closed subscheme exact sequence** corresponding to $X \hookrightarrow Y$.

Date: Monday, January 28, 2008. Mild correction Feb. 19 (thanks Nathan!).

2. Invertible sheaves (line bundles) and divisors

We next develop some mechanism of understanding invertible sheaves (line bundles) on a given scheme X. Recall that $\operatorname{Pic} X$ is the group of invertible sheaves on X. Our goal will be to develop convenient and powerful ways of describing and working with invertible sheaves.

We begin by describing invertible sheaves on projective space (over a field). We then discuss sheaves of ideals that happen to be invertible (effective Cartier divisors). Partially motivated by this insight that invertible sheaves correspond to "codimension 1 information", we will discuss the theory of Weil divisors, and use this to actually compute $\operatorname{Pic} X$ in a number of circumstances.

3. Some line bundles on projective space

We now describe a family of invertible sheaves on projective space over a field k.

As a warm-up, we begin with the invertible sheaf $\mathcal{O}_{\mathbb{P}^1_k}(1)$ on $\mathbb{P}^1_k = \operatorname{Proj} k[x_0, x_1]$. (The subscript $\mathbb{P}^1_k$ refers to the space on which the sheaf lives, and is often omitted when it is clear from the context.) We describe the invertible sheaf $\mathcal{O}(1)$ using transition functions. It is trivial on the usual affine open sets $U_0 = D(x_0) = \operatorname{Spec} k[x_{1/0}]$ and $U_1 = D(x_1 = \operatorname{Spec} k[x_{0/1}]$. (We continue to use the convention $x_{i/j}$ for describing coordinates on patches of projective space.) Thus the data of a section over U_0 is a polynomial in $x_{1/0}$. The transition function from U_0 to U_1 is multiplication by $x_{0/1} = x_{1/0}^{-1}$. The transition function from U_1 to U_0 is hence multiplication by $x_{1/0} = x_{0/1}^{-1}$.

This information is summarized below:

open cover	$U_0 = \operatorname{Spec} k[x_{1/0}]$	$U_1 = \operatorname{Spec} k[x_{0/1}]$
trivialization and transition functions	$k[x_{1/0}] \underset{\times x_{1/0} = x_{0/1}^{-1}}{\overset{\times x_{0/1} = x_{1/0}^{-1}}{\rightleftarrows}}$	$k[x_{0/1}]$

To test our understanding, let's compute the global sections of $\mathcal{O}(1)$. This will be analogous to our hands-on calculation that $\Gamma(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}) \cong k$. A global section is a polynomial $f(x_{1/0}) \in k[x_{1/0}]$ and a polynomial $g(x_{0/1}) ink[x_{0/1}]$ such that $f(1/x_{0/1})x_{0/1} = g(x_{0/1})$. A little thought will show that f must be linear: $f(x_{1/0}) = a x_{1/0} + b$, and hence $f(x_{0/1}) = a + b x_{0/1}$. Thus

$$\dim \Gamma(\mathbb{P}^1_k, \mathcal{O}(1)) = 2 \neq 1 = \dim \Gamma(\mathbb{P}^1_k, \mathcal{O}).$$

Thus $\mathcal{O}(1)$ is not isomorphic to $\mathcal{O}$, and we have constructed our first (proved) example of a nontrivial line bundle!

We next define more generally $\mathcal{O}_{\mathbb{P}^1_k}(n)$ on $\mathbb{P}^1_k$. It is defined in the same way, except that the transition functions are the nth powers of those for $\mathcal{O}(1)$.

open cover	$U_0 = \operatorname{Spec} k[x_{1/0}]$ $\quad$ $U_1 = \operatorname{Spec} k[x_{0/1}]$
trivialization and transition functions	$k[x_{1/0}] \underset{\times x_{1/0}^n = x_{0/1}^{-n}}{\overset{\times x_{0/1}^n = x_{1/0}^{-n}}{\rightleftarrows}} k[x_{0/1}]$

In particular, thanks to the explicit transition functions, we see that $\mathcal{O}(n) = \mathcal{O}(1)^{\otimes n}$ (with the obvious meaning if n is negative: $(\mathcal{O}(1)^{\otimes(-n)})^{\vee}$). Clearly also $\mathcal{O}(m) \otimes \mathcal{O}(n) = \mathcal{O}(m + n)$.

3.A. IMPORTANT EXERCISE. Show that $\Gamma(\mathbb{P}^1, \mathcal{O}(n)) = n + 1$ if $n \geq 0$, and 0 otherwise.

Long ago, we warned that sheafification was necessary when tensoring $\mathcal{O}_X$-modules: if $\mathcal{F}$ and $\mathcal{G}$ are two $\mathcal{O}_X$-modules on a ringed space, then it is not necessarily true that $\mathcal{F}(X) \otimes_{\mathcal{O}_X(X)} \mathcal{G}(X) \cong (\mathcal{F} \otimes \mathcal{G})(X)$. We now have an example: let $X = \mathbb{P}^1_k$, $\mathcal{F} = \mathcal{O}(1)$, $\mathcal{G} = \mathcal{O}(-1)$.

3.B. EXERCISE. Show that if $m \neq n$, then $\mathcal{O}(m) \not\cong \mathcal{O}(n)$. Hence conclude that we have an injection of groups $\mathbb{Z} \hookrightarrow \operatorname{Pic} \mathbb{P}^1_k$ given by $n \mapsto \mathcal{O}(n)$.

It is useful to identify the global sections of $\mathcal{O}(n)$ with the homogeneous polynomials of degree n in x_0 and x_1, i.e. with the degree n part of $k[x_0, x_1]$. Can you see this from your solution to Exercise 3.A? We will see that this identification is natural in many ways. For example, we will later see that the definition of $\mathcal{O}(n)$ doesn't depend on a choice of affine cover, and this polynomial description is also independent of cover. As an immediate check of the usefulness of this point of view, ask yourself: where does the section $x_0^3 - x_0 x_1^2$ of $\mathcal{O}(3)$ vanish? The section $x_0 + x_1$ of $\mathcal{O}(1)$ can be multiplied by the section x_0^2 of $\mathcal{O}(2)$ to get a section of $\mathcal{O}(3)$. Which one? Where does the rational section $x_0^4(x_1 + x_0)/x_1^7$ of $\mathcal{O}(-2)$ have zeros and poles, and to what order? (We will rigorously define the meaning of zeros and poles shortly, but you should already be able to intuitively answer these questions.)

We now define the invertible sheaf $\mathcal{O}_{\mathbb{P}^m_k}(n)$ on the projective space $\mathbb{P}^m_k$. On the usual affine open set $U_i = \operatorname{Spec} k[x_{0/i}, \dots, x_{m/i}]/(x_{i/i} - 1) = \operatorname{Spec} A_i$, it is trivial, so sections (as an A_i-module) are isomorphic to A_i. The transition function from U_i to U_j is multiplication by $x_{i/j}^n = x_{j/i}^{-n}$. Note that these transition functions clearly satisfy the cocycle condition.

$$U_i = \operatorname{Spec} k[x_{0/i}, \dots, x_{m/i}]/(x_{i/i} - 1) \qquad U_j = \operatorname{Spec} k[x_{0/j}, \dots, x_{m/j}]/(x_{j/j} - 1)$$

$$k[x_{0/i}, \dots, x_{m/i}]/(x_{i/i} - 1) \underset{\times x_{j/i}^n = x_{i/j}^{-n}}{\overset{\times x_{i/j}^n = x_{j/i}^{-n}}{\rightleftarrows}} \operatorname{Spec} k[x_{0/j}, \dots, x_{m/j}]/(x_{j/j} - 1)$$

3.C. ESSENTIAL EXERCISE. Show that $\dim_k \Gamma(\mathbb{P}^m_k, \mathcal{O}_{\mathbb{P}^m_k}(n)) = \binom{m+n}{n}$.

You will notice that, as in the $\mathbb{P}^1$ case, sections of $\mathcal{O}(m)$ on $\mathbb{P}^n_k$ are naturally identified with forms degree m polynomials in our $n+1$ variables. Thus $x+y+2z$ is a section of $\mathcal{O}(1)$ on $\mathbb{P}^2$. It isn't a function, but I can say where this section vanishes — precisely where $x+y+2z=0$.

Also, notice that for fixed n, $\binom{m+n}{n}$ is a polynomial in m of degree n *for* $m \geq 0$ (or better: for $m \geq -n-1$). This should be telling you that this function "wants to be a polynomial" but has not succeeded. We will later define $h^0(\mathbb{P}^n_k, \mathcal{O}(m)) := \Gamma(\mathbb{P}^n_k, \mathcal{O}(m))$, and later still we will define higher cohomology groups, and we will define the *Euler characteristic* $\chi(\mathbb{P}^n_k, \mathcal{O}(m)) := \sum_{i=0}^{\infty}(-1)^i h^i(\mathbb{P}^n_k, \mathcal{O}(m))$ (cohomology will vanish in degree higher than n). We will discover the moral that the Euler characteristic is better-behaved than h^0, and so we should now suspect (and later prove) that this polynomial is in fact the Euler characteristic, and the reason that it agrees with h^0 for $m \geq 0$ because all the other cohomology groups should vanish.

We finally note that we can define $\mathcal{O}(n)$ on $\mathbb{P}^m_A$ for any ring A: the above definition applies without change.

4. EFFECTIVE CARTIER DIVISORS "=" INVERTIBLE IDEAL SHEAVES

In the previous section, we produced a number of interesting invertible sheaves on projective space by explicitly giving transition functions. We now give a completely different means of describing invertible sheaves on a scheme.

Suppose $D \hookrightarrow X$ is a closed subscheme such that corresponding ideal sheaf $\mathcal{I}$ is an invertible sheaf. Then D is called an *effective Cartier divisor*. Suppose D is an effective Cartier divisor. Then $\mathcal{I}$ is locally trivial; suppose U is a trivializing affine open set $\operatorname{Spec} A$. Then the closed subscheme exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_D \to 0$$

corresponds to

$$0 \to I \to A \to A/I \to 0$$

with $I \cong A$ as an A-module. Thus I is generated by a single element, say a, and this exact sequence starts as

$$0 \longrightarrow A \xrightarrow{\times a} A$$

As multiplication by a is injective, a is not a zero-divisor. We conclude that D is locally cut out by a single equation, that is not a zero-divisor. This was the definition of effective Cartier divisor given before. This argument is clearly reversible, so we now have a quick new definition of effective Cartier divisor (that $\mathcal{I}$ is invertible).

4.A. EASY EXERCISE. Show that a is unique up to multiplication by a unit.

In the case where X is locally Noetherian, and we can use the language of associated points, we can restate this definition as: D is locally cut out by a single equation, not vanishing at any associated point of X.

We now define an invertible sheaf corresponding to D. The seemingly obvious definition would be to take $\mathcal{I}_D$, but instead we define the invertible sheaf $\mathcal{O}(D)$ corresponding to an effective Cartier divisor to be the *dual*: $\mathcal{I}_D^\vee$. The ideal sheaf itself is sometimes denoted $\mathcal{O}(-D)$. We have an exact sequence

$$0 \to \mathcal{O}(-D) \to \mathcal{O} \to \mathcal{O}_D \to 0.$$

The invertible sheaf $\mathcal{O}(D)$ has a canonical section s_D: Tensoring $0 \to \mathcal{I} \to \mathcal{O}$ with $\mathcal{I}^\vee$ gives us $\mathcal{O} \to \mathcal{I}^\vee$. (Easy unimportant fact to check: instead of tensoring $\mathcal{I} \to \mathcal{O}$ with $\mathcal{I}^\vee$, we could have dualized $\mathcal{I} \to \mathcal{O}$, and we would get the same section.)

4.B. SURPRISINGLY TRICKY EXERCISE. Recall that a section of a locally free sheaf on X cuts out a closed subscheme of X. Show that this section s_D cuts out D.

This construction has a converse.

4.C. EXERCISE. Suppose $\mathcal{L}$ is an invertible sheaf, and s is a section that is not locally a zero divisor (make sense of this!). Show that $s = 0$ cuts out an effective Cartier divisor D, and $\mathcal{O}(D) \cong \mathcal{L}$. (Again, if X is locally Noetherian, "not locally a zero divisor" translate to "does not vanish at an associated point".)

4.D. EXERCISE. Suppose $\mathcal{I}$ and $\mathcal{J}$ are invertible ideal sheaves (hence corresponding to effective Cartier divisors, say D and D' respectively). Show that $\mathcal{I}\mathcal{J}$ is an invertible ideal sheaf. (First make sense of this notation!) We define the corresponding Cartier divisor to be $D + D'$. Verify that $\mathcal{O}(D + D') \cong \mathcal{O}(D) \otimes \mathcal{O}(D')$.

Thus the effective Cartier divisors form a semigroup. Thus we have a map of semigroups, from effective Cartier divisors to invertible sheaves with sections not locally zero-divisors (and hence also to the Picard group of invertible sheaves).

Hence we can get a bunch of invertible sheaves, by taking differences of these two. In fact we "usually get them all"! It is very hard to describe an invertible sheaf on a finite type k-scheme that is not describable in such a way. For example, we will see soon that there are none if the scheme is nonsingular or even factorial. We will see later that there are none if X is quasiprojective. over a field.

We thus have an important correspondence between *effective Cartier divisors* (closed subschemes whose ideal sheaves are invertible, or equivalently locally cut out by one non-zero-divisor, or in the locally Noetherian case locally cut out by one equation not vanishing at an associated point) and ordered pairs $(\mathcal{L}, s)$ where $\mathcal{L}$ is an invertible sheaf, and s is a section that is not locally a zero-divisor (or in the locally Noetherian case, not vanishing at an associated point). This is an isomorphism of semigroups.

An effective Cartier divisor is pure codimension 1 by Krull's Principal Ideal Theorem. This correspondence of "invertible sheaf with section" with "codimension one information" is a powerful theme that we will explore further in the next section.

E-mail address: vakil@math.stanford.edu