FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 26

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CONTENTS

1. MODULE-LIKE CONSTRUCTIONS

In a similar way, basically any nice construction involving modules extends to quasicoherent sheaves.

As an important example, we consider tensor products.

1.A. EXERCISE. If $\mathcal{F}$ and $\mathcal{G}$ are quasicoherent sheaves, show that $\mathcal{F} \otimes \mathcal{G}$ is a quasicoherent sheaf described by the following information: If $\operatorname{Spec} A$ is an affine open, and $\Gamma(\operatorname{Spec} A, \mathcal{F}) = M$ and $\Gamma(\operatorname{Spec} A, \mathcal{G}) = N$, then $\Gamma(\operatorname{Spec} A, \mathcal{F} \otimes \mathcal{G}) = M \otimes N$, and the restriction map $\Gamma(\operatorname{Spec} A, \mathcal{F} \otimes \mathcal{G}) \to \Gamma(\operatorname{Spec} A_f, \mathcal{F} \otimes \mathcal{G})$ is precisely the localization map $M \otimes_A N \to (M \otimes_A N)_f \cong M_f \otimes_{A_f} N_f$. (We are using the algebraic fact that $(M \otimes_R N)_f \cong M_f \otimes_{R_f} N_f$. You can prove this by universal property if you want, or by using the explicit construction.)

Note that thanks to the machinery behind the distinguished affine base, sheafification is taken care of. This is a feature we will use often: constructions involving quasicoherent sheaves that involve sheafification for general sheaves don't require sheafification when considered on the distinguished affine base. Along with the fact that injectivity, surjectivity, kernels and so on may be computed on affine opens, this is the reason that it is particularly convenient to think about quasicoherent sheaves in terms of affine open sets.

1.B. EASY EXERCISE. Show that the stalk of the tensor product of quasicoherent sheaves at a point is the tensor product of the stalks.

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Given a section s of $\mathcal{F}$ and a section t of $\mathcal{G}$, we have a section $s \otimes t$ of $\mathcal{F} \otimes \mathcal{G}$. If either $\mathcal{F}$ or $\mathcal{G}$ is an invertible sheaf, this section is denoted st.

1.1. Tensor algebra constructions.

For the next exercises, recall the following. If M is an A-module, then the *tensor algebra* $T^*(M)$ is a non-commutative algebra, graded by $\mathbb{Z}^{\geq 0}$, defined as follows. $T^0(M) = A$, $T^n(M) = M \otimes_A \cdots \otimes_A M$ (where n terms appear in the product), and multiplication is what you expect. The *symmetric algebra* $\operatorname{Sym}^* M$ is a symmetric algebra, graded by $\mathbb{Z}^{\geq 0}$, defined as the quotient of $T^*(M)$ by the (two-sided) ideal generated by all elements of the form $x \otimes y - y \otimes x$ for all $x, y \in M$. Thus $\operatorname{Sym}^n M$ is the quotient of $M \otimes \cdots \otimes M$ by the relations of the form $m_1 \otimes \cdots \otimes m_n - m_1' \otimes \cdots \otimes m_n'$ where $(m_1', \ldots, m_n')$ is a rearrangement of $(m_1, \ldots, m_n)$. The *exterior algebra* $\wedge^* M$ is defined to be the quotient of $T^* M$ by the (two-sided) ideal generated by all elements of the form $x \otimes y + y \otimes x$ for all $x, y \in M$. Thus $\wedge^n M$ is the quotient of $M \otimes \cdots \otimes M$ by the relations of the form $m_1 \otimes \cdots \otimes m_n - (-1)^{\operatorname{sgn}(\sigma)} m_{\sigma(1)} \otimes \cdots \otimes m_{\sigma(n)}$ where σ is a permutation of $\{1, \ldots, n\}$. It is a "skew-commutative" A-algebra. It is most correct to write $T^*_A(M)$, $\operatorname{Sym}^*_A(M)$, and $\wedge^*_A(M)$, but the "base ring" A is usually omitted for convenience. (Better: both Sym and $\wedge$ are defined by universal properties. For example, $\operatorname{Sym}^n_A(M)$ is universal among modules such that any map of A-modules $M^{\otimes n} \to N$ that is symmetric in the n entries factors uniquely through $\operatorname{Sym}^n_A(M)$.)

1.C. EXERCISE. Suppose $\mathcal{F}$ is a quasicoherent sheaf. Define the quasicoherent sheaves $\operatorname{Sym}^n \mathcal{F}$ and $\wedge^n \mathcal{F}$. (One possibility: describe them on each affine open set.) If $\mathcal{F}$ is locally free of rank m, show that $T^n \mathcal{F}$, $\operatorname{Sym}^n \mathcal{F}$, and $\wedge^n \mathcal{F}$ are locally free, and find their ranks.

You can also define the sheaf of non-commutative algebras $T^* \mathcal{F}$, the sheaf of commutative algebras $\operatorname{Sym}^* \mathcal{F}$, and the sheaf of skew-commutative algebras $\wedge^* \mathcal{F}$.

1.D. EXERCISE (POSSIBLE HELP FOR LATER PROBLEMS). Suppose $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is a short exact sequence of locally free sheaves on X. Suppose $U = \operatorname{Spec} A$ is an affine open set where $\mathcal{F}'$, $\mathcal{F}''$ are free, say $\mathcal{F}'|_{\operatorname{Spec} A} = \tilde{A}^a$, $\mathcal{F}''|_{\operatorname{Spec} A} = \tilde{A}^b$. Show that $\mathcal{F}$ is also free, and that $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ can be interpreted as coming from the tautological exact sequence $0 \to A^a \to A^{a+b} \to A^b \to 0$. Show that given such an open cover, the transition matrices of $\mathcal{F}$ may be interpreted as block upper-diagonal matrices, where thet top $a \times a$ block are transition matrices for $\mathcal{F}'$, and the bottom $b \times b$ blocks are transition matrices for $\mathcal{F}''$.

1.E. IMPORTANT EXERCISE. Suppose $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of locally free sheaves. Show that for any r, there is a filtration of $\operatorname{Sym}^r \mathcal{F}$

$$\operatorname{Sym}^r \mathcal{F} = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^r \supset F^{r+1} = 0$$

with subquotients

$$F^p / F^{p+1} \cong (\operatorname{Sym}^p \mathcal{F}') \otimes (\operatorname{Sym}^{r-p} \mathcal{F}'').$$

(Possible hint for this, and Exercise 1.G: It suffices to consider a small enough affine open set $\operatorname{Spec} A$, where $\mathcal{F}'$, $\mathcal{F}$, $\mathcal{F}''$ are free, and to show that your construction behaves well with respect to localization at an element $f \in A$. In such an open set, the sequence is $0 \to A^p \to A^{p+q} \to A^q \to 0$ by the Exercise 1.D. Let $e_1, \dots, e_n$ be the standard basis of A^n, and $f_1, \dots, f_q$ be the the standard basis of A^q. Let $e'_1, \dots, e'_p$ be denote the images of e_1, $\dots$, e_p in A^{p+q}. Let $f'_1, \dots, f'_q$ be any lifts of $f_1, \dots, f_q$ to A^{p+q}. Note that f'_i is well-defined modulo $e'_1, \dots, e'_p$. Note that

$$\operatorname{Sym}^s \mathcal{F}|_{\operatorname{Spec} A} \cong \oplus_{i=0}^s \operatorname{Sym}^i \mathcal{F}'|_{\operatorname{Spec} A} \otimes_{\mathcal{O}_{\operatorname{Spec} A}} \operatorname{Sym}^{s-i} \mathcal{F}''|_{\operatorname{Spec} A}.$$

Show that $\mathcal{F}^p := \oplus_{i=p}^s \operatorname{Sym}^i \mathcal{F}'|_{\operatorname{Spec} A} \otimes_{\mathcal{O}_{\operatorname{Spec} A}} \operatorname{Sym}^{s-i} \mathcal{F}''|_{\operatorname{Spec} A}$ gives a well-defined (locally free) subsheaf that is independent of the choices made, e.g. of the basis $e_1, \dots, e_p$ (this is in $GL_p(A)$), $f_1, \dots, f_q$ (this is in $GL_q(A)$), and the lifts $f'_1, \dots, f'_q$.)

1.F. EXERCISE. Suppose $\mathcal{F}$ is locally free of rank n. Then $\wedge^n \mathcal{F}$ is called the **determinant (line) bundle** or (perhaps better) **determinant locally free sheaf**. Show that $\wedge^r \mathcal{F} \times \wedge^{n-r} \mathcal{F} \to \wedge^n \mathcal{F}$ is a perfect pairing for all r.

1.G. EXERCISE. Suppose $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of locally free sheaves. Show that for any r, there is a filtration of $\wedge^r \mathcal{F}$:

$$\wedge^r \mathcal{F} = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^r \supset F^{r+1} = 0$$

with subquotients

$$F^p / F^{p+1} \cong (\wedge^p \mathcal{F}') \otimes (\wedge^{r-p} \mathcal{F}'')$$

for each p. In particular, $\det \mathcal{F} = (\det \mathcal{F}') \otimes (\det \mathcal{F}'')$. In fact we only need that $\mathcal{F}''$ is locally free.

1.H. EXERCISE (DETERMINANT LINE BUNDLES BEHAVE WELL IN EXACT SEQUENCES). Suppose $0 \to \mathcal{F}_1 \to \cdots \to \mathcal{F}_n \to 0$ is an exact sequence of finite rank locally free sheaves on X. Show that "the alternating product of determinant bundles is trivial":

$$\det(\mathcal{F}_1) \otimes \det(\mathcal{F}_2)^\vee \otimes \det(\mathcal{F}_3) \otimes \det(\mathcal{F}_4)^\vee \otimes \cdots \cong \mathcal{O}_X.$$

1.2. Torsion-free sheaves (a stalk-local condition). Recall that an A-module M is torsion-free if $rm = 0$ implies $r = 0$ or $m = 0$. An $\mathcal{O}_X$-module $\mathcal{F}$ is said to be **torsion-free** if $\mathcal{F}_p$ is a torsion-free $\mathcal{O}_{X,p}$-module for all p.

1.I. EXERCISE. Show that if M is a torsion-free A-module, then so is any localization of M. Hence show that $\tilde{M}$ is a torsion free sheaf on $\operatorname{Spec} A$.

1.J. UNIMPORTANT EXERCISE (TORSION-FREENESS IS NOT AFFINE LOCAL FOR STUPID REASONS). Find an example on a two-point space showing that $M := A$ might not be torsion-free on $\operatorname{Spec} A$ even though $\mathcal{O}_{\operatorname{Spec} A} = \tilde{M}$ is torsion-free.

2. Finiteness conditions on quasicoherent sheaves: finite type quasicoherent sheaves, and coherent sheaves

There are some natural finiteness conditions on an A-module M. I will tell you three. In the case when A is a Noetherian ring, which is the case that almost all of you will ever care about, they are all the same.

The first is the most naive: a module could be **finitely generated**. In other words, there is a surjection $A^p \to M \to 0$.

The second is reasonable too. It could be finitely presented — it could have a finite number of generators with a finite number of relations: there exists a **finite presentation**

$$A^q \to A^p \to M \to 0.$$

The third notion is frankly a bit surprising, and I'll justify it soon. We say that an A-module M is **coherent** if (i) it is finitely generated, and (ii) whenenver we have a map $A^p \to M$ (not necessarily surjective!), the kernel is finitely generated.

Clearly coherent implies finitely presented, which in turn implies finitely generated.

2.1. *Proposition. — If A is Noetherian, then these three definitions are the same.*

Before proving this, we take this as an excuse to develop some algebraic background.

2.2. *Noetherian conditions for modules.* If A is any ring, not necessarily Noetherian, we say an A-module is Noetherian if it satisfies the ascending chain condition for submodules. Thus for example A is a Noetherian ring if and only if it is a Noetherian A-module.

2.A. Exercise. Show that if M is a Noetherian A-module, then any submodule of M is a finitely generated A-module.

2.B. Exercise. If $0 \to M' \to M \to M'' \to 0$ is exact, show that M' and M'' are Noetherian if and only if M is Noetherian. (Hint: Given an ascending chain in M, we get two simultaneous ascending chains in M' and M''. Possible further hint: prove that if $M' \longrightarrow M \xrightarrow{\phi} M''$ is exact, and $N, N' \subset M$, and $N \cap M' = N' \cap M'$ and $\phi(N) = \phi(N')$, then $N = N'$.)

2.C. Exercise. Show that if A is a Noetherian ring, then A^n is a Noetherian A-module.

2.D. Exercise. Show that if A is a Noetherian ring and M is a finitely generated A-module, then M is a Noetherian module. Hence by Exercise 2.A, any submodule of a finitely generated module over a Noetherian ring is finitely generated.

Proof of Proposition 2.1. As we observed earlier, coherent implies finitely presented implies finitely generated. So suppose M is finitely generated. Take any $A^p \xrightarrow{\alpha} M$. Then $\ker \alpha$ is a submodule of a finitely generated module over A, and is thus finitely generated by Exercise 2.D. Thus M is coherent. □

Hence most normal people can think of these three notions as the same thing.

2.3. *Proposition. — The coherent* A*-modules form an abelian subcategory of the category of* A*-modules.*

The proof in general is given in §3 in a series of short exercises.

Proof if A *is Noetherian.* Recall from our discussion a few classes ago that we must check three things:

(i) The 0-sheaf is coherent.
(ii) The category of coherent modules is closed under finite sums.
(iii) The category of coherent modules is closed under kernels and cokernels

The first two are clear. For (iii), suppose that $f : M \to N$ is a map of finitely generated modules. Then $\operatorname{coker} f$ is finitely generated (it is the image of N), and $\ker f$ is too (it is a submodule of a finitely generated module over a Noetherian ring, Exercise 2.D). □

2.E. EASY EXERCISE (ONLY IMPORTANT FOR NON-NOETHERIAN PEOPLE). Show A is coherent as an A-module if and only if the notion of finitely presented agrees with the notion of coherent.

2.F. EXERCISE. If $f \in A$, show that if M is a finitely generated (resp. finitely presented, coherent) A-module, then M_f is a finitely generated (resp. finitely presented, coherent) A_f-module. (The "coherent" case is the tricky one.)

2.G. EXERCISE. If $(f_1, \ldots, f_n) = A$, and M_{f_i} is finitely generated (resp. coherent) A_{f_i}-module for all i, then M is a finitely generated (resp. coherent) A-module.

Definition. A quasicoherent sheaf $\mathcal{F}$ is **finite type** (resp. **coherent**) if for every affine open $\operatorname{Spec} A$, $\Gamma(\operatorname{Spec} A, \mathcal{F})$ is a finitely generated (resp. coherent) A-module.

Thanks to the affine communication lemma, and the two previous exercises 2.F and 2.G, it suffices to check this on the open sets in a single affine cover.

I want to say a few words on the notion of coherence. I see Proposition 2.3 as a good motivation for this definition: it gives a small (in a non-technical sense) abelian category in which we can think about vector bundles.

There are two sorts of people who should care about the details of this definition (as opposed to working in a Noetherian world and always thinking that coherent equals finite type). Complex geometers should care. They consider complex-analytic spaces with the classical topology. One can define the notion of coherent $\mathcal{O}_X$-module in a way analogous to this. Then Oka's theorem states that the structure sheaf is coherent, and this is very hard.

The second sort of people who should care are the sort of arithmetic people who sometimes are forced to consider non-Noetherian rings. For example, the ring of *adeles* is non-Noetherian.

Warning: it is common in the later literature to incorrectly define coherent as finitely generated. Please only use the correct definition, as the wrong definition only causes confusion. I will try to be scrupulous about this. Besides doing this for the reason of honesty, it will also help you see what hypotheses are actually necessary to prove things. And that always helps you remember what the proofs are — and hence why things are true.

3. Coherent modules over non-Noetherian rings $\star\star$

This section is intended for people who might work with non-Noetherian rings, or who otherwise might want to understand coherent sheaves in a more general setting. Read this only if you really want to!

Suppose A is a ring. Recall that an A-module M is *finitely generated* if there is a surjection $A^n \to M \to 0$. It is *finitely presented* if there is a presentation $A^m \to A^n \to M \to 0$. And M is *coherent* if (i) M is finitely generated, and (ii) every map $A^n \to M$ has a finitely generated kernel. The reason we like this third definition is that coherent modules form an abelian category.

Here are some quite accessible exercises working out why these notions behave well. Some repeat earlier discussion in order to keep this section self-contained.

3.A. Exercise. Show that coherent implies finitely presented implies finitely generated. (This was discussed in the previous section.)

3.B. Exercise. Show that 0 is coherent.

Suppose for problems 3.C–3.I that

$$0 \to M \to N \to P \to 0 \tag{1}$$

is an exact sequence of A-modules. In thise series of problems, we will show that if two of (1) are coherent, the third is as well, whiich will prove very useful.

Hint $\star$**.** Here is a *hint* which applies to several of the problems: try to write

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^p & \longrightarrow & A^{p+q} & \longrightarrow & A^q & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & P & \longrightarrow & 0 \end{array}$$

and possibly use the Snake Lemma.

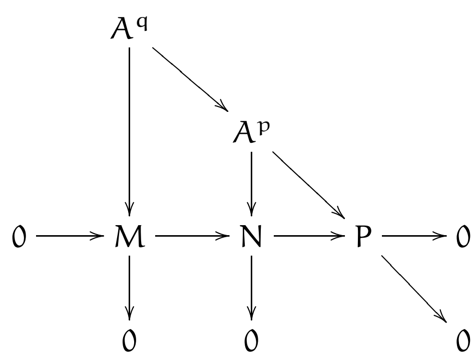
3.C. EXERCISE. Show that N finitely generated implies P finitely generated. (You will only need right-exactness of (1).)

3.D. EXERCISE. Show that M, P finitely generated implies N finitely generated. (Possible hint: $\star$.) (You will only need right-exactness of (1).)

3.E. EXERCISE. Show that N, P finitely generated need not imply M finitely generated. (Hint: if I is an ideal, we have $0 \to I \to A \to A/I \to 0$.)

3.F. EXERCISE. Show that N coherent, M finitely generated implies M coherent. (You will only need left-exactness of (1).)

3.G. EXERCISE. Show that N, P coherent implies M coherent. Hint for (i):



(You will only need left-exactness of (1).)

3.H. EXERCISE. Show that M finitely generated and N coherent implies P coherent. (Hint for (ii): $\star$.)

3.I. EXERCISE. Show that M, P coherent implies N coherent. (Hint: $\star$.)

3.J. EXERCISE. Show that a finite direct sum of coherent modules is coherent.

3.K. EXERCISE. Suppose M is finitely generated, N coherent. Then if $\phi : M \to N$ is any map, then show that $\operatorname{Im} \phi$ is coherent.

3.L. EXERCISE. Show that the kernel and cokernel of maps of coherent modules are coherent.

At this point, we have verified that coherent A-modules form an abelian subcategory of the category of A-modules. (Things you have to check: 0 should be in this set; it should be closed under finite sums; and it should be closed under taking kernels and cokernels.)

3.M. EXERCISE. Suppose M and N are coherent submodules of the coherent module P. Show that $M + N$ and $M \cap N$ are coherent. (Hint: consider the right map $M \oplus N \to P$.)

3.N. EXERCISE. Show that if A is coherent (as an A-module) then finitely presented modules are coherent. (Of course, if finitely presented modules are coherent, then A is coherent, as A is finitely presented!)

3.O. EXERCISE. If M is finitely presented and N is coherent, show that $\mathrm{Hom}(M, N)$ is coherent. (Hint: Hom is left-exact in its first entry.)

3.P. EXERCISE. If M is finitely presented, and N is coherent, show that $M \otimes N$ is coherent.

3.Q. EXERCISE. If $f \in A$, show that if M is a finitely generated (resp. finitely presented, coherent) A-module, then M_f is a finitely generated (resp. finitely presented, coherent) A_f-module. (Hint: localization is exact.) (This exercise appeared earlier as Exercise 2.F.)

3.R. EXERCISE. Suppose $(f_1, \dots, f_n) = A$. Show that if M_{f_i} is finitely generated for all i, then M is too. (Hint: Say M_{f_i} is generated by $m_{ij} \in M$ as an A_{f_i}-module. Show that the m_{ij} generate M. To check surjectivity $\oplus_{i,j} A \to M$, it suffices to check "on $D(f_i)$" for all i.)

3.S. EXERCISE. Suppose $(f_1, \dots, f_n) = A$. Show that if M_{f_i} is coherent for all i, then M is too. (Hint: if $\phi : A^2 \to M$, then $(\ker \phi)_{f_i} = \ker(\phi_{f_i})$, which is finitely generated for all i. Then apply the previous exercise.)

3.T. EXERCISE. Show that the ring $A := k[x_1, x_2, \dots]$ is not coherent over itself. (Hint: consider $A \to A$ with $x_1, x_2, \dots \mapsto 0$.) Thus we have an example of a finitely presented module that is not coherent; a surjection of finitely presented modules whose kernel is not even finitely generated; hence an example showing that finitely presented modules don't form an abelian category.

4. PLEASANT PROPERTIES OF FINITE TYPE AND COHERENT SHEAVES

4.A. EXERCISE. Suppose $\mathcal{F}$ is a coherent sheaf on X, and $\mathcal{G}$ is a quasicoherent sheaf on X. Show that $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})$. (Hint: Describe it on affine open sets, and show that it behaves well with respect to localization with respect to f. To show that $\mathrm{Hom}_A(M, N)_f \cong$

$\mathrm{Hom}_{A_f}(M_f, N_f)$, take a presentation $A^q \to A^p \to M \to 0$, and apply $\mathrm{Hom}(\cdot, N)$ and localize. You will use the fact that p and q are finite.) If further $\mathcal{G}$ is coherent, show that $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})$ is also coherent. Show that $\underline{\mathrm{Hom}}$ is a left-exact functor in both variables.

Recall that $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{O}_X)$ is called the *dual* of $\mathcal{F}$, and is denoted $\mathcal{F}^\vee$.

4.B. USEFUL EXERCISE: GEOMETRIC NAKAYAMA. Suppose X is a scheme, and $\mathcal{F}$ is a finite type quasicoherent sheaf. Show that if $x \in U \subset X$ is a neighborhood of x in X and $a_1, \dots, a_n \in \mathcal{F}(U)$ so that the images $\overline{a}_1, \dots, \overline{a}_n \in \mathcal{F}_x$ generate $\mathcal{F} \otimes k(x)$, then there is a neighborhood $x \subset V \subset U$ of x so that $a_1|_V, \dots, a_n|_V$ generate $\mathcal{F}|_V$. In particular, if $\mathcal{F}_x \otimes k(x) = 0$, then there exists V such that $\mathcal{F}|_V = 0$.

4.C. LESS IMPORTANT EXERCISE. Suppose $\mathcal{F}$ and $\mathcal{G}$ are finite type sheaves such that $\mathcal{F} \otimes \mathcal{G} \cong \mathcal{O}_X$. Then $\mathcal{F}$ and $\mathcal{G}$ are both invertible (Hint: Nakayama.) This is the reason for the adjective "invertible": these sheaves are the invertible elements of the "monoid of finite type sheaves".

4.1. The support of a finite type sheaf is closed. Recall the definition of *support* of a section of a sheaf, and of a sheaf.

Suppose $\mathcal{F}$ is a sheaf of abelian groups (resp. $\mathcal{O}_X$-module) on a topological space X (resp. ringed space $(X, \mathcal{O}_X)$). Define the **support** of a section s of $\mathcal{F}$ to be

$$\mathrm{Supp}\, s = \{p \in X : s_p \neq 0 \text{ in } \mathcal{F}_p\}.$$

I think of this as saying where s "lives". Define the **support** of $\mathcal{F}$ as

$$\mathrm{Supp}\, \mathcal{F} = \{p \in X : \mathcal{F}_p \neq 0\}.$$

It is the union of "all the supports of sections on various open sets". I think of this as saying where $\mathcal{F}$ "lives". *Caution.* This is where the *germ(s)* are nonzero, not where the *value(s)* are nonzero.

Support is a stalk-local notion, and hence behaves well with respect to restriction to open sets, or to stalks.

4.D. EXERCISE. The support of a finite type quasicoherent sheaf on a scheme X is a closed subset. (Hint: Reduce to the case X affine. Choose a finite set of generators of the corresponding module.) Show that the support of a quasicoherent sheaf need not be closed. (Hint: If $A = \mathbb{C}[t]$, then $\mathbb{C}[t]/(t-a)$ is an A-module supported at a. Consider $\oplus_{a \in \mathbb{C}} \mathbb{C}[t]/(t-a)$. Warning: this example won't work if $\oplus$ is replaced by $\prod$, so be careful!)

4.2. Rank of a finite type sheaf at a point.

The **rank** $\mathcal{F}$ of a finite type sheaf at a point p is $\dim_k \mathcal{F}_p / \mathfrak{m}\mathcal{F}_p$ where $\mathfrak{m}$ is the maximal ideal corresponding to p. More explicitly, on any affine set $\mathrm{Spec}\, A$ where $p = [\mathfrak{p}]$ and $\mathcal{F}(\mathrm{Spec}\, A) = M$, then the rank is $\dim_{FF(A/\mathfrak{p})} M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p}$. The rank is finite because of the finite type hypothesis. By Nakayama's lemma (again using the finite type condition), this is the minimal number of generators of $M_{\mathfrak{p}}$ as an $A_{\mathfrak{p}}$-module.

If $\mathcal{F}$ is quasicoherent (not necessarily finite type), then $\mathcal{F}_p/\mathfrak{m}\mathcal{F}_p$ can be interpreted as the fiber of the sheaf at the point. A section of $\mathcal{F}$ over an open set containing p can be said to take on a value at that point, which is an element of $\mathcal{F}_p/\mathfrak{m}\mathcal{F}_p$.

4.E. EXERCISE. Show that at any point, $\operatorname{rank}(\mathcal{F} \oplus \mathcal{G}) = \operatorname{rank}(\mathcal{F}) + \operatorname{rank}(\mathcal{G})$ and $\operatorname{rank}(\mathcal{F} \otimes \mathcal{G}) = \operatorname{rank} \mathcal{F} \operatorname{rank} \mathcal{G}$ at any point. (Hint: Show that direct sums and tensor products commute with ring quotients and localizations, i.e. $(M \oplus N) \otimes_R (R/I) \cong M/IM \oplus N/IN$, $(M \otimes_R N) \otimes_R (R/I) \cong (M \otimes_R R/I) \otimes_{R/I} (N \otimes_R R/I) \cong M/IM \otimes_{R/I} N/IM$, etc.)

4.F. EXERCISE. Show that $\operatorname{rank}(\mathcal{F})$ is an upper semicontinuous function on X. (Hint: Generators at P are generators nearby.)

Note that this definition of rank is consistent with the notion of rank of a locally free sheaf. In the locally free case, the rank is a (locally) constant function of the point. The converse is sometimes true, as is shown in Exercise 4.G below.

4.G. IMPORTANT HARD EXERCISE. (a) If X is reduced, $\mathcal{F}$ is coherent, and the rank is constant, show that $\mathcal{F}$ is locally free. (Hint: choose a point $p \in X$, and choose generators of the stalk $\mathcal{F}_p$. Let U be an open set where the generators are sections, so we have a map $\phi : \mathcal{O}_U^{\oplus n} \to \mathcal{F}|_U$. The cokernel and kernel of ϕ are supported on closed sets by Exercise 4.D. Show that these closed subsets don't include p. Make sure you use the reduced hypothesis!) Thus (as rank is uppersemicontinuous, Exercise 4.F) coherent sheaves are locally free on a dense open set. Hint: Reduce to the case where X is affine, say $\operatorname{Spec} A$, so the closed points are dense. Then show it in a neighborhood of a closed point $[\mathfrak{m}]$. Choose $m_1, \dots, m_n$ generators of $M/\mathfrak{m}M$, and lift them to elements of M. Then they generate M, by Nakayama's Lemma. Let $\phi : A^n \to M$ with $(r_1, \dots, r_n) \mapsto \sum r_i m_i$. Let K be the cokernel, which is finitely generated. Then $K_{\mathfrak{m}} = 0$ (because $\otimes A_{\mathfrak{m}}$ is right-exact), so there is an $f \in A$ such that $K_f = 0$ (take the product of the annihilators of a finite generating set of K). Replace A by A_f. We now have that $\operatorname{coker} \phi = 0$, and we want to prove $\ker \phi = 0$. Otherwise, say $(r_1, \dots, r_n)$ is in the kernel, with $r_1 \neq 0$. As $r_1 \neq 0$, there is some $\mathfrak{p}$ where $r_1 \notin \mathfrak{p}$ — here we use the reduced hypothesis. Then r_1 is invertible in $A_{\mathfrak{p}}$, so $M_{\mathfrak{p}}$ has fewer than n generators, contradicting the constancy of rank.
(b) Show that part (a) can be false without the condition of X being reduced. (Hint: $\operatorname{Spec} k[x]/x^2$, $M = k$.)

You can use the notion of rank to help visualize finite type sheaves, or even quasicoherent sheaves. I drew some pictures in class, but I haven't figured out yet how to latex them up.

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