FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 24

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CONTENTS

Quasicoherent and coherent sheaves are natural generalizations of the notion of a vector bundle. In order to help motivate them, we first discuss vector bundles, and how they can be interpreted in terms of locally free shaves.

In a nutshell, a **free sheaf** on X is an $\mathcal{O}_X$-module isomorphic to $\mathcal{O}_X^{\oplus I}$ where the sum is over some index set I. A **locally free sheaf** X is an $\mathcal{O}_X$-module locally isomorphic to a free sheaf. This corresponds to the notion of a vector bundle. A **quasicoherent sheaf on** X may be defined as an $\mathcal{O}_X$-module which may be locally written as the cokernel of a map of free sheaves. These definitions are useful for ringed spaces in general. We will instead start with two other definitions of quasicoherent sheaf which better highlight the parallel between this notion and that of modules over a ring, and make it easy to work with a scheme by considering an affine cover.

1. VECTOR BUNDLES AND LOCALLY FREE SHEAVES

As motivation, we discuss vector bundles on real manifolds. Examples to keep in mind are the tangent bundle to a manifold, and the Möbius strip over a circle.

Arithmetically-minded readers shouldn't tune out! Fractional ideals of the ring of integers in a number field will turn out to be an example of a "line bundle on a smooth curve".

A *rank* n *vector bundle on a manifold* M is a fibration $\pi : V \to M$ with the structure of an n-dimensional real vector space on $\pi^{-1}(x)$ for each point $x \in M$, such that for every $x \in M$, there is an open neighborhood U and a homeomorphism

$$\phi : U \times \mathbb{R}^n \to \pi^{-1}(U)$$

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over U (so that the diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xleftrightarrow{\;\cong\;} & U \times \mathbb{R}^n \\ & {\scriptstyle \pi|_{\pi^{-1}(U)}} \searrow \qquad \swarrow {\scriptstyle \text{projection to first factor}} & \\ & U & \end{array} \tag{1}$$

commutes) that is an isomorphism of vector spaces over each $y \in U$.

An isomorphism (1) is called a **trivialization over** U.

In this definition, n is called the **rank** of the vector bundle. A rank 1 vector bundle is called a **line bundle**. (It is sometimes convenient to be agnostic about the rank of the vector bundle, so it can have different ranks on different connected components. It is also sometimes convenient to consider infinite-rank vector bundles.)

1.1. Transition functions. Given trivializations over U_1 and U_2, over their intersection, the two trivializations must be related by an element T_{ij} of $GL(n)$ with entries consisting of functions on $U_1 \cap U_2$. If $\{U_i\}$ is a cover of M, and we are given trivializations over each U_i, then the $\{T_{ij}\}$ must satisfy the *cocycle condition*:

$$f_{ij}|_{U_i \cap U_j \cap U_k} \circ f_{jk}|_{U_i \cap U_j \cap U_k} = f_{ik}|_{U_i \cap U_j \cap U_K}. \tag{2}$$

Note that this implies $T_{ij} = T_{ji}^{-1}$. The data of the T_{ij} are called **transition functions** for the trivialization.

Conversely, given the data of a cover $\{U_i\}$ and transition functions T_{ij} (an element of $GL(n)$ with entries that are functions on $U_i \cap U_j$), we can recover the vector bundle (up to unique isomorphism) by "gluing together the $U_i \times \mathbb{R}^n$ along over $U_i \cap U_j$ using f_{ij}".

1.2. Sheaf of sections. Fix a rank n vector bundle $V \to M$. The sheaf of sections $\mathcal{F}$ of V is an $\mathcal{O}_M$-module — given any open set U, we can multiply a section over U by a function on U and get another section.

Moreover, given a U and a trivialization, the sections over U are naturally identified with n-tuples of functions of U.

$$\begin{array}{c} U \times \mathbb{R}^n \\ {\scriptstyle \pi} \downarrow \uparrow {\scriptstyle \mathbf{f} = \text{an } n\text{-tuple of functions}} \\ U \end{array}$$

Thus given a trivialization, over each open set U_i, we have an isomorphism $\mathcal{F}|_{U_i} \cong \mathcal{O}_{U_i}^{\oplus n}$. We say that $\mathcal{F}$ is a **locally free sheaf of rank** n. (As stated earlier, a sheaf $\mathcal{F}$ is **free of rank** n if $\mathcal{F} \cong \mathcal{O}^{\oplus n}$.)

1.3. Transition functions for the sheaf of sections. Suppose we have a vector bundle on M, along with a trivialization over an open cover U_i. Suppose we have a section of the vector bundle over M. (This discussion will apply with M replaced by any open subset.) Then over each U_i, the section corresponds to an n-tuple functions over U_i, say $\mathbf{f}_i$.

1.A. EXERCISE. Show that over $U_i \cap U_j$, the vector-valued function $\mathbf{f}_i$ is related to $\mathbf{f}_j$ by the transition functions:

$$T_{ij}\mathbf{f}_i = \mathbf{f}_j$$

Given a locally free sheaf $\mathcal{F}$ with rank n, and a trivializing neighborhood of $\mathcal{F}$ (an open cover $\{U_i\}$ such that over each U_i, $\mathcal{F}|_{U_i} \cong \mathcal{O}_{U_i}^{\oplus n}$ as $\mathcal{O}$-modules), we have transition functions $T_{ij} \in GL(n, \mathcal{O}(U_i \cap U_j))$ satisfying the cocycle condition (2). Thus in conclusion the data of a locally free sheaf of rank n is equivalent to the data of a vector bundle of rank n.

A rank 1 locally free sheaf is called an **invertible sheaf**. We'll see later why it is called invertible; but it is still a somewhat heinous term for something so fundamental.

1.4. Locally free sheaves on schemes.

Suitably motivated, we now become rigorous and precise. We can generalize the notion of locally free sheaves to schemes without change. A **locally free sheaf of rank n on a scheme** X is an $\mathcal{O}_X$-module $\mathcal{F}$ that is locally trivial of rank n. Precisely, there is an open cover $\{U_i\}$ of X such that for each U_i, $\mathcal{F}|_{U_i} \cong \mathcal{O}_{U_i}^{\oplus n}$. A locally free sheaf may be described in terms of transition functions: the data of a cover $\{U_i\}$ of X, and functions $T_{ij} \in GL(n, \mathcal{O}(U_i \cap U_j))$ satisfying the cocycle condition (2). As before, given this data, we can find the sections over any open set U. Informally, they are sections of the free sheaves over each $U \cap U_i$ that agree on overlaps. More formally, for each i, they are $\vec{s}^i = \begin{pmatrix} s_1^i \\ \vdots \\ s_n^i \end{pmatrix} \in \Gamma(U \cap U_i, \mathcal{O}_X)^n$, satisfying $T_{ij}\vec{s}^i = \vec{s}^j$ on $U \cap U_i \cap U_j$.

You should think of these "as" vector bundles, but just keep in mind that they are not the "same", just equivalent notions. We will later define the "total space" of the vector bundle $V \to X$ (a scheme over X) in terms of the sheaf version of Spec (precisely, $\underline{\operatorname{Spec}} \operatorname{Sym} \mathbf{V}^\bullet$). But the locally free sheaf perspective will prove to be more useful. As one example: the definition of a locally free sheaf is much shorter than that of a vector bundle.

As in our motivating discussion, it is sometimes convenient to let the rank vary among connected components, or to consider infinite rank locally free sheaves.

1.5. Useful constructions.

We now give some useful constructions in the form of a series of exercises. Most will later generalize readily to quasicoherent sheaves.

1.B. EXERCISE. Suppose s is a section of a locally free sheaf $\mathcal{F}$ on a scheme X. Define the notion of the **subscheme cut out by** $s = 0$. (Hint: given a trivialization over an open set

U, s corresponds to a number of functions $f_1, \dots$ on U; on U, take the scheme cut out by these functions.)

1.C. EXERCISE. Suppose $\mathcal{F}$ and $\mathcal{G}$ are locally free sheaves on X of rank m and n respectively. Show that $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})$ is a locally free sheaf of rank mn.

1.D. EXERCISE. If $\mathcal{E}$ is a locally free sheaf of rank n, show that $\mathcal{E}^\vee := \underline{\mathrm{Hom}}(\mathcal{E}, \mathcal{O})$ is also a locally free sheaf of rank n. This is called the **dual** of $\mathcal{E}$. Given transition functions for $\mathcal{E}$, describe transition functions for $\mathcal{E}^\vee$. (Note that if $\mathcal{E}$ is rank 1 (i.e. invertible), the transition functions of the dual are the inverse of the transition functions of the original.) Show that $\mathcal{E} \cong \mathcal{E}^{\vee\vee}$. (Caution: your argument showing that if there is a canonical isomorphism $(\mathcal{F}^\vee)^\vee \cong \mathcal{F}$ better not also show that there is a canonical isomorphism $\mathcal{F}^\vee \cong \mathcal{F}$! We'll see an example soon of a locally free $\mathcal{F}$ that is not isomorphic to its dual. The example will be the line bundle $\mathcal{O}(1)$ on $\mathbb{P}^1$.)

1.E. EXERCISE. If $\mathcal{F}$ and $\mathcal{G}$ are locally free sheaves, show that $\mathcal{F} \otimes \mathcal{G}$ is a locally free sheaf. (Here $\otimes$ is tensor product as $\mathcal{O}_X$-modules, defined last quarter) If $\mathcal{F}$ is an invertible sheaf, show that $\mathcal{F} \otimes \mathcal{F}^\vee \cong \mathcal{O}_X$.

1.F. EXERCISE. Recall that tensor products tend to be only right-exact in general. Show that tensoring by a locally free sheaf is exact. More precisely, if $\mathcal{F}$ is a locally free sheaf, and $\mathcal{G}' \to \mathcal{G} \to \mathcal{G}''$ is an exact sequence of $\mathcal{O}_X$-modules, then then so is $\mathcal{G}' \otimes \mathcal{F} \to \mathcal{G} \otimes \mathcal{F} \to \mathcal{G}'' \otimes \mathcal{F}$.

1.G. EXERCISE. If $\mathcal{E}$ is a locally free sheaf, and $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{O}_X$-modules, show that $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G} \otimes \mathcal{E}) \cong \underline{\mathrm{Hom}}(\mathcal{F} \otimes \mathcal{E}^\vee, \mathcal{G})$.

1.H. EXERCISE AND IMPORTANT DEFINITION. Show that the invertible sheaves on X, up to isomorphism, form an abelian group under tensor product. This is called the **Picard group** of X, and is denoted $\operatorname{Pic} X$. (For arithmetic people: this group, for the Spec of the ring of integers R in a number field, is the class group of R.)

1.6. Random concluding remarks.

We define **rational and regular sections of a locally free sheaf** on a scheme X.

1.I. LESS IMPORTANT EXERCISE. Show that locally free sheaves on Noetherian normal schemes satisfy "Hartogs' theorem": sections defined away from a set of codimension at least 2 extend over that set.

1.7. *Remark.* Based on your intuition for line bundles on manifolds, you might hope that every point has a "small" open neighborhood on which all invertible sheaves (or locally free sheaves) are trivial. Sadly, this is not the case. We will eventually see that for the curve $y^2 - x^3 - x = 0$ in $\mathbb{A}^2_{\mathbb{C}}$, every nonempty open set has nontrivial invertible sheaves. (This will use the fact that it is an open subset of an *elliptic curve*.)

1.J. EXERCISE (FOR ARITHMETICALLY-MINDED PEOPLE ONLY — I WON'T DEFINE MY TERMS). Prove that a fractional ideal on a ring of integers in a number field yields an invertible sheaf. Show that any two that differ by a principal ideal yield the same invertible sheaf. Show that two that yield the same invertible sheaf differ by a principal ideal. The *class group* is defined to be the group of fractional ideals modulo the principal ideals. This exercises shows that the class group is (isomorphic to) the Picard group. (This discussion applies to the ring integers in any global field.)

1.8. The problem with locally free sheaves.

Recall that $\mathcal{O}_X$-modules form an abelian category: we can talk about kernels, cokernels, and so forth, and we can do homological algebra. Similarly, vector spaces form an abelian category. But locally free sheaves (i.e. vector bundles), along with reasonably natural maps between them (those that arise as maps of $\mathcal{O}_X$-modules), don't form an abelian category. As a motivating example in the category of differentiable manifolds, consider the map of the trivial line bundle on $\mathbb{R}$ (with co-ordinate t) to itself, corresponding to multiplying by the co-ordinate t. Then this map jumps rank, and if you try to define a kernel or cokernel you will get yourself confused.

This problem is resolved by enlarging our notion of nice $\mathcal{O}_X$-modules in a natural way, to quasicoherent sheaves.

$\mathcal{O}_X$-modules	$\supset$	quasicoherent sheaves	$\supset$	locally free sheaves
abelian category		abelian category		not an abelian category

Similarly, finite rank locally free sheaves will sit in a nice smaller abelian category, that of *coherent sheaves*.

quasicoherent sheaves	$\supset$	coherent sheaves	$\supset$	finite rank locally free sheaves
abelian category		abelian category		not an abelian category

2. TOWARD QUASICOHERENT SHEAVES: THE DISTINGUISHED AFFINE BASE

Schemes generalize and geometrize the notion of "ring". It is now time to define the corresponding analogue of "module", which is a quasicoherent sheaf.

One version of this notion is that of an $\mathcal{O}_X$-module. They form an abelian category, with tensor products.

We want a better one — a subcategory of $\mathcal{O}_X$-modules. Because these are the analogues of modules, we're going to define them in terms of affine open sets of the scheme. So let's think a bit harder about the structure of affine open sets on a general scheme X. I'm going to define what I'll call the *distinguished affine base* of the Zariski topology. This won't be a base in the sense that you're used to. (For experts: it is a first example of a *Grothendieck topology*.)

The open sets are the affine open subsets of X. We've already observed that this forms a base. But forget about that.

We like distinguished open sets $\operatorname{Spec} A_f \hookrightarrow \operatorname{Spec} A$, and we don't really understand open immersions of one random affine open subset in another. So we just remember the "nice" inclusions.

Definition. The **distinguished affine base** of a scheme X is the data of the affine open sets and the distinguished inclusions.

In other words, we are remembering only some of the open sets (the affine open sets), and only some of the morphisms between them (the distinguished morphisms). For experts: if you think of a topology as a category (the category of open sets), we have described a subcategory.

We can define a sheaf on the distinguished affine base in the obvious way: we have a set (or abelian group, or ring) for each affine open set, and we know how to restrict to distinguished open sets.

Given a sheaf $\mathcal{F}$ on X, we get a sheaf on the distinguished affine base. You can guess where we're going: we'll show that all the information of the sheaf is contained in the information of the sheaf on the distinguished affine base.

As a warm-up, we can recover stalks as follows. (We will be implicitly using only the following fact. We have a collection of open subsets, and *some* subsets, such that if we have any $x \in U, V$ where U and V are in our collection of open sets, there is some W containing x, and contained in U and V such that $W \hookrightarrow U$ and $W \hookrightarrow V$ are both in our collection of inclusions. In the case we are considering here, this is the key fact that given any two affine open sets $\operatorname{Spec} A$, $\operatorname{Spec} B$ in X, $\operatorname{Spec} A \cap \operatorname{Spec} B$ could be covered by affine open sets that were simultaneously distinguished in $\operatorname{Spec} A$ and $\operatorname{Spec} B$. This is a *cofinal* condition.)

The stalk $\mathcal{F}_x$ is the direct limit $\varinjlim(f \in \mathcal{F}(U))$ where the limit is over all open sets contained in U. We compare this to $\varinjlim(f \in \mathcal{F}(U))$ where the limit is over all affine open sets, and all distinguished inclusions. You can check that the elements of one correspond to elements of the other. (Think carefully about this! It corresponds to the fact that the basic elements are cofinal in this directed system.)

2.A. EXERCISE. Show that a section of a sheaf on the distinguished affine base is determined by the section's germs.

2.1. *Theorem.* —

(a) *A sheaf on the distinguished affine base* $\mathcal{F}^b$ *determines a unique sheaf* $\mathcal{F}$*, which when restricted to the affine base is* $\mathcal{F}^b$*. (Hence if you start with a sheaf, and take the sheaf on the distinguished affine base, and then take the induced sheaf, you get the sheaf you started with.)*
(b) *A morphism of sheaves on a distinguished affine base uniquely determines a morphism of sheaves.*
(c) *An* $\mathcal{O}_X$*-module "on the distinguished affine base" yields an* $\mathcal{O}_X$*-module.*

This proof is identical to our argument showing that sheaves are (essentially) the same as sheaves on a base, using the "sheaf of compatible germs" construction. The main reason for repeating it is to let you see that all that is needed is for the open sets to form a cofinal system (or better, that the category of open sets and inclusions we are considering is cofinal).

For experts: (a) and (b) are describing an equivalence of categories between sheaves on the Zariski topology of X and sheaves on the distinguished affine base of X.

Proof. (a) Suppose $\mathcal{F}^b$ is a sheaf on the distinguished affine base. Then we can define stalks.

For *any* open set U of X, define the sheaf of compatible germs

$$\mathcal{F}(U) := \{(f_x \in \mathcal{F}^b_x)_{x \in U} : \forall x \in U, \exists U_x \text{ with } x \subset U_x \subset U, F^x \in \mathcal{F}^b(U_x) : F^x_y = f_y \forall y \in U_x\}$$

where each U_x is in our base, and F^x_y means "the germ of F^x at y". (As usual, those who want to worry about the empty set are welcome to.)

This is a sheaf: convince yourself that we have restriction maps, identity, and gluability, really quite easily.

I next claim that if U is in our base, that $\mathcal{F}(U) = \mathcal{F}^b(U)$. We clearly have a map $\mathcal{F}^b(U) \to \mathcal{F}(U)$. This is an isomorphism on stalks, and hence an isomorphism by an Exercise from last quarter.

2.B. EXERCISE. Prove (b).

2.C. EXERCISE. Prove (c). □

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