FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 22

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CONTENTS

Last day, we discussed the Zariski tangent space, and saw that it was often quite computable. We proved the key inequality $\dim A \leq \dim_k \mathfrak{m}/\mathfrak{m}^2$ for Noetherian local rings $(A, \mathfrak{m})$. When equality holds, we said that the ring was regular (or nonsingular), and we defined the notion of (non)singularity for locally Noetherian schemes.

1. DISCRETE VALUATION RINGS: DIMENSION 1 NOETHERIAN REGULAR LOCAL RINGS

The case of dimension 1 is important, because if you understand how primes behave that are separated by dimension 1, then you can use induction to prove facts in arbitrary dimension. This is one reason why Krull's Principal Ideal Theorem is so useful.

A dimension 1 Noetherian regular local ring can be thought of as a "germ of a smooth curve" (see Figure 1). Two examples to keep in mind are $k[x]_{(x)} = \{f(x)/g(x) : x \not| \, g(x)\}$ and $\mathbb{Z}_{(5)} = \{a/b : 5 \not| \, b\}$.

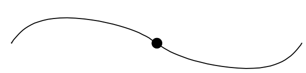

FIGURE 1. A germ of a curve

The purpose of this section is to give a long series of equivalent definitions of these rings. We will eventually have seven equivalent definitions, (a) through (g).

1.1. *Theorem. — Suppose* $(A, \mathfrak{m})$ *is a Noetherian local ring of dimension* 1*. Then the following are equivalent.*

(a) $(A, \mathfrak{m})$ *is regular.*
(b) $\mathfrak{m}$ *is principal.*

Date: Monday, January 14, 2008.

Here is why (a) implies (b). If A is regular, then $\mathfrak{m}/\mathfrak{m}^2$ is one-dimensional. Choose any element $t \in \mathfrak{m} - \mathfrak{m}^2$. Then t generates $\mathfrak{m}/\mathfrak{m}^2$, so generates $\mathfrak{m}$ by Nakayama's lemma. We call such an element a **uniformizer**. Warning: we need to know that $\mathfrak{m}$ is finitely generated to invoke Nakayama — but fortunately we do, thanks to the Noetherian hypothesis.

Conversely, if $\mathfrak{m}$ is generated by one element t over A, then $\mathfrak{m}/\mathfrak{m}^2$ is generated by one element t over $A/\mathfrak{m} = k$. Note that $t \notin \mathfrak{m}^2$, as otherwise $\mathfrak{m} = \mathfrak{m}^2$ and hece $\mathfrak{m} = 0$ by Nakayama's Lemma. $\square$

We will soon use a useful fact, and we may as well prove it in much more generality than we need, because the proof is so short.

1.2. *Proposition. — If* $(A, \mathfrak{m})$ *is a Noetherian local ring, then* $\cap_i \mathfrak{m}^i = 0$.

The geometric intuition for this is that any function that is analytically zero at a point (vanishes to all orders) actually vanishes at that point.

It is tempting to argue that $\mathfrak{m}(\cap_i \mathfrak{m}^i) = \cap_i \mathfrak{m}^i$, and then to use Nakayama's lemma to argue that $\cap_i \mathfrak{m}^i = 0$. Unfortunately, it is not obvious that this first equality is true: product does not commute with infinite intersections in general.

Proof. Let $I = \cap_i \mathfrak{m}^i$. We wish to show that $I \subset \mathfrak{m}I$; then as $\mathfrak{m}I \subset I$, we have $I = \mathfrak{m}I$, and hence by Nakayama's Lemma, $I = 0$. Fix a primary decomposition of $\mathfrak{m}I$. It suffices to show that $\mathfrak{q}$ contains I for any $\mathfrak{q}$ in this primary decomposition, as then I is contained in all the primary ideals in the decomposition of $\mathfrak{m}I$, and hence $\mathfrak{m}I$. Let $\mathfrak{p} = \sqrt{\mathfrak{q}}$.

If $\mathfrak{p} \neq \mathfrak{m}$, then choose $x \in \mathfrak{m} - \mathfrak{p}$. Now x is not nilpotent in $A/\mathfrak{q}$, and hence is not a zero-divisor. (Recall that $\mathfrak{q}$ is primary if and only if in $A/\mathfrak{q}$, each zero-divisor is nilpotent.) But $xI \subset \mathfrak{m}I \subset \mathfrak{q}$, so $I \subset \mathfrak{q}$.

On the other hand, if $\mathfrak{p} = \mathfrak{m}$, then as $\mathfrak{m}$ is finitely generated, and each generator is in $\sqrt{\mathfrak{q}} = \mathfrak{m}$, there is some a such that $\mathfrak{m}^a \subset \mathfrak{q}$. But $I \subset \mathfrak{m}^a$, so we are done. $\square$

1.3. *Proposition. — Suppose* $(A, \mathfrak{m})$ *is a Noetherian regular local ring of dimension* 1 *(i.e. satisfying (a) above). Then* A *is an integral domain.*

Proof. Suppose $xy = 0$, and $x, y \neq 0$. Then by Proposition 1.2, $x \in \mathfrak{m}^i \setminus \mathfrak{m}^{i+1}$ for some $i \geq 0$, so $x = at^i$ for some $a \notin \mathfrak{m}$. Similarly, $y = bt^j$ for some $j \geq 0$ and $b \notin \mathfrak{m}$. As $a, b \notin \mathfrak{m}$, a and b are invertible. Hence $xy = 0$ implies $t^{i+j} = 0$. But as nilpotents don't affect dimension,

$$\dim A = \dim A/(t) = \dim A/\mathfrak{m} = \dim k = 0, \qquad (1)$$

contradicting $\dim A = 1$. $\square$

1.4. *Theorem. — Suppose* $(A, \mathfrak{m})$ *is a Noetherian local ring of dimension* 1*. Then (a) and (b) are equivalent to:*

(c) *all ideals are of the form* $\mathfrak{m}^n$ *or* 0.

Proof. Assume (a): suppose $(A, \mathfrak{m}, k)$ is a Noetherian regular local ring of dimension 1. Then I claim that $\mathfrak{m}^n \neq \mathfrak{m}^{n+1}$ for any n. Otherwise, by Nakayama's lemma, $\mathfrak{m}^n = 0$, from which $t^n = 0$. But A is a domain, so $t = 0$, from which $A = A/\mathfrak{m}$ is a field, which can't have dimension 1, contradiction.

I next claim that $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ is dimension 1. Reason: $\mathfrak{m}^n = (t^n)$. So $\mathfrak{m}^n$ is generated as as a A-module by one element, and $\mathfrak{m}^n/(\mathfrak{m}\mathfrak{m}^n)$ is generated as a $(A/\mathfrak{m} = k)$-module by 1 element (non-zero by the previous paragraph), so it is a one-dimensional vector space.

So we have a chain of ideals $A \supset \mathfrak{m} \supset \mathfrak{m}^2 \supset \mathfrak{m}^3 \supset \cdots$ with $\cap \mathfrak{m}^i = (0)$ (Proposition 1.2). We want to say that there is no room for any ideal besides these, because "each pair is "separated by dimension 1", and there is "no room at the end". Proof: suppose $I \subset A$ is an ideal. If $I \neq (0)$, then there is some n such that $I \subset \mathfrak{m}^n$ but $I \not\subset \mathfrak{m}^{n+1}$. Choose some $u \in I - \mathfrak{m}^{n+1}$. Then $(u) \subset I$. But u generates $\mathfrak{m}^n/\mathfrak{m}^{n+1}$, hence by Nakayama it generates $\mathfrak{m}^n$, so we have $\mathfrak{m}^n \subset I \subset \mathfrak{m}^n$, so we are done. Conclusion: in a Noetherian local ring of dimension 1, regularity implies all ideals are of the form $\mathfrak{m}^n$ or (0).

We now show that (c) implies (a). Assume (a) is false: suppose we have a dimension 1 Noetherian local domain that is not regular, so $\mathfrak{m}/\mathfrak{m}^2$ has dimension at least 2. Choose any $u \in \mathfrak{m} - \mathfrak{m}^2$. Then $(u, \mathfrak{m}^2)$ is an ideal, but $\mathfrak{m} \subsetneq (u, \mathfrak{m}^2) \subsetneq \mathfrak{m}^2$. $\square$

1.A. EASY EXERCISE. Suppose $(A, \mathfrak{m})$ is a Noetherian dimension 1 local ring. Show that (a)–(c) above are equivalent to:

(d) A *is a principal ideal domain.*

1.5. Discrete valuation rings. We next define the notion of a discrete valuation ring. Suppose K is a field. A **discrete valuation** on K is a **surjective homomorphism** $v : K^* \to \mathbb{Z}$ (homomorphism: $v(xy) = v(x) + v(y)$) satisfying

$$v(x + y) \geq \min(v(x), v(y))$$

except if $x + y = 0$ (in which case the left side is undefined). (Such a valuation is called **non-archimedean**, although we will not use that term.) It is often convenient to say $v(0) = \infty$. More generally, a **valuation** is a surjective homomorphism $v : K^* \to G$ to a totally ordered group G, although this isn't so important to us. (Not every valuation is discrete. Consider the ring of Puisseux series over a field k, $K = \cup_{n \geq 1} k((x^{1/n}))$, with $v : K^* \to \mathbb{Q}$ given by $v(x^q) = q$.)

Examples.

(i) *(the* 5*-adic valuation)* $K = \mathbb{Q}$, $v(r)$ is the "power of 5 appearing in r", e.g. $v(35/2) = 1$, $v(27/125) = -3$.
(ii) $K = k(x)$, $v(f)$ is the "power of x appearing in f."

(iii) $K = k(x)$, $\nu(f)$ is the negative of the degree. This is really the same as (ii), with x replaced by $1/x$.

Then $0 \cup \{x \in K^* : \nu(x) \geq 0\}$ is a ring, which we denote $\mathcal{O}_\nu$. It is called the **valuation ring** of ν.

1.B. EXERCISE. Describe the valuation rings in the three examples above. (You will notice that they are familiar-looking dimension 1 Noetherian local rings. What a coincidence!)

1.C. EXERCISE. Show that $0 \cup \{x \in K^* : \nu(x) \geq 1\}$ is the unique maximal ideal of the valuation ring. (Hint: show that everything in the complement is invertible.) Thus the valuation ring is a local ring.

An integral domain A is called a **discrete valuation ring** (or **DVR**) if there exists a discrete valuation ν on its fraction field $K = FF(A)$ for which $\mathcal{O}_\nu = A$.

Now if A is a Noetherian regular local ring of dimension 1, and t is a uniformizer (a generator of $\mathfrak{m}$ as an ideal, or equivalently of $\mathfrak{m}/\mathfrak{m}^2$ as a k-vector space) then any non-zero element r of A lies in some $\mathfrak{m}^n - \mathfrak{m}^{n+1}$, so $r = t^n u$ where u is a unit (as t^n generates $\mathfrak{m}^n$ by Nakayama, and so does r), so $FF(A) = A_t = A[1/t]$. So any element of $FF(A)$ can be written uniquely as ut^n where u is a unit and $n \in \mathbb{Z}$. Thus we can define a valuation $\nu(ut^n) = n$.

1.D. EXERCISE. Show that ν is a discrete valuation.

Thus (a)–(d) implies (e).

Conversely, suppose $(A, \mathfrak{m})$ is a discrete valuation ring.

1.E. EXERCISE. Show that $(A, \mathfrak{m})$ is a Noetherian regular local ring of dimension 1. (Hint: Show that the ideals are all of the form (0) or $I_n = \{r \in A : \nu(r) \geq n\}$, and I_1 is the only prime of the second sort. Then we get Noetherianness, and dimension 1. Show that I_1/I_2 is generated by the image of any element of $I_1 - I_2$.)

Hence we have proved:

1.6. *Theorem. — An integral domain A is a Noetherian local ring of dimension 1 satisfying (a)–(d) if and only if*

(e) A *is a discrete valuation ring.*

1.F. EXERCISE. Show that there is only one discrete valuation on a discrete valuation ring.

Thus any Noetherian regular local ring of dimension 1 comes with a unique valuation on its fraction field. If the valuation of an element is $n > 0$, we say that the element has a **zero of order** n. If the valuation is $-n < 0$, we say that the element has a **pole of order** n. We'll come back to this shortly, after dealing with (f) and (g).

1.7. *Theorem. — Suppose* $(A, \mathfrak{m})$ *is a Noetherian local ring of dimension* 1*. Then (a)–(e) are equivalent to:*

(f) A *is a unique factorization domain,*
(g) A *is integrally closed in its fraction field* $K = FF(A)$*.*

Proof. (a)–(e) clearly imply (f), because we have the following stupid unique factorization: each non-zero element of r can be written uniquely as ut^n where $n \in \mathbb{Z}^{\geq 0}$ and u is a unit. Also, (f) implies (b), by an earlier easy Proposition, that in a unique factorization domain all codimension 1 prime ideals are principal. (In fact, we could just have (b) $\Longleftrightarrow$ (f) from the harder Proposition we proved, which showed that this property characterizes unique factorization domains.)

(f) implies (g), because unique factorization domains are integrally closed in their fraction fields (an earlier exercise).

It remains to check that (g) implies (a)–(e). We'll show that (g) implies (b).

Suppose $(A, \mathfrak{m})$ is a Noetherian local domain of dimension 1, integrally closed in its fraction field $K = FF(A)$. Choose any nonzero $r \in \mathfrak{m}$. Then $S = A/(r)$ is a Noetherian local ring of dimension 0 — its only prime is the image of $\mathfrak{m}$, which we denote $\mathfrak{n}$ to avoid confusion. Then $\mathfrak{n}$ is finitely generated, and each generator is nilpotent (the intersection of all the prime ideals in any ring are the nilpotents). Then $\mathfrak{n}^N = 0$, where N is the maximum of the nilpotence order of the finite set of generators. Hence there is some n such that $\mathfrak{n}^n = 0$ but $\mathfrak{n}^{n-1} \neq 0$.

Thus in A, $\mathfrak{m}^n \subseteq (r)$ but $\mathfrak{m}^{n-1} \not\subset (r)$. Choose $s \in \mathfrak{m}^{n-1} - (r)$. Consider $x = r/s$. Then $x^{-1} \notin A$, so as A is integrally closed, x^{-1} is not integral over A.

Now $x^{-1}\mathfrak{m} \not\subset \mathfrak{m}$ (or else $x^{-1}\mathfrak{m} \subset \mathfrak{m}$ would imply that $\mathfrak{m}$ is a faithful $A[x^{-1}]$-module, contradicting an Exercise from the Nakayama section that I promised we'd use). But $x^{-1}\mathfrak{m} \subset A$. Thus $x^{-1}\mathfrak{m} = A$, from which $\mathfrak{m} = xA$, so $\mathfrak{m}$ is principal. $\square$

(At some point I'd like a different proof using a more familiar version of Nakayama, rather than this version which people might not remember.)

1.8. Geometry of normal Noetherian schemes. Suppose X is a locally Noetherian scheme. Then for any regular codimension 1 points (i.e. any point p where $\mathcal{O}_{X,p}$ is a regular local ring of dimension 1), we have a discrete valuation ν. If f is any non-zero element of the fraction field of $\mathcal{O}_{X,p}$ (e.g. if X is integral, and f is a non-zero element of the function field of X), then if $\nu(f) > 0$, we say that the element has a **zero of order** $\nu(f)$, and if $\nu(f) < 0$, we say that the element has a **pole of order** $-\nu(f)$. (We aren't yet allowed to discuss order of vanishing at a point that is not regular codimension 1. One can make a definition, but it doesn't behave as well as it does when have you have a discrete valuation.)

So we can finally make precise the fact that the function $(x-2)^2x/(x-3)^4$ on $\mathbb{A}^1_{\mathbb{C}}$ has a double zero at $x = 2$ and a quadruple pole at $x = 3$. Furthermore, we can say that 75/34 has a double zero at 5, and a single pole at 2! (What are the zeros and poles of $x^3(x+y)/(x^2+xy)^3$ on $\mathbb{A}^2$?)

1.G. EXERCISE. Suppose X is an integral Noetherian scheme, and $f \in FF(X)^*$ is a non-zero element of its function field. Show that f has a finite number of zeros and poles. (Hint: reduce to $X = \operatorname{Spec} A$. If $f = f_1/f_2$, where $f_i \in A$, prove the result for f_i.)

Suppose A is an Noetherian integrally closed domain. Then it is *regular in codimension* 1 (translation: all its codimension at most 1 points are regular). If A is dimension 1, then obviously A is nonsingular.

For example, $\operatorname{Spec} \mathbb{Z}[i]$ is nonsingular, because it is dimension 1 (proved earlier — e.g. it is integral over $\operatorname{Spec} \mathbb{Z}$), and $\mathbb{Z}[i]$ is a unique factorization domain. Hence $\mathbb{Z}[i]$ is normal, so all its closed (codimension 1) points are nonsingular. Its generic point is also nonsingular, as $\mathbb{Z}[i]$ is a domain.

Remark. A (Noetherian) scheme can be singular in codimension 2 and still be normal. For example, you have shown that the cone $x^2 + y^2 = z^2$ in $\mathbb{A}^3$ is normal (an earlier exercise), but it is clearly singular at the origin (the Zariski tangent space is visibly three-dimensional).

But singularities of normal schemes are not so bad. For example, we've already seen Hartogs' Theorem for Noetherian normal schemes, which states that you could extend functions over codimension 2 sets.

Remark: We know that for Noetherian rings we have implications

$$\text{unique factorization domain} \Longrightarrow \text{integrally closed} \Longrightarrow \text{regular in codimension 1}.$$

Hence for locally Noetherian schemes, we have similar implications:

$$\text{regular in codimension 1} \Longrightarrow \text{normal} \Longrightarrow \text{factorial}.$$

Here are two examples to show you that these inclusions are strict.

1.H. EXERCISE. Let A be the subring $k[x^3, x^2, xy, y] \subset k[x, y]$. (Informally, we allow all polynomials that don't include a non-zero multiple of the monomial x.) Show that A is not integrally closed (hint: consider the "missing x"). Show that it is regular in codimension 1 (hint: show it is dimension 2, and when you throw out the origin you get something nonsingular, by inverting x^2 and y respectively, and considering A_{x^2} and A_y).

1.I. EXERCISE. You have checked that $k[w,x,y,z]/(wz-xy)$ is integrally closed (at least if k is algebraically closed of characteristic not 2, an earlier exercise). Show that it is not a unique factorization domain. (One possibility is to do this "directly". This might be hard to do rigorously — how do you know that x is irreducible in $k[w,x,y,z]/(wz-xy)$? Another possibility, faster but less intuitive, is to use the intermediate result that in a unique factorization domain, any height 1 prime is principal, and considering the exercise from last day that the cone over a ruling is not principal.)

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