FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 20

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CONTENTS

This is the last class of the quarter! We will finish with dimension theory today.

1. DIMENSION AND TRANSCENDENCE DEGREE

We now prove an alternative interpretation for dimension for irreducible varieties.

1.1. *Theorem (dimension = transcendence degree). — Suppose* A *is a finitely-generated domain over a field* k*. Then* $\dim \operatorname{Spec} A$ *is the transcendence degree of the fraction field* $FF(A)$ *over* k*.*

By "finitely generated domain over k", we mean "a finitely generated k-algebra that is an integral domain".

In case you haven't seen the notion of transcendence degree, here is a quick summary of the relevant facts. Suppose K/k is a finitely generated field extension. Then any two maximal sets of algebraically independent elements of K over k (i.e. any set with no algebraic relation) have the same size (a non-negative integer or ∞). If this size is finite, say n, and $x_1, \dots, x_n$ is such a set, then $K/k(x_1, \dots, x_n)$ is necessarily a finitely generated algebraic extension, i.e. a finite extension. (Such a set $x_1, \dots, x_n$ is called a transcendence basis, and n is called the *transcendence degree*.)

In particular, we see that $\dim \mathbb{A}^n_k = n$. However, our proof of Theorem 1.1 will go *through* this fact, so it isn't really a Corollary.

Date: Wednesday, December 6, 2007.

1.2. Sample consequences. We will prove Theorem 1.1 shortly. But we first show that it is useful by giving some immediate consequences. We begin with a proof of the Nullstellensatz, promised earlier.

1.A. EXERCISE: NULLSTELLENSATZ FROM DIMENSION THEORY.
(a) Suppose $A = k[x_1, \dots, x_n]/I$, where k is an algebraically closed field and I is some ideal. Then the maximal ideals are precisely those of the form $(x_1 - a_1, \dots, x_n - a_n)$, where $a_i \in k$. This version (the "weak Nullstellensatz") was stated earlier.
(b) Suppose $A = k[x_1, \dots, x_n]/I$ where k is not necessarily algebraically closed. Show that every maximal ideal of A has a residue field that is a finite extension of k. This version was stated in earlier. (Hint for both parts: the maximal ideals correspond to dimension 0 points, which correspond to transcendence degree 0 extensions of k, i.e. finite extensions of k. If $k = \overline{k}$, the maximal ideals correspond to surjections $f : k[x_1, \dots, x_n] \to k$. Fix one such surjection. Let $a_i = f(x_i)$, and show that the corresponding maximal ideal is $(x_1 - a_1, \dots, x_n - a_n)$.)

1.3. *Points of* $\mathbb{A}^2_k$. We can now confirm that we have named all the primes of $k[x, y]$ where k is algebraically closed (as promised earlier when $k = \mathbb{C}$). Recall that we have discovered the primes (0), $f(x, y)$ where f is irreducible, and $(x - a, y - b)$ where $a, b \in k$. As $\mathbb{A}^2_k$ is irreducible, there is only one irreducible subset of codimension 0. By the Proposition from last day about UFDs, all codimension 1 primes are principal. By the inequality $\dim X + \operatorname{codim}_Y X = \dim Y$, there are no primes of codimension greater than 2, and any prime of codimension 2 must be maximal. We have identified all the maximal ideals of $k[x, y]$ by the Nullstellensatz.

1.B. IMPORTANT EXERCISE. Suppose X is an irreducible variety. Show that $\dim X$ is the transcendence degree of the function field (the stalk at the generic point) $\mathcal{O}_{X,\eta}$ over k. Thus (as the generic point lies in all non-empty open sets) the dimension can be computed in any open set of X. (This is not true in general, see §3.4.)

Here is an application that you might reasonably have wondered about before thinking about algebraic geometry.

1.C. EXERCISE. Suppose $f(x, y)$ and $g(x, y)$ are two complex polynomials ($f, g \in \mathbb{C}[x, y]$). Suppose f and g have no common factors. Show that the system of equations $f(x, y) = g(x, y) = 0$ has a finite number of solutions. (This isn't essential for what follows. But it is a basic fact, and very believable.)

1.D. EXERCISE. Suppose $X \subset Y$ is an inclusion of irreducible k-varieties, and η is the generic point of X. Show that $\dim X + \dim \mathcal{O}_{Y,\eta} = \dim Y$. Hence show that $\dim X + \operatorname{codim}_Y X = \dim Y$. Thus for varieties, the inequality $\dim X + \operatorname{codim}_Y X \leq \dim Y$ is always an equality.

1.E. EXERCISE. Show that $\operatorname{Spec} k[w,x,y,z]/(wz-xy, wy-x^2, xz-y^2)$ is an integral *surface*. You might expect it to be a curve, because it is cut out by three equations in 4-space. (You may recognize this as the affine cone over the twisted cubic.) It turns out that you actually need three equations to cut out this surface. The first equation cuts out a threefold in four-space (by Krull's theorem 3.2, see later). The second equation cuts out a surface: our surface, along with another surface. The third equation cuts out our surface, and removes the "extraneous component". One last aside: notice once again that the cone over the quadric surface $k[w,x,y,z]/(wz-xy)$ makes an appearance.)

1.4. Noether Normalization.

Hopefully you are now motivated to understand the proof of Theorem 1.1 on dimension and transcendence degree. To set up the argument, we introduce another important and ancient result, Noether's Normalization Lemma.

1.5. *Noether Normalization Lemma. — Suppose* A *is an integral domain, finitely generated over a field* k. *If* $\operatorname{tr.deg.}_k A = n$, *then there are elements* $x_1, \dots, x_n \in A$, *algebraically independent over* k, *such that* A *is a finite (hence integral) extension of* $k[x_1, \dots, x_n]$.

The geometric content behind this result is that given any integral affine k-scheme X, we can find a surjective finite morphism $X \to \mathbb{A}^n_k$, where n is the transcendence degree of the function field of X (over k). Surjectivity follows from the Going-Up Theorem.

Nagata's proof of Noether normalization $\star$. Suppose we can write $A = k[y_1, \dots, y_m]/\mathfrak{p}$, i.e. that A can be chosen to have m generators. Note that $m \geq n$. We show the result by induction on m. The base case $m = n$ is immediate.

Assume now that $m > n$, and that we have proved the result for smaller m. We will find $m-1$ elements $z_1, \dots, z_{m-1}$ of A such that A is finite over $A' := k[z_1, \dots, z_{m-1}]$ (i.e. the subring of A generated by $z_1, \dots, z_{m-1}$). Then by the inductive hypothesis, A' is finite over some $k[x_1, \dots, x_n]$, and A is finite over A', so A is finite over $k[x_1, \dots, x_n]$.

$$\begin{array}{c} A \\ \Big|\,{\scriptstyle \text{finite}} \\ A' = k[z_1, \dots, z_{m-1}]/\mathfrak{p} \\ \Big|\,{\scriptstyle \text{finite}} \\ k[x_1, \dots, x_n] \end{array}$$

As $y_1, \dots, y_m$ are algebraically dependent, there is some non-zero algebraic relation $f(y_1, \dots, y_m) = 0$ among them (where f is a polynomial in m variables).

Let $z_1 = y_1 - y_m^{r_1}$, $z_2 = y_2 - y_m^{r_2}$, $\dots$, $z_{m-1} = y_{m-1} - y_m^{r_{m-1}}$, where $r_1, \dots, r_{m-1}$ are positive integers to be chosen shortly. Then

$$f(z_1 + y_m^{r_1}, z_2 + y_m^{r_2}, \dots, z_{m-1} + y_m^{r_{m-1}}, y_m) = 0.$$

Then upon expanding this out, each monomial in f (as a polynomial in m variables) will yield a single term in that is a constant times a power of y_m (with no z_i factors). By choosing the r_i so that $0 \ll r_1 \ll r_2 \ll \cdots \ll r_{m-1}$, we can ensure that the powers of y_m appearing are all distinct, and so that in particular there is a leading term y_m^N, and all other terms (including those with z_i-factors) are of smaller degree in y_m. Thus we have described an integral dependence of y_m on $z_1, \ldots, z_{m-1}$ as desired. □

1.6. *Aside: the geometric idea behind Nagata's proof.* There is some geometric intuition behind this. Suppose we have an m-dimensional variety in $\mathbb{A}_k^n$ with $m < n$, for example $xy = 1$ in $\mathbb{A}^2$. One approach is to project it to a hyperplane via a finite morphism. In the case of $xy = 1$, if we projected to the x-axis, it wouldn't be finite, roughly speaking because the asymptote $x = 0$ prevents the map from being closed. But if we projected to a line, we might hope that we would get rid of this problem, and indeed we usually can: this problem arises for only a finite number of directions. But we might have a problem if the field were finite: perhaps the finite number of directions in which to project each have a problem. (The reader may show that if k is an infinite field, then the substitution in the above proof $z_i = y_i - y_m^{r_i}$ can be replaced by the linear substitution $z_i = y_i - a_i y_m$ where $a_i \in k$, and that for a non-empty Zariski-open choice of a_i, we indeed obtain a finite morphism.) Nagata's trick in general is to "jiggle" the variables in a non-linear way, and that this is enough to prevent non-finiteness of the map.

Proof of Theorem 1.1 on dimension and transcendence degree. Suppose X is an integral affine k-scheme. We show that $\dim X$ equals the transcendence degree n of its function field, by induction on n. Fix X, and assume the result is known for all transcendence degrees less than n.

By Noether normalization, there exists a surjective finite morphism map $X \to \mathbb{A}_k^n$. By the Going-Up theorem, $\dim X = \dim \mathbb{A}_k^n$. If $n = 0$, we are done, as $\dim \mathbb{A}_k^0 = 0$.

We now show that $\dim \mathbb{A}_k^n = n$ for $n > 0$, by induction. Clearly $\dim \mathbb{A}_k^n \geq n$, as we can describe a chain of irreducible subsets of length $n+1$: if $x_1, \ldots, x_n$ are coordinates on $\mathbb{A}^n$, consider the chain of ideals

$$(0) \subset (x_1) \subset \cdots \subset (x_1, \ldots, x_n)$$

in $k[x_1, \ldots, x_n]$. Suppose we have a chain of prime ideals of length at least n:

$$(0) = \mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_m.$$

where $\mathfrak{p}_1$ is a codimension 1 prime ideal. Then $\mathfrak{p}_1$ is principal (as $k[x_1, \ldots, x_n]$ is a unique factorization domain, a Proposition proved on Monday) say $\mathfrak{p}_1 = (f(x_1, \ldots, x_n))$, where f is an irreducible polynomial. Then $k[x_1, \ldots, x_n]/(f(x_1, \ldots, x_n))$ has transcendence degree $n-1$, so by induction,

$$\dim k[x_1, \ldots, x_n]/(f) = n - 1.$$

□

We can now prove Chevalley's Theorem 2.1, discussed earlier.

2.1. *Chevalley's Theorem.* — *Suppose* $f : X \to Y$ *is a morphism of finite type of Noetherian schemes. Then the image of any constructable set is constructable.*

The proof will use Noether normalization. This is remarkable: Noether normalization is about finitely generated algebras over a field, but there is no field in the statement of Chevalley's theorem. Hence if you prefer to work over arbitrary rings (or schemes), this shows that you still care about facts about finite type schemes over a field. Conversely, even if you are interested in finite type schemes over a given field (like $\mathbb{C}$), the field that comes up in the proof of Chevalley's theorem is *not* that field, so even if you prefer to work over $\mathbb{C}$, this argument shows that you still care about working over arbitrary fields, not necessarily algebraically closed.

2.A. HARD EXERCISE. Reduce the proof of Chevalley's theorem 2.1 to the following statement: suppose $f : X = \operatorname{Spec} A \to Y = \operatorname{Spec} B$ is a dominant morphism, where A and B are domains, and f corresponds to $\phi : B \to B[x_1, \dots, x_n]/I \cong A$. Then the image of f contains a dense open subset of $\operatorname{Spec} B$. (Hint: Make a series of reductions. The notion of constructable is local, so reduce to the case where Y is affine. Then X can be expressed as a finite union of affines; reduce to the case where X is affine. X can be expressed as the finite union of irreducible components; reduce to the case where X is irreducible. Reduce to the case where X is reduced. By considering the closure of the image of the generic point of X, reduce to the case where Y also is integral (irreducible and reduced), and $X \to Y$ is dominant. Use Noetherian induction in some way on Y.)

Proof. We prove the statement given in the previous exercise. Let $K := FF(B)$. Now $A \otimes_B K$ is a localization of A with respect to B^* (interpreted as a subset of A), so it is a domain, and it is finitely generated over K (by x_1, ..., x_n), so it has finite transcendence degree r over K. Thus by Noether normalization, we can find a subring $K[y_1, \dots, y_r] \subset A \otimes_B K$, so that $A \otimes_B K$ is integrally dependent on $K[y_1, \dots, y_r]$. We can choose the y_i to be in A: each is in $(B^*)^{-1}A$ to begin with, so we can replace each y_i by a suitable K-multiple.

Sadly A is not necessarily integrally dependent on $A[y_1, \dots, y_r]$ (as this would imply that $\operatorname{Spec} A \to \operatorname{Spec} B$ is surjective by the Going-Up Theorem). However, each x_i satisfies some integral equation

$$x_i^n + f_1(y_1, \dots, y_r)x_i^{n-1} + \cdots + f_n(y_1, \dots, y_r) = 0$$

where f_j are polynomials with coefficients in $K = FF(B)$. Let g be the product of the denominators of all the coefficients of all these polynomials (a finite set). Then A_g is integral over $B_g[y_1, \dots, y_r]$, and hence $\operatorname{Spec} A_g \to \operatorname{Spec} B_g$ is surjective; $\operatorname{Spec} B_g$ is our open subset. □

3. FUN IN CODIMENSION ONE: KRULL'S PRINCIPAL IDEAL THEOREM, ALGEBRAIC HARTOGS' LEMMA, AND MORE

In this section, we'll explore a number of results related to codimension one.

Codimension one primes of $\mathbb{Z}$ and $k[x, y]$ correspond to prime numbers and irreducible polynomials respectively. We will make this link precise for unique factorization domains. Then we introduce two results that apply in more general situations, and link functions and the codimension one points where they vanish, Krull's Principal Ideal Theorem 3.2, and Algebraic Hartogs' Lemma 3.6. We will find these two theorems very useful. For example, Krull's Principal Ideal Theorem will help us compute codimensions, and will show us that codimension can behave oddly, and Algebraic Hartogs' Lemma will give us a useful characterization of Unique Factorization Domains (Proposition 3.8).

The results in this section will require (locally) Noetherian hypotheses.

3.1. Krull's Principal Ideal Theorem. As described earlier in the chapter, in analogy with linear algebra, we have the following.

3.2. *Krull's Principal Ideal Theorem (geometric version). — Suppose X is a Noetherian scheme, and f is a function. Then the irreducible components of $V(f)$ are codimension 0 or 1.*

This is clearly equivalent to the following algebraic statement.

3.3. *Krull's Principal Ideal Theorem (algebraic version). — Suppose A is a Noetherian ring, and $f \in A$. Then every minimal prime $\mathfrak{p}$ containing f has codimension at most 1. If furthermore f is not a zero-divisor, then every minimal prime $\mathfrak{p}$ containing f has codimension precisely 1.*

The full proof is technical, so I'll postpone it to §4, and you shouldn't read it unless you really want to.

But this immediately useful. For example, consider the scheme $\operatorname{Spec} k[w, x, y, z]/(wx - yz)$. What is its dimension? It is cut out by one non-zero equation $wx - yz$ in $\mathbb{A}^4$, so it is a threefold.

3.A. EXERCISE. What is the dimension of $\operatorname{Spec} k[w, x, y, z]/(wz - xy, y^{17} + z^{17})$? (Be careful to check they hypotheses before invoking Krull!)

3.B. EXERCISE. Show that an irreducible homogeneous polynomial in $n + 1$ variables over a field k describes an integral scheme of dimension $n - 1$.

3.C. EXERCISE (IMPORTANT FOR LATER). (a) *(Hypersurfaces meet everything of dimension at least 1 in projective space — unlike in affine space.)* Suppose X is a closed subset of $\mathbb{P}^n_k$ of dimension at least 1, and H a nonempty hypersurface in $\mathbb{P}^n_k$. Show that H meets X. (Hint:

consider the affine cone, and note that the cone over H contains the origin. Use Krull's Principal Ideal Theorem 3.3.)
(b) (Definition: Subsets in $\mathbb{P}^n$ cut out by linear equations are called **linear subspaces**. Dimension 1, 2 linear subspaces are called **lines** and **planes** respectively.) Suppose $X \hookrightarrow \mathbb{P}^n_k$ is a closed subset of dimension r. Show that any codimension r linear space meets X. Hint: Refine your argument in (a). (In fact any two things in projective space that you might expect to meet for dimensional reasons do in fact meet. We won't prove that here.)
(c) Show further that there is an intersection of $r+1$ hypersurfaces missing X. (The key step: show that there is a hypersurface of sufficiently high degree that doesn't contain every generic point of X. Show this by induction on the number of generic points. To get from n to $n+1$: take a hypersurface not vanishing on $p_1, \dots, p_n$. If it doesn't vanish on p_{n+1}, we're done. Otherwise, call this hypersurface f_{n+1}. Do something similar with $n+1$ replaced by i ($1 \leq i \leq n$). Then consider $\sum_i f_1 \cdots \widehat{f_i} \cdots f_{n+1}$.)

3.4. Pathologies of the notion of "codimension". We can use Krull's Principal Ideal Theorem to produce the long-promised example of pathology in the notion of codimension. Let $A = k[x]_{(x)}[t]$. In other words, elements of A are polynomials in t, whose coefficients are quotients of polynomials in x, where no factors of x appear in the denominator. (Warning: A is not isomorphic to $k[x,t]_{(x)}$.) Clearly, A is a domain, and $(xt-1)$ is not a zero divisor. You can verify that $A/(xt-1) \cong k[x]_{(x)}[1/x] \cong k(x)$ — "in $k[x]_{(x)}$, we may divide by everything but x, and now we are allowed to divide by x as well" — so $A/(xt-1)$ is a field. Thus $(xt-1)$ is not just prime but also maximal. By Krull's theorem, $(xt-1)$ is codimension 1. Thus $(0) \subset (xt-1)$ is a maximal chain. However, A has dimension at least 2: $(0) \subset (t) \subset (x,t)$ is a chain of primes of length 3. (In fact, A has dimension precisely 2, although we don't need this fact in order to observe the pathology.) Thus we have a codimension 1 prime in a dimension 2 ring that is dimension 0. Here is a picture of this lattice of ideals.

$$\begin{tikzcd}
(x,t) & & \\
(t) \arrow[u, dash] & & (xt-1) \\
& (0) \arrow[ul, dash] \arrow[ur, dash] &
\end{tikzcd}$$

This example comes from geometry; it is enlightening to draw a picture see Figure 1. $\operatorname{Spec} k[x]_{(x)}$ corresponds to a germ of $\mathbb{A}^1_k$ near the origin, and $\operatorname{Spec} k[x]_{(x)}[t]$ corresponds to "this $\times$ the affine line". You may be able to see from the picture some motivation for this pathology — note that $V(xt-1)$ doesn't meet $V(x)$, so it can't have any specialization on $V(x)$, and there nowhere else for $V(xt-1)$ to specialize.

It is disturbing that this misbehavior turns up even in a relative benign-looking ring.

3.D. UNIMPORTANT EXERCISE. Show that it is false that if X is an integral scheme, and U is a non-empty open set, then $\dim U = \dim X$.

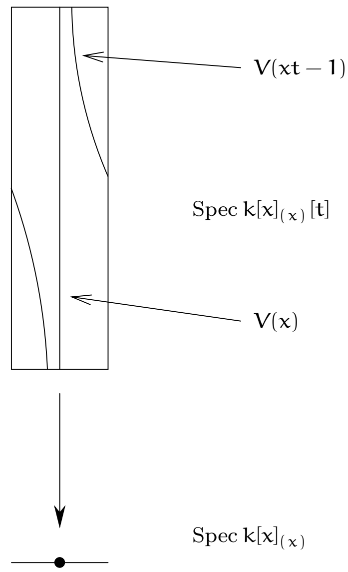

FIGURE 1. Dimension and codimension behave oddly on the surface $\operatorname{Spec} k[x]_{(x)}[t]$

3.5. Algebraic Hartogs' Lemma for Noetherian normal schemes.

Hartogs' Lemma in several complex variables states (informally) that a holomorphic function defined away from a codimension two set can be extended over that. We now describe an algebraic analog, for Noetherian normal schemes.

3.6. *Algebraic Hartogs' Lemma. — Suppose A is a Noetherian normal domain.*

$$A = \cap_{\mathfrak{p} \text{ codimension } 1} A_{\mathfrak{p}}.$$

The equality takes place inside $FF(A)$; recall that any localization of a domain A is naturally a subset of $FF(A)$. Warning: No one else calls this Algebraic Hartogs' Lemma. I've called it this because I find the that it parallels the statement in complex geometry. The proof is technical, so we postpone it to §3.9. (One can state Algebraic Hartogs' Lemma more generally in the case that $\operatorname{Spec} A$ is a Noetherian normal scheme, meaning that A is a product of Noetherian normal domains; the reader may wish to do so. A more general statement is that if A is a Noetherian domain, then $\cap_{\operatorname{codim} P=1} A_P$ is the integral closure of A (Atiyah-Macdonald, Cor. 5.22). We won't need this. And this "domain" condition can also be relaxed.)

One might say that if $f \in FF(A)$ does not lie in $A_{\mathfrak{p}}$ where $\mathfrak{p}$ has codimension 1, then f has a pole at $[\mathfrak{p}]$, and if $f \in FF(A)$ lies in $\mathfrak{p}A_{\mathfrak{p}}$ where $\mathfrak{p}$ has codimension 1, then f has a zero at $[\mathfrak{p}]$. It is worth interpreting Algebraic Hartogs' Lemma as saying that a rational function on a normal scheme with no poles is in fact regular (an element of A). More generally, if X is a Noetherian normal scheme, we can define zeros and poles of rational functions on X. (We will soon define the *order* of a zero or a pole.)

3.E. EXERCISE. Suppose f is an element of a normal domain A, and f is contained in no codimension 1 primes. Show that f is a unit.

3.F. EXERCISE. Suppose f and g are two global sections of a Noetherian normal scheme, not vanishing at any associated point, with the same poles and zeros. Show that each is a unit times the other.

3.7. A useful characterization of unique factorization domains.

We can use Algebraic Hartogs' Lemma 3.6 to prove one of the four things you need to know about unique factorization domains.

3.8. *Proposition. — Suppose that* A *is a Noetherian domain. Then* A *is a Unique Factorization Domain if and only if all codimension* 1 *primes are principal.*

This contains the Proposition last day showing that in a UFD, all height 1 primes are principal, and (in some sense) its converse.

Proof. We have already shown in last day (in the Proposition mentioned in the previous sentence) that if A is a Unique Factorization Domain, then all codimension 1 primes are principal. Assume conversely that all codimension 1 primes of A are principal. I claim that the generators of these ideals are irreducible, and that we can uniquely factor any element of A into these irreducibles, and a unit. First, suppose (f) is a codimension 1 prime ideal $\mathfrak{p}$. Then if $f = gh$, then either $g \in \mathfrak{p}$ or $h \in \mathfrak{p}$. As $\operatorname{codim} \mathfrak{p} > 0$, $\mathfrak{p} \neq 0$, so by Nakayama's Lemma (as $\mathfrak{p}$ is finitely generated), $\mathfrak{p} \neq \mathfrak{p}^2$. Thus both g and h cannot be in $\mathfrak{p}$. Say $g \notin \mathfrak{p}$. Then g is contained in no codimension 1 primes (as f was contained in only one, namely $\mathfrak{p}$), and hence is a unit by Exercise 3.E.

Finally, we show that any non-zero element f of A can be factored into irreducibles. Now $V(f)$ is contained in a finite number of codimension 1 primes, as (f) as a finite number of associated primes, and hence a finite number of minimal primes. We show that any nonzero f can be factored into irreducibles by induction on the number of codimension 1 primes containing f. In the base case where there are none, then f is a unit by Exercise 3.E. For the general case where there is at least one, say $f \in \mathfrak{p} = (g)$. Then $f = g^n h$ for some $h \notin (g)$. (Reason: otherwise, we have an ascending chain of ideals $(f) \subset (f/g) \subset (f/g^2) \subset \cdots$, contradicting Noetherianness.) Thus $f/g^n \in A$, and is contained in one fewer codimension 1 primes. $\square$

3.9. *Proof of Algebraic Hartogs' Lemma 3.6* $\star$. This proof does not shed light on any of the other discussion in this section, and thus should not be read. However, you should sleep soundly at night knowing that the proof is this short. Obviously the right side is contained in the left. Assume we have some x in all $A_{\mathfrak{p}}$ but not in A. Let I be the "ideal of denominators":

$$I := \{r \in A : rx \in A\}.$$

(The ideal of denominators arose in an earlier discussion about normality, when we proved the stalk-locality of normality.) We know that $I \neq A$, so choose $\mathfrak{q}$ a minimal prime containing I.

Observe that this construction behaves well with respect to localization (i.e. if $\mathfrak{p}$ is any prime, then the ideal of denominators x in $A_{\mathfrak{p}}$ is the $I_{\mathfrak{p}}$, and it again measures the failure of 'Algebraic Hartogs' Lemma for x,' this time in $A_{\mathfrak{p}}$). But Hartogs' Theorem is vacuously true for dimension 1 rings, so hence no codimension 1 prime contains I. Thus $\mathfrak{q}$ has codimension at least 2. By localizing at $\mathfrak{q}$, we can assume that A is a local ring with maximal ideal $\mathfrak{q}$, and that $\mathfrak{q}$ is the only prime containing I. Thus $\sqrt{I} = \mathfrak{q}$, so there is some n with $I \subset \mathfrak{q}^n$. Take a minimal such n, so $I \not\subset \mathfrak{q}^{n-1}$, and choose any $y \in \mathfrak{q}^{n-1} - \mathfrak{q}^n$. Let $z = yx$. Then $z \notin A$ (so $\mathfrak{q}z \notin \mathfrak{q}$), but $\mathfrak{q}z \subset A$: $\mathfrak{q}z$ is an ideal of A.

I claim $\mathfrak{q}z$ is not contained in $\mathfrak{q}$. Otherwise, we would have a finitely-generated A-module (namely $\mathfrak{q}$) with a faithful $A[z]$-action, forcing z to be integral over A (and hence in A) by an Exercise in the Nakayama section last day.

Thus $\mathfrak{q}z$ is an ideal of A not contained in $\mathfrak{q}$, so it must be A! Thus $\mathfrak{q}z = A$ from which $\mathfrak{q} = A(1/z)$, from which $\mathfrak{q}$ is principal. But then $\operatorname{codim} Q = \dim A \leq \dim_{A/Q} Q/Q^2 \leq 1$ by Nakayama's lemma, contradicting the fact that $\mathfrak{q}$ has codimension at least 2. $\square$

4. Proof of Krull's Principal Ideal Theorem 3.3 $\star\star$

The details of this proof won't matter much to us, so you should probably not read it. It is included so you can glance at it and believe that the proof is fairly short, and you could read it if you really wanted to.

4.1. *Lemma. — If A is a Noetherian ring with one prime ideal. Then A is Artinian, i.e., it satisfies the descending chain condition for ideals.*

The notion of Artinian rings is very important, but we will get away without discussing it much.

Proof. If A is a ring, we define more generally an *Artinian A-module*, which is an A-module satisfying the descending chain condition for submodules. Thus A is an Artinian ring if it is Artinian over itself as a module.

If $\mathfrak{m}$ is a maximal ideal of R, then any finite-dimensional $(R/\mathfrak{m})$-vector space (interpreted as an R-module) is clearly Artinian, as any descending chain

$$M_1 \supset M_2 \supset \cdots$$

must eventually stabilize (as $\dim_{R/\mathfrak{m}} M_i$ is a non-increasing sequence of non-negative integers).

4.A. EXERCISE. Show that for any n, $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ is a finitely-dimensional $A/\mathfrak{m}$-vector space. (Hint: show it for $n = 0$ and $n = 1$. Use the dimension for $n = 1$ to bound the dimension for general n.) Hence $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ is an Artinian A-module.

As $\sqrt{0}$ is prime, it must be $\mathfrak{m}$.

4.B. EXERCISE. Prove that $\mathfrak{m}^n = 0 = 0$ for some n. (Hint: suppose $\mathfrak{m}$ can be generated by m elements, each of which has kth power 0, and show that $\mathfrak{m}^{m(k-1)+1} = 0$.)

4.C. EXERCISE. Show that if $0 \to M' \to M \to M'' \to 0$ is an exact sequence of modules. then M is Artinian if and only if M' and M'' are Artinian. (Hint: think about the corresponding question about Noetherian modules, which we've seen before.)

Thus as we have a finite filtration

$$A \supset \mathfrak{m} \supset \cdots \supset \mathfrak{m}^n = 0$$

all of whose quotients are Artinian, so A is Artinian as well. This completes the proof of the Lemma. □

Proof of Krull's principal ideal theorem 3.3. Suppose we are given $x \in A$, with $\mathfrak{p}$ a minimal prime containing x. By localizing at $\mathfrak{p}$, we may assume that A is a local ring, with maximal ideal $\mathfrak{p}$. Suppose $\mathfrak{q}$ is another prime strictly contained in $\mathfrak{p}$.

$$\begin{tikzcd}
x \arrow[dr, hook] & & \\
& \mathfrak{p} \arrow[r, hook] & A \\
\mathfrak{q} \arrow[ur, hook] & &
\end{tikzcd}$$

For the first part of the theorem, we must show that $A_{\mathfrak{q}}$ has dimension 0. The second part follows from our earlier work: if any minimal primes are height 0 (minimal primes of A), f is a zero-divisor, as minimal primes of A are all associated primes of A, and elements of associated primes of A are zero-divisors.

Now $\mathfrak{p}$ is the only prime ideal containing (x), so $A/(x)$ has one prime ideal. By Lemma 4.1, $A/(x)$ is Artinian.

We invoke a useful construction, the n*th symbolic power of a prime ideal*: if A is a ring, and $\mathfrak{q}$ is a prime ideal, then define

$$\mathfrak{q}^{(n)} := \{r \in A : rs \in \mathfrak{q}^n \text{ for some } s \in A - \mathfrak{q}\}.$$

We have a descending chain of ideals in A

$$\mathfrak{q}^{(1)} \supset \mathfrak{q}^{(2)} \supset \cdots ,$$

so we have a descending chain of ideals in $A/(x)$

$$\mathfrak{q}^{(1)} + (x) \supset \mathfrak{q}^{(2)} + (x) \supset \cdots$$

which stabilizes, as $A/(x)$ is Artinian. Say $\mathfrak{q}^{(n)} + (x) = \mathfrak{q}^{(n+1)} + (x)$, so

$$\mathfrak{q}^{(n)} \subset \mathfrak{q}^{(n+1)} + (x).$$

Hence for any $f \in \mathfrak{q}^{(n)}$, we can write $f = ax + g$ with $g \in \mathfrak{q}^{(n+1)}$. Hence $ax \in \mathfrak{q}^{(n)}$. As $\mathfrak{p}$ is minimal over x, $x \notin \mathfrak{q}$, so $a \in \mathfrak{q}^{(n)}$. Thus

$$\mathfrak{q}^{(n)} = (x)\mathfrak{q}^{(n)} + \mathfrak{q}^{(n+1)}.$$

As x is in the maximal ideal $\mathfrak{p}$, the second version of Nakayama's lemma gives $\mathfrak{q}^{(n)} = \mathfrak{q}^{(n+1)}$.

We now shift attention to the local ring $A_\mathfrak{q}$, which we are hoping is dimension 0. We have $\mathfrak{q}^{(n)}A_\mathfrak{q} = \mathfrak{q}^{(n+1)}A_\mathfrak{q}$ (the symbolic power construction clearly construction commutes with respect to localization). For any $r \in \mathfrak{q}^n A_\mathfrak{q} \subset \mathfrak{q}^{(n)}A_\mathfrak{q}$, there is some $s \in A_\mathfrak{q} - \mathfrak{q}A_\mathfrak{q}$ such that $rs \in \mathfrak{q}^{n+1}A_\mathfrak{q}$. As s is invertible, $r \in \mathfrak{q}^{n+1}A_\mathfrak{q}$ as well. Thus $\mathfrak{q}^n A_\mathfrak{q} \subset \mathfrak{q}^{n+1}A_\mathfrak{q}$, but as $\mathfrak{q}^{n+1}A_\mathfrak{q} \subset \mathfrak{q}^n A_\mathfrak{q}$, we have $\mathfrak{q}^n A_\mathfrak{q} = \mathfrak{q}^{n+1}A_\mathfrak{q}$. By Nakayama's Lemma version 4,

$$\mathfrak{q}^n A_\mathfrak{q} = 0.$$

Finally, any local ring $(R, \mathfrak{m})$ such that $\mathfrak{m}^n = 0$ has dimension 0, as $\operatorname{Spec} A$ consists of only one point: $[\mathfrak{m}] = V(\mathfrak{m}) = V(\mathfrak{m}^n) = V(0) = \operatorname{Spec} A$. $\square$

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