FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 17

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CONTENTS

Hi everyone — Welcome back! We had last introduced the algebraic analogue of Hausdorffness, called *separation* or *separatedness*. This is a bit weird, but frankly, it is because the notion of Hausdorff involves some mild contortions, and it is easy to forget that.

1. REVIEW OF EARLIER DISCUSSION ON SEPARATION

Let me remind you how it works. Our motivating example of what we are ejecting from civilized discourse is the line with the doubled origin.

We said that a morphism $X \to Y$ is **separated** if the diagonal morphism $\delta : X \to X \times_Y X$ is a closed immersion. An A-scheme X is said to be **separated over** A if the structure morphism $X \to \operatorname{Spec} A$ is separated.

A **variety** over a field k, or k**-variety**, is a reduced, separated scheme of finite type over k. For example, a reduced finite type affine k-scheme is a variety. In other words, to check if $\operatorname{Spec} k[x_1, \dots, x_n]/(f_1, \dots, f_r)$ is a variety, you need only check reducedness.

As diagonals are always locally closed immersions, a morphism is separated if and only if the diagonal is closed. This is reminiscent of a definition of Hausdorff., as the next exercise shows.

We saw that the following types of morphisms are separated:

- open and closed immersions (more generally, monomorphisms)
- morphisms of affine schemes

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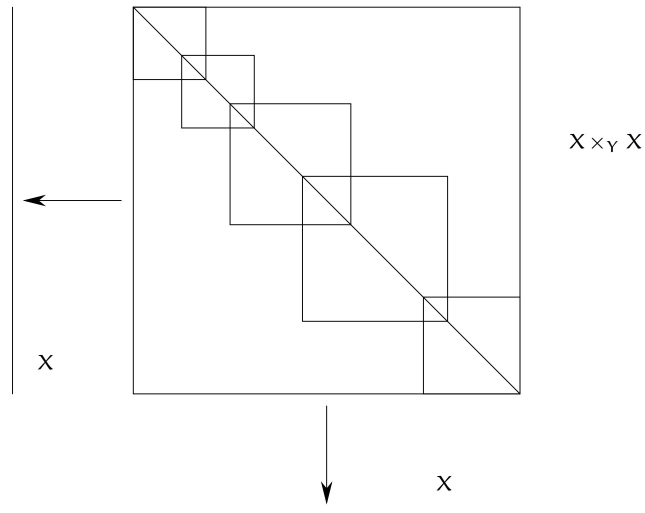


FIGURE 1. A neighborhood of the diagonal is covered by $U_{ij} \times_{V_j} U_{ij}$

- projective A-schemes (over A)

In the course of proving the projective fact, we showed:

1.1. *Small Proposition. — If U and V are open subsets of an A-scheme X, then $\Delta \cap (U \times_A V) \cong U \cap V$.*

We used this to show a handy consequence of separatedness.

1.2. *Proposition. — Suppose $X \to \operatorname{Spec} A$ is a separated morphism to an affine scheme, and U and V are affine open sets of X. Then $U \cap V$ is an affine open subset of X.*

2. QUASISEPARATED MORPHISMS (AND QUASISEPARATED SCHEMES)

We now define a handy relative of separation, that is also given in terms of a property of the diagonal morphism, and has similar properties. The reason it is less famous is because it automatically holds for the sorts of schemes that people usually deal with. We say a morphism $f : X \to Y$ is **quasiseparated** if the diagonal morphism $\delta : X \to X \times_Y X$ is quasicompact. I'll give a more insightful translation shortly, in Exercise 2.A.

Most algebraic geometers will only see quasiseparated morphisms, so this may be considered a very weak assumption. Here are two large classes of morphisms that are quasiseparated. (a) As closed immersions are quasicompact (easy, and an earlier exercise), separated implies quasiseparated. (b) If X is a Noetherian scheme, then any morphism to another scheme is quasicompact (easy, an earlier exercise), so any $X \to Y$ is quasiseparated. Hence those working in the category of Noetherian schemes need never worry about this issue.

The following characterization makes quasiseparation a useful hypothesis in proving theorems.

2.A. EXERCISE. Show that $f : X \to Y$ is quasiseparated if and only if for any affine open $\operatorname{Spec} A$ of Y, and two affine open subsets U and V of X mapping to $\operatorname{Spec} A$, $U \cap V$ is a *finite* union of affine open sets. (Hint: compare this to Proposition 1.2.)

In particular, a morphism $f : X \to Y$ is quasicompact and quasiseparated if and only if the preimage of any affine open subset of Y is a *finite* union of affine open sets in X, whose pairwise intersections are all *also* finite unions of affine open sets. The condition of quasiseparation is often paired with quasicompactness in hypotheses of theorems.

2.B. EXERCISE (A NONQUASISEPARATED SCHEME). Let $X = \operatorname{Spec} k[x_1, x_2, \dots]$, and let U be $X - [\mathfrak{m}]$ where $\mathfrak{m}$ is the maximal ideal $(x_1, x_2, \dots)$. Take two copies of X, glued along U. Show that the result is not quasiseparated. (This open immersion $U \hookrightarrow X$ came up earlier, as an example of a nonquasicompact open subset of an affine scheme.)

3. BACK TO SEPARATION

3.1. *Theorem. — Both separatedness and quasiseparatedness are preserved by base change.*

Proof. Suppose

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z \end{array}$$

is a fiber square. We will show that if $Y \to Z$ is separated or quasiseparated, then so is $W \to X$. The reader should verify that

$$\begin{array}{ccc} W & \xrightarrow{\delta_W} & W \times_X W \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\delta_Y} & Y \times_Z Y \end{array}$$

is a fiber diagram. (This is a categorical fact, and holds true in any category with fibered products.) As the property of being a closed immersion is preserved by base change, if δ_Y is a closed immersion, so is δ_X.

Quasiseparatedness follows in the identical manner, as quasicompactness is also preserved by base change. $\square$

3.2. *Proposition. — The condition of being separated is local on the target. Precisely, a morphism* $f : X \to Y$ *is separated if and only if for any cover of* Y *by open subsets* U_i, $f^{-1}(U_i) \to U_i$ *is separated for each* i.

3.3. Hence affine morphisms are separated, as maps from affine schemes to affine schemes are separated by an exercise from last day. In particular, finite morphisms are separated.

Proof. If $X \to Y$ is separated, then for any $U_i \hookrightarrow Y$, $f^{-1}(U_i) \to U_i$ is separated, as separatedness is preserved by base change (Theorem 3.1). Conversely, to check if $\Delta \hookrightarrow X \times_Y X$ is a closed subset, it suffices to check this on an open cover. If $g : X \times_Y X \to Y$ is the natural morphism, our open cover U_i of Y induces an open cover $f^{-1}(U_i) \times_{U_i} f^{-1}(U_i)$ of $X \times_Y X$. Then $f^{-1}(U_i) \to U_i$ separated implies $f^{-1}(U_i) \to f^{-1}(U_i) \times_{U_i} f(U_i)$ is a closed immersion by definition of separatedness. □

3.A. EXERCISE. Prove that the condition of being quasiseparated is local on the target. (Hint: the condition of being quasicompact is local on the target; use a similar argument.)

3.4. *Proposition. — (a) The condition of being separated is closed under composition. In other words, if* $f : X \to Y$ *is separated and* $g : Y \to Z$ *is separated, then* $g \circ f : X \to Z$ *is separated.*
(b) The condition of being quasiseparated is closed under composition.

Proof. (a) We are given that $\delta_f : X \hookrightarrow X \times_Y X$ and $\delta_g : Y \to Y \times_Z Y$ are closed immersions, and we wish to show that $\delta_h : X \to X \times_Z X$ is a closed immersion. Consider the diagram

$$\begin{array}{ccccc} X & \xrightarrow{\delta_f} & X \times_Y X & \xrightarrow{c} & X \times_Z X \\ & & \downarrow & & \downarrow \\ & & Y & \xrightarrow{\delta_g} & Y \times_Z Y. \end{array}$$

The square is the magic fibered diagram I've discussed before. As δ_g is a closed immersion, c is too (closed immersions are preserved by base change). Thus $c \circ \delta_f$ is a closed immersion (the composition of two closed immersions is also a closed immersion, an earlier exercise).

(b) The identical argument (with "closed immersion" replaced by "quasicompact") shows that the condition of being quasiseparated is closed under composition. □

3.5. *Proposition. — Any quasiprojective* A*-scheme is separated over* A.

As a corollary, any reduced quasiprojective k-scheme is a k-variety.

Proof. Suppose $X \to \operatorname{Spec} A$ is a quasiprojective A-scheme. The structure morphism can be factored into an open immersion composed with a closed immersion followed by $\mathbb{P}^n_A \to A$. Open immersions and closed immersions are separated (an earlier exercise, from last day I think), and $\mathbb{P}^n_A \to A$ is separated (a Proposition from last day). Separated morphisms are separated (Proposition 3.4), so we are done. □

3.6. *Proposition. — Suppose* $f : X \to Y$ *and* $f' : X' \to Y'$ *are separated (resp. quasiseparated) morphisms of* S*-schemes (where* S *is a scheme). Then the product morphism* $f \times f' : X \times_S X' \to Y \times_S Y'$ *is separated (resp. quasiseparated).*

Proof. An earlier exercise showed that the product of two morphisms having a property has the same property, so long as that property is preserved by base change, and composition. □

3.7. Applications.

As a first application, we define the *graph morphism*.

3.8. *Definition.* Suppose $f : X \to Y$ is a morphism of Z-schemes. The morphism $\Gamma_f : X \to X \times_Z Y$ given by $\Gamma_f = (\mathrm{id}, f)$ is called the **graph morphism**. Then f factors as $\mathrm{pr}_2 \circ \Gamma_f$, where pr_2 is the second projection (see Figure 2).

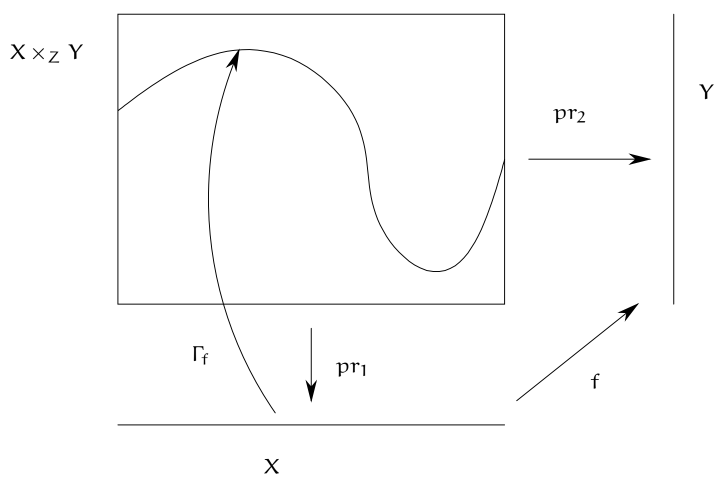


FIGURE 2. The graph morphism

3.9. *Proposition. — The graph morphism* Γ *is always a locally closed immersion. If* Y *is a separated* Z*-scheme (i.e. the structure morphism* $Y \to Z$ *is separated), then* Γ *is a closed immersion.*

This will be generalized in Exercise 3.B.

Proof by Cartesian diagram.

$$\begin{array}{ccc} X & \longrightarrow & X \times_Z Y \\ \big\downarrow & & \big\downarrow \\ Y & \xrightarrow{\delta} & Y \times_Z Y \end{array}$$

The notions of locally closed immersion and closed immersion are preserved by base change, so if the bottom arrow δ has one of these properties, so does the top. □

We now come to a very useful, but bizarre-looking, result.

3.10. *Cancellation Theorem for a Property* P *of Morphisms. — Let* P *be a class of morphisms that is preserved by base change and composition. Suppose*

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle g} & \\ & Z & \end{array}$$

is a commuting diagram of schemes.

(a) *Suppose that the diagonal morphism* $\delta_g : Y \to Y \times_Z Y$ *is in* P *and* $h : X \to Z$ *is in* P. *The* $f : X \to Y$ *is in* P.
(b) *In particular, suppose that closed immersions are in* P. *Then if* h *is in* P *and* g *is separated, then* f *is in* P.

When you plug in different P, you get very different-looking (and non-obvious) consequences.

For example, locally closed immersions are separated, so by part (a), if you factor a locally closed immersion $X \to Z$ into $X \to Y \to Z$, then $X \to Y$ *must* be a locally closed immersion.

Possibilities for P in case (b) include: finite morphisms, morphisms of finite type, closed immersions, affine morphisms.

Proof of (a). By the fibered square

$$\begin{array}{ccc} X & \xrightarrow{\Gamma_f} & X \times_Z Y \\ {\scriptstyle f}\big\downarrow & & \big\downarrow \\ Y & \xrightarrow{\delta_g} & Y \times_Z Y \end{array}$$

we see that the graph morphism $\Gamma : X \to X \times_Z Y$ is in P (Definition 3.8), as P is closed under base change. By the fibered square

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{h'} & Y \\ \downarrow & & \downarrow g \\ X & \xrightarrow{h} & Z \end{array}$$

the projection $h' : X \times_Z Y \to Y$ is in P as well. Thus $f = h' \circ \Gamma$ is in P □

Here now are some fun and useful exercises.

3.B. EXERCISE. Suppose $\pi : Y \to X$ is a morphism, and $s : X \to Y$ is a *section* of a morphism, i.e. $\pi \circ s$ is the identity on X. Show that s is a locally closed immersion. Show that if π is separated, then s is a closed immersion. (This generalizes Proposition 3.9.) Give an example to show that s needn't be a closed immersion if π isn't separated.

3.C. EXERCISE. Show that a A-scheme is separated (over A) if and only if it is separated over $\mathbb{Z}$. (In particular, a complex scheme is separated over $\mathbb{C}$ if and only if it is separated over $\mathbb{Z}$, so complex geometers and arithmetic geometers can communicate about separated schemes without confusion.)

3.D. USEFUL EXERCISE: THE LOCUS WHERE TWO MORPHISMS AGREE. Suppose f and g are two morphisms $X \to Y$, over some scheme Z. We can now give meaning to the phrase 'the locus where f and g agree', and that in particular there is a smallest locally closed subscheme where they agree. Suppose $h : W \to X$ is some morphism (perhaps a locally closed immersion). We say that f and g agree on h if $f \circ h = g \circ h$. Show that there is a locally closed subscheme $i : V \hookrightarrow X$ such that any morphism $h : W \to X$ on which f and g agree factors uniquely through i, i.e. there is a unique $j : W \to V$ such that $h = i \circ j$. (You may recognize this as a universal property statement.) Show further that if $V \to Z$ is separated, then $i : V \hookrightarrow X$ is a closed immersion. Hint: define V to be the following fibered product:

$$\begin{array}{ccc} V & \longrightarrow & Y \\ \downarrow & & \downarrow \delta \\ X & \xrightarrow{(f,g)} & Y \times_Z Y. \end{array}$$

As δ is a locally closed immersion, $V \to X$ is too. Then if $h : W \to X$ is any scheme such that $g \circ h = f \circ h$, then h factors through V.

Minor Remarks. 1) In the previous exercise, we are describing $V \hookrightarrow X$ by way of a universal property. Taking this as the definition, it is not a priori clear that V is a locally closed subscheme of X, or even that it exists.)

2) In the case of reduced finite type k-schemes, the locus where f and g agree can be interpreted as follows. f and g agree at x if $f(x) = g(x)$, and the two maps of residue fields are the same.

3) Notice that Z arises as part of the hypothesis, but is not present in the conclusion!

3.E. EXERCISE. Show that the line with doubled origin X is not separated, by finding two morphisms $f_1, f_2 : W \to X$ whose domain of agreement is not a closed subscheme. (Another argument was given in an exercise, I believe last day.)

3.F. LESS IMPORTANT EXERCISE. Suppose P is a class of morphisms such that closed immersions are in P, and P is closed under fibered product and composition. Show that if $f : X \to Y$ is in P then $f^{\text{red}} : X^{\text{red}} \to Y^{\text{red}}$ is in P. (Two examples are the classes of separated morphisms and quasiseparated morphisms.) Hint:

$$\begin{array}{ccccc} X^{\text{red}} & \longrightarrow & X \times_Y Y^{\text{red}} & \longrightarrow & Y^{\text{red}} \\ & \searrow & \downarrow & & \downarrow \\ & & X & \longrightarrow & Y \end{array}$$

4. RATIONAL MAPS

This is a historically ancient topic. It has appeared late for us because we have just learned about separatedness. Informally: a rational map is a "morphism $X \to Y$ defined almost everywhere". We will see that in good situations that where a rational map is defined, it is uniquely defined.

When discussing rational maps, unless otherwise stated, *we will assume* X *and* Y *to be integral and separated*, although the notions we will introduce can be useful in more general circumstances. The reader interested in more general notions should consider first the case where the schemes in question are reduced and separated, but not necessarily irreducible. Many notions can make sense in more generality (without reducedness hypotheses for example), but I'm not sure if there is a widely accepted definition.

A key example will be irreducible varieties, and the language of rational maps is most often used in this case.

A **rational map** from X to Y, denoted $X \dashrightarrow Y$, is a morphism on a dense open set, with the equivalence relation: $(f : U \to Y) \sim (g : V \to Y)$ if there is a dense open set $Z \subset U \cap V$ such that $f|_Z = g|_Z$. (In a moment, we will improve this to: if $f|_{U \cap V} = g|_{U \cap V}$.) People often use the word "map" for "morphism", which is quite reasonable. But then a rational map need not be a map. So to avoid confusion, when one means "rational map", one should never just say "map".

An obvious example of a rational map is a morphism. Another example is the following.

4.A. EASY EXERCISE. Interpret rational functions on a separated integral scheme as rational maps to $\mathbb{A}^1_{\mathbb{Z}}$. (This is analogous to functions corresponding to morphisms to $\mathbb{A}^1_{\mathbb{Z}}$, an earlier exercise.)

4.1. *Important Theorem. — Two* S*-morphisms* $f_1, f_2 : U \to Z$ *from a reduced scheme to a separated* S*-scheme agreeing on a dense open subset of* U *are the same.*

4.B. EXERCISE. Give examples to show how this breaks down when we give up reducedness of the base or separatedness of the target. Here are some possibilities. For the first, consider the two maps $\operatorname{Spec} k[x,y]/(y^2, xy) \to \operatorname{Spec} k[t]$, where we take f_1 given by $t \mapsto x$ and f_2 given by $t \mapsto x + y$; f_1 and f_2 agree on the distinguished open set $D(x)$. (See Figure 3.) For the second, consider the two maps from $\operatorname{Spec} k[t]$ to the line with the doubled origin, one of which maps to the "upper half", and one of which maps to the "lower half". these to morphisms agree on the dense open set $D(f)$. (See Figure 4.)

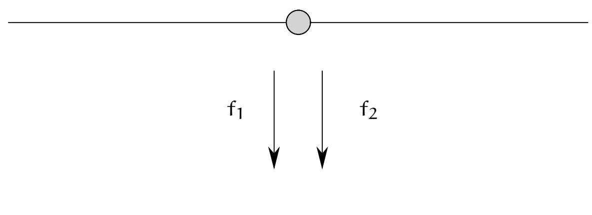


FIGURE 3. Two different maps from a nonreduced scheme agreeing on an open set

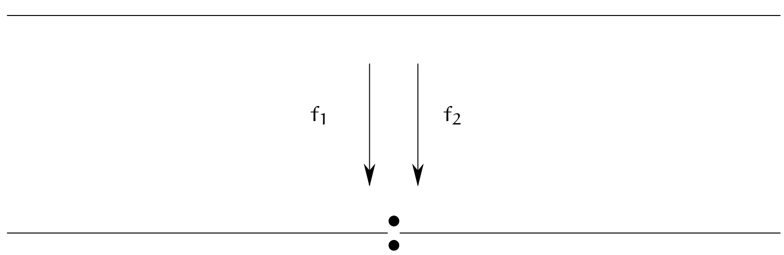


FIGURE 4. Two different maps to a nonseparated scheme agreeing on an open set

Proof. Let V be the locus where f_1 and f_2 agree. It is a closed subscheme of U by Exercise 3.D, which contains the generic point. But the only closed subscheme of a reduced scheme U containing the generic point is all of U. □

Consequence 1. Hence (as X is reduced and Y is separated) if we have two morphisms from open subsets of X to Y, say $f : U \to Y$ and $g : V \to Y$, and they agree on a dense open subset $Z \subset U \cap V$, then they necessarily agree on $U \cap V$.

Consequence 2. Also: a rational map has a largest **domain of definition** on which $f : U \dashrightarrow Y$ is a morphism, which is the union of all the domains of definition.

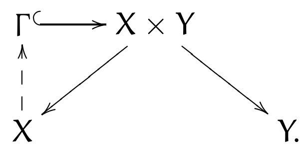
In particular, a rational function from a reduced scheme has a largest domain of definition.

4.2. *The graph of a rational map.*

Define the **graph** of a rational map $f : X \dashrightarrow Y$ as follows. Let (U, f') be any representative of this rational map (so $f' : U \to Y$ is a morphism). Let Γ_f be the scheme-theoretic closure of $\Gamma_{f'} \hookrightarrow U \times Y \hookrightarrow X \times Y$, where the first map is a closed immersion, and the second is an open immersion.

4.C. EXERCISE. Show that the graph of a rational map is independent of the choice of representative of the rational map.

In analogy with graphs of morphisms (e.g. Figure 2), the following diagram of a graph of a rational map can be handy.



5. DOMINANT AND BIRATIONAL MAPS

A rational map $f : X \dashrightarrow Y$ is **dominant** if for some (and hence every) representative $U \to Y$, the image is dense in Y. Equivalently, f is dominant if it sends the generic point of X to the generic point of Y.

5.A. EXERCISE. Show that you can compose two rational maps $f : X \dashrightarrow Y$, $g : Y \dashrightarrow Z$ if f is dominant.

In particular, integral separated schemes and dominant rational maps between them form a category which is geometrically interesting.

5.B. EASY EXERCISE. Show that dominant rational maps give morphisms of function fields in the opposite direction.

It is not true that morphisms of function fields give dominant rational maps, or even rational maps. For example, $\operatorname{Spec} k[x]$ and $\operatorname{Spec} k(x)$ have the same function field ($k(x)$), but there is no rational map $\operatorname{Spec} k[x] \dashrightarrow \operatorname{Spec} k(x)$. Reason: that would correspond to a morphism from an open subset U of $\operatorname{Spec} k[x]$, say $k[x, 1/f(x)]$, to $k(x)$. But there is no map of rings $k(x) \to k[x, 1/f(x)]$ for any one $f(x)$.

However, maps of function fields indeed give dominant rational maps in the case of varieties, see Proposition 5.1 below.

A rational map $f : X \to Y$ is said to be **birational** if it is dominant, and there is another rational map (a "rational inverse") that is also dominant, such that $f \circ g$ is (in the same equivalence class as) the identity on Y, and $g \circ f$ is (in the same equivalence class as) the identity on X. This is the notion of isomorphism in the category of integral separated schemes and dominant rational maps.

A *morphism* is **birational** if it is birational as a rational map. We say X and Y are **birational** (to each other) if there exists a birational map $X \dashrightarrow Y$. Birational maps induce isomorphisms of function fields. Proposition 5.1 will imply that a map between k-varieties that induces an isomorphism of function fields is birational.

We now prove a Proposition promised earlier.

5.1. *Proposition. — Suppose* X, Y *are irreducible varieties, and we are given* $f^{\#} : FF(Y) \hookrightarrow FF(Y)$*. Then there exists a dominant rational map* $f : X \dashrightarrow Y$ *inducing* $f^{\#}$*.*

Proof. By replacing Y with an affine open set, we may assume Y is affine, say $Y = \operatorname{Spec} k[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$. Then we have $x_1, \ldots, x_n \in K(X)$. Let U be an open subset of the domains of definition of these rational functions. Then we get a morphism $U \to \mathbb{A}^n_k$. But this morphism factors through $Y \subset \mathbb{A}^n$, as $x_1, \ldots, x_n$ satisfy the relations $f_1, \ldots, f_r$. $\square$

5.C. EXERCISE. Let K be a finitely generated field extension of k. Show there exists an irreducible k-variety with function field K. (Hint: let $x_1, \ldots, x_n$ be generators for K over k. Consider the map $k[t_1, \ldots, t_n] \to K$ given by $t_i \mapsto x_i$, and show that the kernel is a prime ideal $\mathfrak{p}$, and that $k[t_1, \ldots, t_n]/\mathfrak{p}$ has fraction field K. This can be interpreted geometrically: consider the map $\operatorname{Spec} K \to \operatorname{Spec} k[t_1, \ldots, t_n]$ given by the ring map $t_i \mapsto x_i$, and take the closure of the image.)

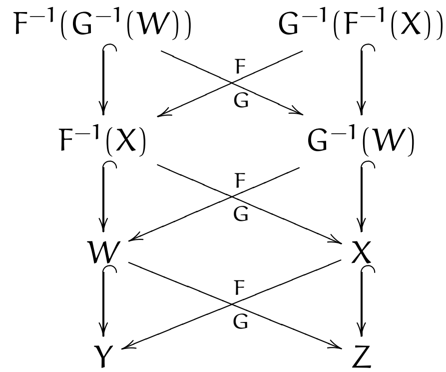
5.2. *Proposition. — Suppose* Y *and* Z *are integral* k*-varieties. Then* Y *and* Z *are birational if and only if there is a dense (=non-empty) open subscheme* U *of* Y *and a dense open subscheme* V *of* Z *such that* $U \cong V$*.*

This gives you a good idea of how to think of birational maps.

Proof. I find this proof kind of surprising and unexpected.

Clearly if Y and Z have isomorphic open sets U and V respectively, then they are birational (with birational maps given by the isomorphisms $U \to V$ and $V \to U$ respectively).

For the other direction, assume that $f : Y \dashrightarrow Z$ is a birational map, with inverse birational map $g : Z \dashrightarrow Y$. Choose representatives for these rational maps $F : W \to Y$ (where W is an open subscheme of Y) and $G : X \to Z$ (where Z is an open subscheme of Z). We will see that $F^{-1}(G^{-1}(W) \subset Y$ and $G^{-1}(F^{-1}(X)) \subset Z$ are isomorphic open subschemes.



The two morphisms $G \circ F$ and the identity from $F^{-1}(G^{-1}(W)) \to W$ represent to the same rational map, so by Theorem 4.1 they are the same morphism. Thus $G \circ F$ gives the identity map from $F^{-1}(G^{-1}(W))$ to itself. Similarly $F \circ G$ gives the identity map on $G^{-1}(F^{-1}(X))$. All that remains is to show that F maps $F^{-1}(G^{-1}(W))$ into $G^{-1}(F^{-1}(X))$, and that G maps $G^{-1}(F^{-1}(X))$ into $F^{-1}(G^{-1}(W))$, and by symmetry it suffices to show the former. Suppose $q \in F^{-1}(G^{-1}(W))$. Then $F(G(F(q)) = F(q) \in X$, from which $F(q) \in G^{-1}(F^{-1}(X))$. $\square$

6. Examples of rational maps

Here are some examples of rational maps. A recurring theme is that domains of definition of rational maps to projective schemes extend over nonsingular codimension one points. We'll make this precise when we discuss curves next quarter.

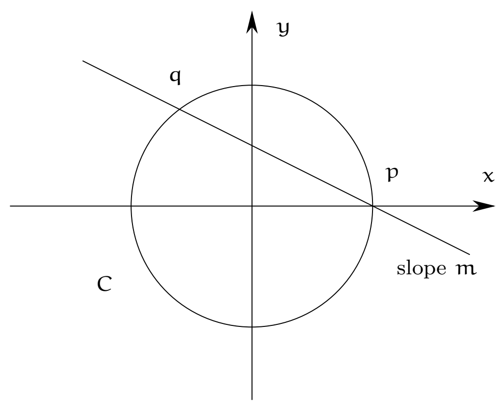


FIGURE 5. Finding primitive Pythagorean triples using geometry

The first example is how you find a formula for Pythagorean triples. Suppose you are looking for rational points on the circle C given by $x^2 + y^2 = 1$ (Figure 5). One rational point is $p = (1, 0)$. If q is another rational point, then pq is a line of rational (non-infinite) slope. This gives a rational map from the conic C to $\mathbb{A}^1$. Conversely, given a line of slope m through p, where m is rational, we can recover q as follows: $y = m(x - 1)$, $x^2 + y^2 = 1$.

We substitute the first equation into the second, to get a quadratic equation in x. We know that we will have a solution $x = 1$ (because the line meets the circle at $(x, y) = (1, 0)$), so we expect to be able to factor this out, and find the other factor. This indeed works:

$$\begin{aligned} x^2 + (m(x-1))^2 &= 1 \\ \Rightarrow \qquad (m^2+1)x^2 + (-2)x + (m^2-1) &= 0 \\ \Rightarrow \qquad (x-1)((m^2+1)x - (m^2-1)) &= 0 \end{aligned}$$

The other solution is $x = (m^2-1)/(m^2+1)$, which gives $y = 2m/(m^2+1)$. Thus we get a birational map between the conic C and $\mathbb{A}^1$ with coordinate m, given by $f : (x, y) \mapsto y/(x-1)$ (which is defined for $x \neq 1$), and with inverse rational map given by $m \mapsto ((m^2-1)/(m^2+1), 2m/(m^2+1))$ (which is defined away from $m^2+1 = 0$).

We can extend this to a rational map $C \dashrightarrow \mathbb{P}^1$ via the inclusion $\mathbb{A}^1 \to \mathbb{P}^1$. Then f is given by $(x, y) \mapsto [y; x-1]$. We then have an interesting question: what is the domain of definition of f? It appears to be defined everywhere except for where $y = x - 1 = 0$, i.e. everywhere but p. But in fact it can be extended over p! Note that $(x, y) \mapsto [x+1; -y]$ (where $(x, y) \neq (-1, y)$) agrees with f on their common domains of definition, as $[x+1; -y] = [y; x-1]$. Hence this rational map can be extended farther than we at first thought. This will be a special case of a result we'll see later .

(For the curious: we are working with schemes over $\mathbb{Q}$. But this works for any scheme over a field of characteristic not 2. What goes wrong in characteristic 2?)

6.A. EXERCISE. Use the above to find a "formula" yielding all Pythagorean triples.

6.B. EXERCISE. Show that the conic $x^2 + y^2 = z^2$ in $\mathbb{P}^2_k$ is isomorphic to $\mathbb{P}^1_k$ for any field k of characteristic not 2. (We've done this earlier in the case where k is algebraically closed, by diagonalizing quadrics.)

In fact, any conic in $\mathbb{P}^2_k$ with a k-valued point (i.e. a point with residue field k) is isomorphic to $\mathbb{P}^1_k$. (This hypothesis is certainly necessary, as $\mathbb{P}^1_k$ certainly has k-valued points. $x^2 + y^2 + z^2 = 0$ over $k = \mathbb{R}$ gives an example of a conic that is not isomorphic to $\mathbb{P}^1_k$.)

6.C. EXERCISE. Find all rational solutions to $y^2 = x^3 + x^2$, by finding a birational map to $\mathbb{A}^1$, mimicking what worked with the conic.

You will obtain a rational map to $\mathbb{P}^1$ that is not defined over the node $x = y = 0$, and *can't* be extended over this codimension 1 set. This is an example of the limits of our future result showing how to extend rational maps to projective space over codimension 1 sets: the codimension 1 sets have to be nonsingular. More on this soon!

6.D. EXERCISE. Use something similar to find a birational map from the quadric $Q = \{x^2 + y^2 = w^2 + z^2\}$ to $\mathbb{P}^2$. Use this to find all rational points on Q. (This illustrates a good way of solving Diophantine equations. You will find a dense open subset of Q that is isomorphic to a dense open subset of $\mathbb{P}^2$, where you can easily find all the rational points. There will be a closed subset of Q where the rational map is not defined, or not an isomorphism, but you can deal with this subset in an ad hoc fashion.)

6.E. IMPORTANT CONCRETE EXERCISE (A FIRST VIEW OF A BLOW-UP). Let k be an algebraically closed field. (We make this hypothesis in order to not need any fancy facts on nonsingularity.) Consider the rational map $\mathbb{A}^2_k \dashrightarrow \mathbb{P}^1_k$ given by $(x, y) \mapsto [x; y]$. I think you have shown earlier that this rational map cannot be extended over the origin. Consider the graph of the birational map, which we denote $\mathrm{Bl}_{(0,0)} \mathbb{A}^2_k$. It is a subscheme of $\mathbb{A}^2_k \times \mathbb{P}^1_k$. Show that if the coordinates on $\mathbb{A}^2$ are x, y, and the coordinates on $\mathbb{P}^1$ are u, v, this subscheme is cut out in $\mathbb{A}^2 \times \mathbb{P}^1$ by the single equation $xv = yu$. Describe the fiber of the morphism $\mathrm{Bl}_{(0,0)} \mathbb{A}^2_k \to \mathbb{P}^1_k$ over each closed point of $\mathbb{P}^1_k$. Describe the fiber of the morphism $\mathrm{Bl}_{(0,0)} \mathbb{A}^2_k \to \mathbb{A}^2_k$. Show that the fiber over $(0, 0)$ is an effective Cartier divisor (a closed subscheme that is locally principal and not a zero-divisor). It is called the *exceptional divisor*.

6.F. EXERCISE (THE CREMONA TRANSFORMATION, A USEFUL CLASSICAL CONSTRUCTION). Consider the rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, given by $[x; y; z] \to [1/x; 1/y; 1/z]$. What is the the domain of definition? (It is bigger than the locus where $xyz \neq 0$!) You will observe that you can extend it over codimension 1 sets. This will again foreshadow a result we will soon prove.

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