FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASSES 15 AND 16

RAVI VAKIL

CONTENTS

This week we discussed fibered products and separatedness.

1. FIBERED PRODUCTS OF SCHEMES EXIST

We will now construct the fibered product in the category of schemes. In other words, given $X, Y \to Z$, we will show that $X \times_Z Y$ exists. (Recall that the *absolute product* in a category is the fibered product over the final object, so $X \times Y = X \times_{\mathbb{Z}} Y$ in the category of schemes, and $X \times Y = X \times_S Y$ if we are implicitly working in the category of S-schemes, for example if S is the spectrum of a field.) Notational warning: lazy people wanting to save chalk and ink will write $\times_k$ for $\times_{\operatorname{Spec} k}$, and similarly for $\times_{\mathbb{Z}}$. It already happened in the paragraph above!

Before we get started, we'll make a few random remarks.

Remark 1. We've made a big deal about schemes being *sets*, endowed with a *topology*, upon which we have a *structure sheaf*. So you might think that we'll construct the product in this order. However, here is a sign that something interesting happens at the level of sets that will mess up this strategy. you should believe that if we take the product of two affine lines (over your favorite algebraically closed field k, say), you should get the affine plane: $\mathbb{A}^1_k \times_k \mathbb{A}^1_k$ should be $\mathbb{A}^2_k$. And we'll see that this is indeed true. But the underlying set of the latter is *not* the underlying set of the former —- we get additional points! Thus products of schemes do something a little subtle on the level of sets.

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1.A. EXERCISE. If k is algebraically closed, describe a natural map of *sets* $\mathbb{A}^1_k \times \mathbb{A}^1_k \to \mathbb{A}^2_k$. Show that this map is not surjective. On the other hand, show that it is a bijection on closed points.

Remark 2. Recall that the diagram of a fibered square

$$\begin{tikzcd} W \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

goes by a number of names, including *fibered diagram*, *Cartesian diagram*, *fibered square*, and *Cartesian square*. Because of its geometric interpretation, in algebraic geometry it is often called a **base change diagram** or a **pullback diagram**, and $W \to X$ is called the **pullback** of $Y \to Z$ by f, and W is called the **pullback** of Y by f.

The reason for the phrase "base change" or "pullback" is the following. If X is a point of Z (i.e. f is the natural map of Spec of the residue field of a point of Z into Z), then W is interpreted as the fiber of the family.

1.B. EXERCISE. Show that in the category of topological spaces, this is true, i.e., if $Y \to Z$ is a continuous map, and X is a point p of Z, then the fiber of Y over p is naturally identified with $X \times_Z Y$.

More generally, for general $X \to Z$, the fiber of $W \to X$ over a point p of X is naturally identified with the fiber of $Y \to Z$ over $f(p)$.

Let's now show that fibered products always exist in the category of schemes.

1.1. *Big Theorem (fibered products always exist). — Suppose $f : X \to Z$ and $g : Y \to Z$ are morphisms of schemes. Then the fibered product*

$$\begin{tikzcd} X \times_Z Y \arrow[r, "f'"] \arrow[d, "g'"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

exists in the category of schemes.

As always when showing that certain objects defined by universal properties exist, we have two ways of looking at the objects in practice: by using the universal property, or by using the details of the construction.

The key idea, roughly, is this: we cut everything up into affine open sets, do fibered products in that category (where it turns out we have seen the concept before in a different guise), and show that everything glues nicely. The conceptually difficult part of the proof comes from the gluing, and realizing that we have to check almost nothing.

The proof will be a little long, but you will notice that we repeat a kind of argument several times. A much shorter proof is possible by interpreting this in the language of representable functors, and we give this proof afterward for experts.

Proof. We have an extended proof by universal property. We divide the proof up into a number of bite-sized pieces. Between bites, we will often take a break for some side comments.

Step 1: everything affine. First, if X, Y, Z are affine schemes, say $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, $Z = \operatorname{Spec} C$, the fibered product exists, and is $\operatorname{Spec} A \otimes_C B$. Here's why. Suppose W is any scheme, along with morphisms $f'' : W \to X$ and $g'' : W \to Y$ such that $f \circ f'' = g \circ g''$ as morphisms $W \to Z$. We hope that there exists a unique $h : W \to \operatorname{Spec} A \otimes_C B$ such that $f'' = g' \circ h$ and $g'' = f' \circ h$.

$$\begin{tikzcd}
W \arrow[dr, "\exists!?"] \arrow[drr, bend left, "g''"] \arrow[ddr, bend right, "f''"'] & & \\
& \operatorname{Spec} A \otimes_C B \arrow[r, "f'"'] \arrow[d, "g'"] & \operatorname{Spec} B \arrow[d, "g"] \\
& \operatorname{Spec} A \arrow[r, "f"] & \operatorname{Spec} C
\end{tikzcd}$$

But maps to affine schemes correspond precisely to maps of global sections in the other direction (earlier exercise):

$$\begin{tikzcd}
\Gamma(W, \mathcal{O}_W) & & \\
& A \otimes_C B \arrow[ul, "\exists!?"] & B \arrow[l, "f'"'] \arrow[ull, bend right, "g''"'] \\
& A \arrow[u, "g'"] \arrow[uul, bend left, "f''"] & C \arrow[l, "f"] \arrow[u, "g"]
\end{tikzcd}$$

But this is precisely the universal property for tensor product! (The tensor product is the cofibered product in the category of rings.)

1.2. *Side remark (cf. Exercise 1.A).* Thus indeed $\mathbb{A}^1 \times \mathbb{A}^1 \cong \mathbb{A}^2$, and more generally $(\mathbb{A}^1)^n \cong \mathbb{A}^n$.

Step 2: fibered products with open immersions. Second, we note that the fibered product with open immersions always exists: if $Y \hookrightarrow Z$ an open immersion, then for any $f : X \to Z$, $X \times_Z Y$ is the open subset $f^{-1}(Y)$. (More precisely, this open subset satisfies the universal property.) This was an earlier exercise (which wasn't hard).

$$\begin{tikzcd}
f^{-1}(Y) \arrow[r] \arrow[d, hook] & Y \arrow[d, hook] \\
X \arrow[r, "f"] & Z
\end{tikzcd}$$

Step 3: fibered products of affine with almost-affine over affine. We can combine steps 1 and 2 as follows. Suppose X and Z are affine, and $Y \to Z$ factors as $Y \overset{i}{\hookrightarrow} Y' \xrightarrow{g} Z$ where i is an open immersion and Y' is affine. Then $X \times_Z Y$ exists. This is because if the two smaller squares of

$$\begin{tikzcd} W \arrow[r] \arrow[d] & Y \arrow[d] \\ W' \arrow[r] \arrow[d] & Y' \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

are fibered diagrams, then the "outside rectangle" is also a fibered diagram. (This was an earlier exercise, although you should be able to see this on the spot.)

Key Step 4: fibered product of affine with arbitrary over affine exists. We now come to the key part of the argument: if X and Z are affine, and Y is arbitrary. This is confusing when you first see it, so we'll first deal with a special case, when Y is the union of two affine open sets $Y_1 \cup Y_2$. Let $Y_{12} = Y_1 \cap Y_2$.

Now for $i = 1, 2$, $X \times_Z Y_i$ exists by Step 1; call this W_i. Also, $X \times_Z Y_{12}$ exists by Step 3 (call it W_{12}), and comes with natural open immersions into W_1 and W_2. Thus we can glue W_1 to W_2 along W_{12}; call this resulting scheme W.

We'll check that this is the fibered product by verifying that it satisfies the universal property. Suppose we have maps $f'' : V \to X$, $g'' : V \to Y$ that compose (with f and g respectively) to the same map $V \to Z$. We need to construct a unique map $h : V \to W$, so that $f' \circ h = g''$ and $g' \circ h = f''$.

$$\begin{tikzcd} V \arrow[dr, "\exists!?"] \arrow[drr, "g''"] \arrow[ddr, "f''"'] & & \\ & W \arrow[r, "f'"'] \arrow[d, "g'"] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"] & Z \end{tikzcd}$$

For $i = 1, 2$, define $V_i := (g'')^{-1}(Y_i)$. Define $V_{12} := (g'')^{-1}(Y_{12}) = V_1 \cap V_2$. Then there is a unique map $V_i \to W_i$ such that the composed maps $V_i \to X$ and $V_i \to Y_i$ are desired (by the universal product of the fibered product $X \times_Z Y_i = W_i$), hence a unique map $h_i : V_i \to W$. Similarly, there is a unique map $h_{12} : V_{12} \to W$ such that the composed maps $V_{12} \to X$ and $V_{12} \to Y$ are as desired. But the restriction of h_i to V_{12} is one such map, so it must be h_{12}. Thus the maps h_1 and h_2 agree on V_{12}, and glue together to a unique map $h : V \to W$. We have shown existence and uniqueness of the desired h. (We are using the fact that "morphisms glue", which corresponds to the fact that maps to a scheme form a sheaf. This leads to a shorter explanation of the proof, which we give at the end of this long proof.)

We have thus shown that if Y is the union of two affine open sets, and X and Z are affine, then $X \times_Z Y$ exists.

We now tackle the general case. (The reader may prefer to first think through the case where "two" is replaced by "three".) We now cover Y with open sets Y_i, as i runs over some index set (not necessarily finite!). As before, we define W_i and W_{ij}. We can glue these together to produce a scheme W along with open sets we identify with W_i (Exercise 4.H in the current revised version of the class 7/8 notes).

As in the two-affine case, we show that W is the fibered product by showing that it satisfies the universal property. Suppose we have maps $f'' : V \to X$, $g'' : V \to Y$ that compose to the same map $V \to Z$. We construct a unique map $h : V \to W$, so that $f' \circ h = g''$ and $g' \circ h = f''$. Define $V_i = (g'')^{-1}(Y_i)$ and $V_{ij} := (g'')^{-1}(Y_{ij}) = V_i \cap V_j$. Then there is a unique map $V_i \to W_i$ such that the composed maps $V_i \to X$ and $V_i \to Y_i$ are desired, hence a unique map $h_i : V_i \to W$. Similarly, there is a unique map $h_{ij} : V_{ij} \to W$ such that the composed maps $V_{ij} \to X$ and $V_{ij} \to Y$ are as desired. But the restriction of h_i to V_{ij} is one such map, so it must be h_{ij}. Thus the maps h_i and h_j agree on V_{ij}. Thus the h_i glue together to a unique map $h : V \to W$. We have shown existence and uniqueness of the desired h, completing this step.

Side remark. One special case of it is called **extending the base field**: if X is a k-scheme, and k' is a field extension (often k' is the algebraic closure of k), then $X \times_{\operatorname{Spec} k} \operatorname{Spec} k'$ (sometimes informally written $X \times_k k'$ or $X_{k'}$) is a k'-scheme. Often properties of X can be checked by verifying them instead on $X_{k'}$. This is the subject of **descent** — certain properties "descend" from $X_{k'}$ to X. We have already seen that the property of being *normal* descends in this way (in an earlier exercise).

Step 5: Z affine, X and Y arbitrary. We next show that if Z is affine, and X and Y are arbitrary schemes, then $X \times_Z Y$ exists. We just follow Step 4, with the roles of X and Y reversed, using the fact that by the previous step, we can assume that the fibered product with an affine scheme with an arbitrary scheme over an affine scheme exists.

Step 6: Z admits an open immersion into an affine scheme Z', X and Y arbitrary. This is akin to Step 3: $X \times_Z Y$ satisfies the universal property of $X \times_{Z'} Y$.

Step 7: the general case. We again employ the trick from Step 4. Say $f : X \to Z$, $g : Y \to Z$ are two morphisms of schemes. Cover Z with affine open subsets Z_i. Let $X_i = f^{-1}X_i$ and $Y_i = g^{-1}Y_i$. Define $Z_{ij} = Z_i \cap Z_j$, and X_{ij} and Y_{ij} analogously. Then $W_i := X_i \times_{Z_i} Y_i$ exists for all i, and has as open sets $W_{ij} := X_{ij} \times_{Z_{ij}} Y_{ij}$ along with gluing information satisfying the cocycle condition (arising from the gluing information for Z from the Z_i and Z_{ij}). Once again, we show that this satisfies the universal property. Suppose V is any scheme, along with maps to X and Y that agree when they are composed to Z. We need to show that there is a unique morphism $V \to W$ completing the diagram

$$\begin{tikzcd}
V \arrow[dr, "\exists!?"] \arrow[drr, bend left, "g''"] \arrow[ddr, bend right, "f''"'] & & \\
& W \arrow[r, "f'"] \arrow[d, "g'"] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"] & Z.
\end{tikzcd}$$

Now break V up into open sets $V_i = g'' \circ f^{-1}(Z_i)$. Then by the universal property for W_i, there is a unique map $V_i \to W_i$ (which we can interpret as $V_i \to W$). Thus we have already shown uniqueness of $V \to W$. These must agree on $V_i \cap V_j$, because there is only one map $V_i \cap V_j$ to W making the diagram commute. Thus all of these morphisms $V_i \to W$ glue together, so we are done. $\square$

1.3. For experts only!: Describing the existence of fibered products using high-falutin' language.

(Thanks to Jarod for suggesting that I include this, and helping me think through how best to present it. If you have suggestions to make this clearer — to experts of course — please let me know!)

The previous proof can be described more cleanly in the language of representable functors. You'll find this enlightening only after you have absorbed the argument above and meditated on it for a long time. For experts, we include the more abstract picture here. You might find that this is most useful to shed light on representable functors, rather than on the existence of the fibered product.

Recall that to each scheme X we have a contravariant functor h^X from the category of schemes **Sch** to the category of **Sets**, taking a scheme Y to $\mathrm{Mor}(Y, X)$. It may be more convenient to think of it as a covariant functor $h^X : \mathbf{Sch}^{\mathrm{opp}} \to \mathbf{Sets}$.

But this functor h^X is better than a functor. We know that if $\{U_i\}$ is an open cover of Y, a morphism $Y \to X$ is determined by its restrictions $U_i \to X$, and given morphisms $U_i \to X$ that agree on the overlap $U_i \cap U_j \to X$, we can glue them together to get a morphism $Y \to X$. (This is roughly our statement that "morphisms glue".) In the language of equalizer exact sequences,

$$\cdot \longrightarrow \mathrm{Hom}(Y, X) \longrightarrow \prod \mathrm{Hom}(U_i, X) \rightrightarrows \prod \mathrm{Hom}(U_i \cap U_j, X) \ .$$

Thus morphisms to X (i.e. the functor h^X) form a sheaf on every scheme X. If this holds, we say that *the functor is a sheaf*. (If you want to impress your friends and frighten your enemies, you can tell them that this is a *sheaf on the big Zariski site*.)

We can repeat this discussion for the category $\mathbf{Sch}_S$ of schemes over a given base scheme S.

Notice that the definition of fibered product also gives a contravariant functor

$$h_{X \times_Z Y} : \mathbf{Sch} \to \mathbf{Sets} :$$

to the scheme W we associate the set of commutative diagrams

$$\begin{tikzcd} W \arrow[d] & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

(What is the image of $W \to W'$ under this functor?) The existence of fibered product is precise the statement that there is a natural isomorphism of functors $h_{X \times_Z Y} \cong h_W$ for some scheme W. In that case, we say that $h_{X \times_Z Y}$ is a **representable functor**, and that it is **representable by** W. The usual universal property argument shows that this determines W up to unique isomorphism.

We can now interpret Key Step 4 of the proof of Theorem 1.1 as follows. Suppose X and Z are affine, and Y_i is an affine open cover of Y. Suppose the covariant functor $F_Y : (\mathbf{Sch}_Y)^{\text{opp}} \to \mathbf{Sets}$ is a sheaf on the category of Y-schemes $\mathbf{Sch}_Y$, and F_{Y_i} is the "restriction of the sheaf to Y_i" (where we include only those Y-schemes that are in fact Y_i-schemes, i.e. those $T \to Y$ whose structure morphisms factor through Y_i, $T \to Y_i \to Y$).

1.C. EXERCISE. Show that if F_{Y_i} is representable, then so is F_Y. (Hint: this is basically just the proofs of Steps 3 and 4.)

We then apply this in the special case where F_Y is given by

$$\left(T \xrightarrow{f} Y \right) \mapsto \left(\begin{tikzcd} T \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd} \right).$$

[I don't see how to make that diagram on the right look good...]

1.D. EXERCISE. Check that this F_Y is a sheaf. (This is not hard once you realize what this is asking.)

Then Steps 5 through 7 are one-liners; you should think these through. (For Step 5, you'll replace Y by X. For Steps 6/7, you'll replace Y by Z.)

We can make this argument slicker still (and not have to repeat three similar arguments) as follows. (This is frighteningly abstract.) One of Grothendieck's insights is that we should hope to treat contravariant functors $\mathbf{Sch} \to \mathbf{Sets}$ as "geometric spaces", even if we don't know if they are representable. For this reason, I'll call such a functor (for this section only!) a functor-space, to emphasize that we are thinking of it as some sort of spaces. Many notions carry over to this more general setting without change, and some notions are easier. For example, a morphism of functor-spaces $h \to h'$ is just a natural transformation of functors. The following exercise shows that this extends the notion of morphisms of schemes.

1.E. EXERCISE. Show that if X and Y are schemes, then there is a natural bijection between morphisms of schemes $X \to Y$ and morphisms of functor spaces $h^X \to h^Y$. (Hint: this has nothing to do with schemes; your argument will work in any category.)

Also, fibered products of functor-spaces always exist: $h \times_{h''} h'$ may be defined by

$$h \times_{h''} h'(W) = h(W) \times_{h''(W)} h'(W)$$

(where the fibered product on the right is a fibered product of sets, and those always exist). Notice that this didn't use any properties of schemes; this works with **Sch** replaced by any category.

We can make some other definitions that extend notions from schemes to functor-spaces. We say that $h \to h'$ express h as an **open subfunctor** of h' if for all representative morphisms h^X and maps $h^X \to h'$, the fibered product $h^X \times_{h'} h$ is representable, by u say, and $h^U \to h^X$ is an open immersion. the following fibered square may help.

$$\begin{tikzcd} h^Y \arrow[r] \arrow[d, "open"'] & h \arrow[d] \\ h^X \arrow[r] & h' \end{tikzcd}$$

Notice that a morphism of representable functor spaces $h^W \to h^Z$ is an open immersion if and only if $W \to Z$ is an open immersion, so this indeed extends the notion of open immersion to these functors.

A collection h_i of open subfunctors of h' is said to **cover** h' if for each map $h^X \to h'$ from a representable subfunctor, the corresponding open subsets $U_i \hookrightarrow X$ cover X.

1.F. KEY EXERCISE. If a functor-space h is a sheaf that has an open cover by representable functor-spaces ("is covered by schemes"), then h is representable.

Given this formalism, we can now give a quick description of the proof of the existence of fibered products. Exercise 1.D showed that $h_{X \times_Z Y}$ is a sheaf.

1.G. EXERCISE. Suppose $(Z_i)_i$ is an affine cover of Z, $(X_{ij})_j$ is an affine cover of the preimage of Z_i in X, and $(Y_{ik})_k$ is an affine cover of the preimage of Z_i in Y. Show that $(h_{X_{ij} \times_{Z_i} Y_{ik}})_{ijk}$ is an open cover of the functor $h_{X \times_Z Y}$. (Hint: use the definition of open covers!)

But $(h_{X_{ij} \times_{Z_i} Y_{ik}})_{ijk}$ is representable (fibered products of affines over and affine exist, Step 1 of the proof of Theorem 1.1), so we are done.

2. COMPUTING FIBERED PRODUCTS IN PRACTICE

Before giving a bunch of examples, we should first see how to actually compute fibered products in practice.

There are four types of morphisms that it is particularly easy to take fibered products with, and all morphisms can be built from these four atomic components.

(1) Base change by open immersions.

We've already done this, and we used it in the proof that fibered products of schemes exist.

$$\begin{tikzcd} f^{-1}(Y) \arrow[r] \arrow[d, hook] & Y \arrow[d, hook] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

I'll describe the remaining three on the level of affine open sets, because we obtain general fibered products by gluing.

(2) Adding an extra variable.

2.A. Easy algebra exercise.. Show that $B \otimes_A A[t] \cong B[t]$.

Hence the following is a fibered diagram.

$$\begin{tikzcd} \operatorname{Spec} B[t] \arrow[r] \arrow[d] & \operatorname{Spec} A[t] \arrow[d] \\ \operatorname{Spec} B \arrow[r] & \operatorname{Spec} A \end{tikzcd}$$

(3) Base change by closed immersions

2.B. Exercise. Suppose $\phi : A \to B$ is a ring homomorphism, and $I \subset A$ is an ideal. Let $I^e := \langle \phi(i) \rangle_{i \in I} \subset B$ be the *extension of* I *to* B. Describe a natural isomorphism $B/I^e \cong B \otimes_A (A/I)$. (Hint: consider $I \to A \to A/I \to 0$, and use the right-exactness of $\otimes_A B$.)

As an immediate consequence: the fibered product with a subscheme is the subscheme of the fibered product in the obvious way. We say that "closed immersions are preserved by base change".

As an application, we can compute tensor products of finitely generated k algebras over k. For example, we have a canonical isomorphism

$$k[x_1, x_2]/(x_1^2 - x_2) \otimes_k k[y_1, y_2]/(y_1^3 + y_2^3) \cong k[x_1, x_2, y_1, y_2]/(x_1^2 - x_2, y_1^3 + y_2^3).$$

2.1. *Example.* We can also use now compute $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$:

$$\begin{aligned} \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} &\cong \mathbb{C} \otimes_{\mathbb{R}} (\mathbb{R}[x]/(x^2+1)) \\ &\cong (\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}[x])/(x^2+1) \qquad \text{by } \textbf{(3)} \\ &\cong \mathbb{C}[x]/(x^2+1) \qquad \text{by } \textbf{(2)} \\ &\cong \mathbb{C}[x]/(x-i)(x+i) \\ &\cong \mathbb{C} \times \mathbb{C} \end{aligned}$$

Thus $\operatorname{Spec}\mathbb{C} \times_{\mathbb{R}} \operatorname{Spec}\mathbb{C} \cong \operatorname{Spec}\mathbb{C} \coprod \operatorname{Spec}\mathbb{C}$. This example is the first example of many different behaviors. Notice for example that two points somehow correspond to the Galois group of $\mathbb{C}$ over $\mathbb{R}$; for one of them, x (the "i" in one of the copies of $\mathbb{C}$) equals i (the "i" in the other copy of $\mathbb{C}$), and in the other, $x = -i$.

(4) Base change of affine schemes by localization.

2.C. EXERCISE. Suppose $\phi : A \to B$ is a ring homomorphism, and $S \subset A$ is a multiplicative subset of A, which implies that $\phi(S)$ is a multiplicative subset of B. Describe a natural isomorphism $\phi(S)^{-1}B \cong B \otimes_A (S^{-1}A)$.

Translation: the fibered product with a localization is the localization of the fibered product in the obvious way. We say that "localizations are preserved by base change". This is handy if the localization is of the form $A \hookrightarrow A_f$ (corresponding to taking distinguished open sets) or $A \hookrightarrow FF(A)$ (from A to the fraction field of A, corresponding to taking generic points), and various things in between.

These four facts let you calculate lots of things in practice, as we will see throughout the rest of this chapter.

2.D. EXERCISE: THE THREE IMPORTANT TYPES OF MONOMORPHISMS OF SCHEMES. Show that the following are monomorphisms: open immersions, closed immersions, and localization of affine schemes. As monomorphisms are closed under composition, compositions of the above are also monomorphisms (e.g. locally closed immersions, or maps from Spec of stalks at points of X to X).

3. PULLING BACK FAMILIES AND FIBERS OF MORPHISMS

3.1. Pulling back families.

We can informally interpret fibered product in the following geometric way. Suppose $Y \to Z$ is a morphism. We interpret this as a "family of schemes parametrized by a **base scheme** (or just plain **base**) Z." Then if we have another morphism $X \to Z$, we interpret the induced map $X \times_Z Y \to X$ as the "pulled back family".

$$\begin{tikzcd} X \times_Z Y \arrow[r] \arrow[d, "\text{pulled back family}"'] & Y \arrow[d, "\text{family}"] \\ X \arrow[r] & Z \end{tikzcd}$$

We sometimes say that $X \times_Z Y$ is the **scheme-theoretic pullback of** Y, **scheme-theoretic inverse image**, or **inverse image scheme of** Y. For this reason, fibered product is often called **base change** or **change of base** or **pullback**.

3.2. Fibers of morphisms.

Suppose $p \to Z$ is the inclusion of a point (not necessarily closed). (If K is the residue field of a point, we mean the canonical map $\operatorname{Spec} K \to Z$.) Then if $g : Y \to Z$ is any morphism, the base change with $p \to Z$ is called the **fiber of** g **above** p or the **preimage of** p, and is denoted $g^{-1}(p)$. If Z is irreducible, the fiber above the generic point is called the **generic fiber**. In an affine open subscheme $\operatorname{Spec} A$ containing p, p corresponds to some prime ideal $\mathfrak{p}$, and the morphism corresponds to the ring map $A \to A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$. this is the composition if localization and closed immersion, and thus can be computed by the tricks above.

(Quick remark: $p \to Z$ is a monomorphism, by Exercise 2.D.)

3.3. *Example.* The following example has many enlightening aspects. Consider the projection of the parabola $y^2 = x$ to the x axis over $\mathbb{Q}$, corresponding to the map of rings $\mathbb{Q}[x] \to \mathbb{Q}[y]$, with $x \mapsto y^2$. (If $\mathbb{Q}$ alarms you, replace it with your favorite field and see what happens.)

Then the preimage of 1 is two points:

$$\begin{aligned}
\operatorname{Spec} \mathbb{Q}[x,y]/(y^2-x) \otimes_{\mathbb{Q}} \operatorname{Spec} \mathbb{Q}[x]/(x-1) &\cong \operatorname{Spec} \mathbb{Q}[x,y]/(y^2-x, x-1) \\
&\cong \operatorname{Spec} \mathbb{Q}[y]/(y^2-1) \\
&\cong \operatorname{Spec} \mathbb{Q}[y]/(y-1) \coprod \operatorname{Spec} \mathbb{Q}[y]/(y+1).
\end{aligned}$$

The preimage of 0 is one nonreduced point:

$$\operatorname{Spec} \mathbb{Q}[x,y]/(y^2-x, x) \cong \operatorname{Spec} \mathbb{Q}[y]/(y^2).$$

The preimage of -1 is one reduced point, but of "size 2 over the base field".

$$\operatorname{Spec} \mathbb{Q}[x,y]/(y^2-x, x+1) \cong \operatorname{Spec} \mathbb{Q}[y]/(y^2+1) \cong \operatorname{Spec} \mathbb{Q}[i].$$

The preimage of the generic point is again one reduced point, but of "size 2 over the residue field", as we verify now.

$$\operatorname{Spec} \mathbb{Q}[x,y]/(y^2-x) \otimes \mathbb{Q}(x) \cong \operatorname{Spec} \mathbb{Q}[y] \otimes \mathbb{Q}(y^2)$$

i.e. you take elements polynomials in y, and you are allowed to invert polynomials in y^2. A little thought shows you that you are then allowed to invert polynomials in y, as if $f(y)$ is any polynomial in y, then

$$\frac{1}{f(y)} = \frac{f(-y)}{f(y)f(-y)},$$

and the latter denominator is a polynomial in y^2. Thus

$$\operatorname{Spec} \mathbb{Q}[x,y]/(y^2-x) \otimes \mathbb{Q}(x) \cong \mathbb{Q}(y)$$

which is a degree 2 field extension of $\mathbb{Q}(x)$.

Notice the following interesting fact: in each case, the number of preimages can be interpreted as 2, where you count to two in several ways: you can count points (as in the case of the preimage of 1); you can get non-reduced behavior (as in the case of the preimage of 0); or you can have a field extension of degree 2 (as in the case of the preimage of -1 or the generic point). In each case, the fiber is an affine scheme whose dimension as a vector space over the residue field of the point is 2. Number theoretic readers may have seen this behavior before. This is going to be symptomatic of a very special and important kind of morphism (a finite flat morphism).

Try to draw a picture of this morphism if you can, so you can develop a pictoral shorthand for what is going on.

Here are some other examples.

3.A. EXERCISE. Prove that $\mathbb{A}^n_R \cong \mathbb{A}^n_{\mathbb{Z}} \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec} R$. Prove that $\mathbb{P}^n_R \cong \mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec} R$.

3.B. EXERCISE. Show that the underlying topological space of the (scheme-theoretic) fiber $X \to Y$ above a point p is naturally identified with the topological fiber of $X \to Y$ above p.

3.C. EXERCISE. Show that for finite-type schemes over $\mathbb{C}$, the closed points (=complex-valued points by the Nullstellensatz) of the fibered product correspond to the fibered product of the complex-valued points. (You will just use the fact that $\mathbb{C}$ is algebraically closed.)

3.4. Here is a definition in common use. The terminology is a bit unfortunate, because it is a second (different) meaning of "points of a scheme". (Sadly, we'll even see a third different meaning soon, §4.2.) If T is a scheme, the **T-valued points of a scheme** X are defined to be the morphism $T \to X$. They are sometimes denoted $X(T)$. If A is a ring (most commonly in this context a field), the **A-valued points of a scheme** X are defined to be the morphism $\operatorname{Spec} A \to X$. They are sometimes denoted $X(A)$. For example, if k is an algebraically closed field, then the k-valued points of a finite type scheme are just the closed points; but in general, things can be weirder. (When we say "points of a scheme", and not A-valued points, we will always mean the usual meaning, not this meaning.)

3.D. EXERCISE. Describe a natural bijection $(X \times_Z Y)(T) \cong X(T) \times_{Z(T)} Y(T)$. (The right side is a fibered product of sets.) In other words, fibered products behaves well with respect to T-valued points. This is one of the motivations for this notion. (This generalizes Exercise 3.C.)

3.E. EXERCISE. Consider the morphism of schemes $X = \operatorname{Spec} k[t] \to Y = \operatorname{Spec} k[u]$ corresponding to $k[u] \to k[t]$, $t = u^2$, where $\operatorname{char} k \neq 2$. Show that $X \times_Y X$ has 2 irreducible components. (What happens if $\operatorname{char} k = 2$?)

3.F. EXERCISE GENERALIZING $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$. Suppose L/K is a finite Galois field extension. What is $L \otimes_K L$?

3.G. HARD BUT FASCINATING EXERCISE FOR THOSE FAMILIAR WITH THE GALOIS GROUP OF $\overline{\mathbb{Q}}$ OVER $\mathbb{Q}$. Show that the points of $\operatorname{Spec} \overline{\mathbb{Q}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ are in natural bijection with $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, and the Zariski topology on the former agrees with the profinite topology on the latter.

3.H. WEIRD EXERCISE. Show that $\operatorname{Spec} \mathbb{Q}(t) \otimes_{\mathbb{Q}} \mathbb{C}$ has closed points in natural correspondence with the transcendental complex numbers. (If the description $\operatorname{Spec} \mathbb{C}[t] \otimes_{\mathbb{Q}[t]} \mathbb{Q}(t)$ is more striking, you can use that instead.) This scheme doesn't come up in nature, but it is certainly neat!

4. PROPERTIES PRESERVED BY BASE CHANGE

We now discuss a number of properties that behave well under base change.

We've already shown that the notion of "open immersion" is preserved by base change. We did this by explicitly describing what the fibered product of an open immersion is: if $Y \hookrightarrow Z$ is an open immersion, and $f : X \to Z$ is any morphism, then we checked that the open subscheme $f^{-1}(Y)$ of X satisfies the universal property of fibered products.

We have also shown that the notion of "closed immersion" is preserved by base change (§2 **(3)**). In other words, given a fiber diagram

$$\begin{tikzcd} W \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r, hook, "\text{cl. imm.}"] & Z \end{tikzcd}$$

where $Y \hookrightarrow Z$ is a closed immersion, $W \to X$ is as well.

4.A. EASY EXERCISE. Show that locally principal closed subschemes pull back to locally principal closed subschemes.

Similarly, other important properties are preserved by base change.

4.B. EXERCISE. Show that the following properties of morphisms are preserved by base change.

(a) quasicompact
(b) quasiseparated
(c) affine morphism
(d) finite
(e) locally of finite type
(f) finite type
(g) locally of finite presentation
(h) finite presentation

4.C. EXERCISE. Show that the notion of "quasifinite morphism" (finite type + finite fibers) is preserved by base change. (Warning: the notion of "finite fibers" is not preserved by base change. $\operatorname{Spec} \overline{\mathbb{Q}} \to \operatorname{Spec} \mathbb{Q}$ has finite fibers, but $\operatorname{Spec} \overline{\mathbb{Q}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} \to \operatorname{Spec} \overline{\mathbb{Q}}$ has one point for each element of $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, see Exercise 3.G.)

4.D. EXERCISE. Show that surjectivity is preserved by base change. (**Surjectivity** has its usual meaning: surjective as a map of sets.) (You may end up using the fact that for any fields k_1 and k_2 containing k_3, $k_1 \otimes_{k_3} k_2$ is non-zero, and also the axiom of choice.)

4.E. EXERCISE. If P is a property of morphisms preserved by base change, and $X \to Y$ and $X' \times Y'$ are two morphisms of S-schemes with property P, show that $X \times_S X' \to Y \times_S Y'$ has property P as well.

4.1. $\star$ Properties not preserved by base change, and how to fix them.

There are some notions that you should reasonably expect to be preserved by pullback based on your geometric intuition. Given a family in the topological category, fibers pull back in reasonable ways. So for example, any pullback of a family in which all the fibers are irreducible will also have this property; ditto for connected. Unfortunately, both of these fail in algebraic geometry, as the Example 2.1 shows:

$$\begin{tikzcd} \operatorname{Spec} \mathbb{C} \coprod \operatorname{Spec} \mathbb{C} \arrow[r] \arrow[d] & \operatorname{Spec} \mathbb{C} \arrow[d] \\ \operatorname{Spec} \mathbb{C} \arrow[r] & \operatorname{Spec} \mathbb{R} \end{tikzcd}$$

The family on the right (the vertical map) has irreducible and connected fibers, and the one on the left doesn't. The same example shows that the notion of "integral fibers" also doesn't behave well under pullback.

4.F. EXERCISE. Suppose k is a field of characteristic p, so $k(u^p)/k(u)$ is an inseparable extension. By considering $k(u^p) \otimes_{k(u)} k(u^p)$, show that the notion of "reduced fibers" does not necessarily behave well under pullback. (The fact that I'm giving you this example should show that this happens only in characteristic p, in the presence of something as strange as inseparability.)

We rectify this problem as follows.

4.2. A **geometric point** of a scheme X is defined to be a morphism $\operatorname{Spec} k \to X$ where k is an algebraically closed field. Awkwardly, this is now the third kind of "point" of a scheme! There are just plain points, which are elements of the underlying set; there are T-valued points, which are maps $T \to X$, §3.4; and there are geometric points. Geometric points are clearly a flavor of a T-valued point, but they are also an enriched version of a (plain) point: they are the data of a point with an inclusion of the residue field of the point in an algebraically closed field.

A **geometric fiber** of a morphism $X \to Y$ is defined to be the fiber over a geometric point of Y. A morphism has **connected** (resp. **irreducible**, **integral**, **reduced**) **geometric fibers** if all its geometric fibers are connected (resp. irreducible, integral, reduced).

4.G. EXERCISE. Show that the notion of "connected (resp. irreducible, integral, reduced)" geometric fibers behaves well under base change.

4.H. EXERCISE FOR THE ARITHMETICALLY-MINDED. Show that for the morphism $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{R}$, all geometric fibers consist of two reduced points. (Cf. Example 2.1.)

4.I. EXERCISE. Recall Example 3.3, the projection of the parabola $y^2 = x$ to the x axis, corresponding to the map of rings $\mathbb{Q}[x] \to \mathbb{Q}[y]$, with $x \mapsto y^2$. Show that the geometric fibers of this map are always two points, except for those geometric fibers over $0 = [(x)]$.

Checking whether a k-scheme is geometrically connected etc. seems annoying: you need to check every single algebraically closed field containing k. However, in each of these four cases, the failure of nice behavior of geometric fibers can already be detected after a finite field extension. For example, $\operatorname{Spec} \mathbb{Q}(i) \to \operatorname{Spec} \mathbb{Q}$ is not geometrically connected, and in fact you only need to base change by $\operatorname{Spec} \mathbb{Q}(i)$ to see this. We make this precise as follows.

4.J. EXERCISE. Suppose X is a k-scheme.

(a) Show that X is geometrically irreducible if and only if $X \times_k k^s$ is irreducible if and only if $X \times_k K$ is irreducible for all field extensions K/k. (Here k^s is the separable closure of k.)
(b) Show that X is geometrically connected if and only if $X \times_k k^s$ is connected if and only if $X \times_k K$ is connected for all field extensions K/k.
(c) Show that X is geometrically reduced if and only if $X \times_k k^p$ is reduced if and only if $X \times_k K$ is reduced for all field extensions K/k. (Here k^p is the perfect closure of k.) Thus if $\operatorname{char} k = 0$, then X is geometrically reduced if and only if it is reduced.
(d) Combining (a) and (c), show that X is geometrically integral if and only if $X \times_k K$ is geometrically integral for all field extensions K/k.

5. PRODUCTS OF PROJECTIVE SCHEMES: THE SEGRE EMBEDDING

I will next describe products of projective A-schemes over A. The case of greatest initial interest is if $A = k$. In order to do this, I need only describe $\mathbb{P}^m_A \times_A \mathbb{P}^n_A$, because any projective scheme has a closed immersion in some $\mathbb{P}^m_A$, and closed immersions behave well under base change, so if $X \hookrightarrow \mathbb{P}^m_A$ and $Y \hookrightarrow \mathbb{P}^n_A$ are closed immersions, then $X \times_A Y \hookrightarrow \mathbb{P}^m_A \times_A \mathbb{P}^n_A$ is also a closed immersion, cut out by the equations of X and Y.

We'll describe $\mathbb{P}^m_A \times_A \mathbb{P}^n_A$, and see that it too is a projective A-scheme.

Before we do this, we'll get some motivation from classical projective spaces (non-zero vectors modulo non-zero scalars) in a special case. Our map will send $[x_0; x_1; x_2] \times [y_0; y_1]$ to a point in $\mathbb{P}^5$, whose co-ordinates we think of as being entries in the "multiplication table"

$$\begin{matrix} [& x_0y_0; & x_1y_0; & x_2y_0; & \\ & x_0y_1; & x_1y_1; & x_2y_1 &] \end{matrix}$$

This is indeed a well-defined map of sets. Notice that the resulting matrix is rank one, and from the matrix, we can read off $[x_0; x_1; x_2]$ and $[y_0; y_1]$ up to scalars. For example, to read off the point $[x_0; x_1; x_2] \in \mathbb{P}^2$, we just take the first row, unless it is all zero, in which case we take the second row. (They can't both be all zero.) In conclusion: in classical projective geometry, given a point of $\mathbb{P}^m$ and $\mathbb{P}^n$, we have produced a point in $\mathbb{P}^{mn+m+n}$, and from this point in $\mathbb{P}^{mn+m+n}$, we can recover the points of $\mathbb{P}^m$ and $\mathbb{P}^n$.

Suitably motivated, we return to algebraic geometry. We define a map

$$\mathbb{P}^m_A \times_A \mathbb{P}^n_A \to \mathbb{P}^{mn+m+n}_A$$

by

$$([x_0; \ldots; x_m], [y_0; \ldots; y_n]) \mapsto [z_{00}; z_{01}; \cdots; z_{ij}; \cdots; z_{mn}]$$

$$= [x_0y_0; x_0y_1; \cdots; x_iy_j; \cdots x_my_n].$$

More explicitly, we consider the map from the affine open set $U_i \times V_j$ (where $U_i = D(x_i)$ and $V_j = D(y_j)$ to the affine open set $W_{ij} = D(z_{ij})$ by

$$(x_{0/i}, \ldots, x_{m/i}, y_{0/j}, \ldots, y_{n/j}) \mapsto (x_{0/i}y_{0/j}; \ldots; x_{i/i}y_{j/j}; \ldots; x_{m/i}y_{n/j})$$

or, in terms of algebras, $z_{ab/ij} \mapsto x_{a/i}y_{b/j}$.

5.A. EXERCISE. Check that these maps glue to give a well-defined morphism $\mathbb{P}^m_A \times_A \mathbb{P}^n_A \to \mathbb{P}^{mn+m+n}_A$.

I claim this morphism is a closed immersion. We can check this on an open cover of the target (the notion of being a closed immersion is affine-local, an earlier exercise). Let's check this on the open set where $z_{ij} \neq 0$. The preimage of this open set in $\mathbb{P}^m_A \times \mathbb{P}^n_A$ is the locus where $x_i \neq 0$ and $y_j \neq 0$, i.e. $U_i \times V_j$. As described above, the map of rings is given by $z_{ab/ij} \mapsto x_{a/i}y_{b/j}$; this is clearly a surjection, as $z_{aj/ij} \mapsto x_{a/i}$ and $z_{ib/ij} \mapsto y_{b/j}$.

This map is called the **Segre morphism** or **Segre embedding**. If A is a field, the image is called the **Segre variety**.

Here are some useful comments.

5.B. EXERCISE. Show that the Segre scheme (the image of the Segre morphism) is cut out by the equations corresponding to

$$\operatorname{rank} \begin{pmatrix} a_{00} & \cdots & a_{0n} \\ \vdots & \ddots & \vdots \\ a_{m0} & \cdots & a_{mn} \end{pmatrix} = 1,$$

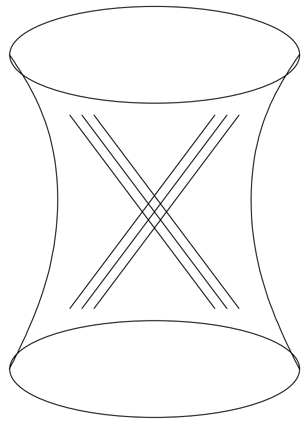

FIGURE 1. The two rulings on the quadric surface $V(wz - xy) \subset \mathbb{P}^3$. One ruling contains the line $V(w, x)$ and the other contains the line $V(w, y)$.

i.e. that all 2×2 minors vanish. (Hint: suppose you have a polynomial in the a_{ij} that becomes zero upon the substitution $a_{ij} = x_i y_j$. Give a recipe for subtracting polynomials of the form monomial times 2×2 minor so that the end result is 0.)

5.1. *Important Example.* Let's consider the first non-trivial example, when $m = n = 1$. We get $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$. We get a single equation

$$\operatorname{rank} \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} = 1,$$

i.e. $a_{00}a_{11} - a_{01}a_{10} = 0$. We get our old friend, the quadric surface! Hence: the nonsingular quadric surface $wz - xy = 0$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ (Figure 1). One family of lines corresponds to the image of $\{x\} \times \mathbb{P}^1$ as x varies, and the other corresponds to the image $\mathbb{P}^1 \times \{y\}$ as y varies.

Since (by diagonalizability of quadratics) all nonsingular quadratics over an algebraically closed field are isomorphic, we have that all nonsingular quadric surfaces over an algebraically closed field are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

Note that this is not true over a field that is not algebraically closed. For example, over $\mathbb{R}$, $w^2 + x^2 + y^2 + z^2 = 0$ is not isomorphic to $\mathbb{P}^1_{\mathbb{R}} \times_{\mathbb{R}} \mathbb{P}^1_{\mathbb{R}}$. Reason: the former has no real points, while the latter has lots of real points.

5.C. EXERCISE: A CO-ORDINATE-FREE DESCRIPTION OF THE SEGRE EMBEDDING. Show that the Segre embedding can be interpreted as $\mathbb{P}V \times \mathbb{P}W \to \mathbb{P}(V \otimes W)$ via the surjective map of graded rings

$$\operatorname{Sym}^\bullet(V^\vee \otimes W^\vee) \twoheadrightarrow \sum_{i=0}^\infty \left(\operatorname{Sym}^i V^\vee\right) \otimes \left(\operatorname{Sym}^i W^\vee\right)$$

"in the opposite direction".

Can you define the Segre embedding for the product of three projective spaces?

6. SEPARATED MORPHISMS

The notion of a separated morphism is fundamentally important. It looks weird the first time you see it, but it is highly motivated.

6.1. Motivation. Separation is the analogue of the Hausdorff condition for manifolds (see Exercise 6.A), so let's review why we like Hausdorffness. Recall that a topological space is *Hausdorff* if for every two points x and y, there are disjoint open neighborhoods of x and y. The real line is Hausdorff, but the "real line with doubled origin" is not. Many proofs and results about manifolds use Hausdorffness in an essential way. For example, the classification of compact one-dimensional real manifolds is very simple, but if the Hausdorff condition were removed, we would have a very wild set.

So armed with this definition, we can cheerfully exclude the line with doubled origin from civilized discussion, and we can (finally) define the notion of a *variety*, in a way that corresponds to the classical definition.

With our motivation from manifolds, we shouldn't be surprised that all of our affine and projective schemes are separated: certainly, in the land of real manifolds, the Hausdorff condition comes for free for "subsets" of manifolds. (More precisely, if Y is a manifold, and X is a subset that satisfies all the hypotheses of a manifold except possibly Hausdorffness, then Hausdorffness comes for free.)

As an unexpected added bonus, a separated morphism to an affine scheme has the property that the intersection of a two affine open sets in the source is affine (Proposition 6.8). This will make Cech cohomology work very easily on (quasicompact) schemes. You should see this as the analogue of the fact that in $\mathbb{R}^n$, the intersection of two convex sets is also convex. In fact affine schemes will be trivial from the point of view of quasicoherent cohomology, just as convex sets in $\mathbb{R}^n$ are, so this metaphor is quite apt.

A lesson arising from the construction is the importance of the diagonal morphism. More precisely given a morphism $X \to Y$, nice consequences can be leveraged from good behavior of the diagonal morphism $\delta : X \to X \times_Y X$, usually through fun diagram chases. This is a lesson that applies across many fields of mathematics. (Another nice gift the diagonal morphism: it will soon give us a good algebraic definition of differentials.)

Grothendieck taught us that one should try to define properties of morphisms, not of objects; then we can say that an object has that property if the morphism to the final object has that property. We saw this earlier with the notion of quasicompact. In this spirit, separation will be a property of morphisms, not schemes.

Before we define separation, we make an observation about all diagonal maps.

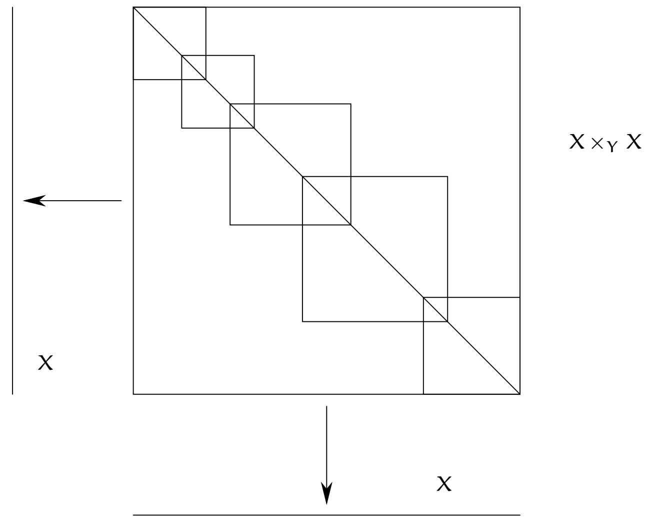

FIGURE 2. A neighborhood of the diagonal is covered by $U_{ij} \times_{V_j} U_{ij}$

6.2. *Proposition. — Let* $X \to Y$ *be a morphism of schemes. Then the diagonal morphism* $\delta : X \to X \times_Y X$ *is a locally closed immersion.*

This locally closed subscheme of $X \times_Y X$ (the diagonal) will be denoted Δ.

Proof. We will describe a union of open subsets of $X \times_Y X$ covering the image of X, such that the image of X is a closed immersion in this union.

6.3. Say Y is covered with affine open sets V_i and X is covered with affine open sets U_{ij}, with $\pi : U_{ij} \to V_i$. Then the diagonal is covered by $U_{ij} \times_{V_i} U_{ij}$. (Any point $p \in X$ lies in some U_{ij}; then $\delta(p) \in U_{ij} \times_{V_i} U_{ij}$. Figure 2 may be helpful.) As a reality check: $U_{ij} \times_{V_i} U_{ij}$ is indeed an affine open subscheme of $X \times_Y X$, by considering the factorization

$$U_{ij} \times_{V_i} U_{ij} \to U_{ij} \times_Y U_{ij} \to U_{ij} \times_Y X \to X \times_Y X$$

where the first arrow is an isomorphism as $V_i \hookrightarrow Y$ is a monomorphism (as it is an open immersion, Exercise 2.D). The second and third arrows are open immersions as open immersions are preserved by base change.

Finally, we'll check that $U_{ij} \to U_{ij} \times_{V_i} U_{ij}$ is a closed immersion. Say $V_i = \operatorname{Spec} S$ and $U_{ij} = \operatorname{Spec} R$. Then this corresponds to the natural ring map $R \times_S R \to R$, which is obviously surjective. $\square$

The open subsets we described may not cover $X \times_Y X$, so we have not shown that δ is a closed immersion.

6.4. *Definition.* A morphism $X \to Y$ is **separated** if the diagonal morphism $\delta : X \to X \times_Y X$ is a closed immersion. An A-scheme X is said to be **separated over** A if the structure morphism $X \to \operatorname{Spec} A$ is separated. When people say that a scheme (rather than a morphism) X is separated, they mean implicitly that some morphism is separated. For example, if they are talking about A-schemes, they mean that X is separated over A.

Thanks to Proposition 6.2, a morphism is separated if and only if the diagonal is closed. This is reminiscent of a definition of Hausdorff, as the next exercise shows.

6.A. EXERCISE (FOR THOSE SEEKING TOPOLOGICAL MOTIVATION). Show that a topological space X is Hausdorff if the diagonal is a closed subset of $X \times X$. (The reason separatedness of schemes doesn't give Hausdorffness — i.e. that for any two open points x and y there aren't necessarily disjoint open neighborhoods — is that in the category of schemes, the topological space $X \times X$ is not in general the product of the topological space X with itself. For example, Exercise 1.A showed that $\mathbb{A}^2_k$ does not have the product topology on $\mathbb{A}^1_k \times_k \mathbb{A}^1_k$.)

6.B. IMPORTANT EASY EXERCISE. Show that open immersions and closed immersions are separated. (Hint: Just do this by hand. Alternatively, show that monomorphisms are separated. Open and closed immersions are monomorphisms, by Exercise 2.D.)

6.C. IMPORTANT EASY EXERCISE. Show that every morphism of affine schemes is separated. (Hint: this was essentially done in Proposition 6.2.)

I'll now give you an example of something separated that is not affine. The following single calculation will imply that all quasiprojective A-schemes are separated (once we know that the composition of separated morphisms are separated, after Thanksgiving).

6.5. *Proposition. — $\mathbb{P}^n_A \to \operatorname{Spec} A$ is separated.*

We give two proofs. The first is by direct calculation. The second requires no calculation, and just requires that you remember some classical constructions described earlier.

Proof 1: direct calculation. We cover $\mathbb{P}^n_A \times_A \mathbb{P}^n_A$ with open sets of the form $U_i \times U_j$, where $U_0, \dots, U_n$ form the "usual" affine open cover. The case $i = j$ was taken care of before, in the proof of Proposition 6.2. If $i \neq j$ then

$$U_i \times_A U_j \cong \operatorname{Spec} A[x_{0/i}, \dots, x_{n/i}, y_{0/j}, \dots, y_{n/j}]/(x_{i/i} - 1, y_{j/j} - 1).$$

Now the restriction of the diagonal Δ is contained in U_i (as the diagonal map composed with projection to the first factor is the identity), and similarly is contained in U_j. Thus the diagonal map over $U_i \times_A U_j$ is $U_i \cap U_j \to U_i \times_A U_j$. This is a closed immersion, as the corresponding map of rings

$$\operatorname{Spec} A[x_{0/i}, \dots, x_{n/i}, y_{0/j}, \dots, y_{n/j}] \to \operatorname{Spec} A[x_{0/i}, \dots, x_{n/i}, x_{j/i}^{-1}]/(x_{i/i} - 1)$$

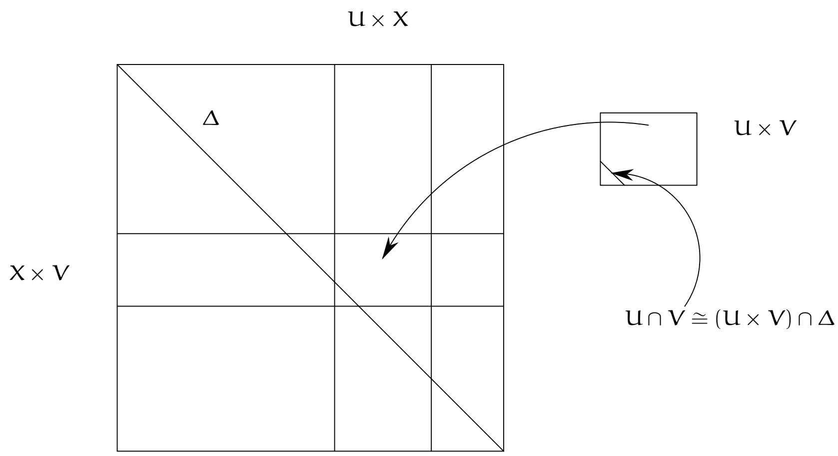

FIGURE 3. Small Proposition 6.6

(given by $x_{k/i} \mapsto x_{k/i}$, $y_{k/j} \mapsto x_{k/i}/x_{j/i}$) is clearly a surjection (as each generator of the ring on the right is clearly in the image — note that $x_{j/i}^{-1}$ is the image of $y_{i/j}$). $\square$

Proof 2: classical geometry, pointed out by Jarod. Note that the diagonal map $\delta : \mathbb{P}^n_A \to \mathbb{P}^n_A \times_A \mathbb{P}^n_A$ followed by the Segre embedding $S : \mathbb{P}^n_A \times_A \mathbb{P}^n_A \to \mathbb{P}^{n^2+n}$ (a closed immersion) can also be factored as the second Veronese map $\nu_2 : \mathbb{P}^n_A \to \mathbb{P}^{\binom{n+2}{2}-1}$ followed by a linear map $L : \mathbb{P}^{\binom{n+2}{2}-1} \to \mathbb{P}^{n^2+n}$ (an earlier exercise, from when we discussed morphisms of projective schemes via morphisms of graded rings), both of which are closed immersions. You should verify this. This forces δ to send closed sets to closed sets (or else $S \circ \delta$ won't, but $L \circ \nu_2$ to).

$$\begin{tikzcd}
 & \mathbb{P}^n_A \times_A \mathbb{P}^n_A \arrow[dr, "S"] & \\
\mathbb{P}^n_A \arrow[ur, "\delta"] \arrow[dr, "\nu_2"'] & & \mathbb{P}^{n^2+n} \\
 & \mathbb{P}^{\binom{n+2}{2}-1} \arrow[ur, "L"'] &
\end{tikzcd}$$

We note for future reference a minor result proved in the course of Proof 1. Figure 3 may help show why this is natural.

6.6. *Small Proposition. — If U and V are open subsets of an A-scheme X, then $\Delta \cap (U \times_A V) \cong U \cap V$.*

6.D. EXERCISE. Show that the line with doubled origin X is not separated, by verifying that the image of the diagonal morphism is not closed.

We finally define then notion of variety!

6.7. *Definition.* A **variety** over a field k, or k**-variety**, is a reduced, separated scheme of finite type over k. For example, a reduced finite type affine k-scheme is a variety. In other words, to check if $\operatorname{Spec} k[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$ is a variety, you need only check reducedness.

Notational caution: In some sources, the additional condition of irreducibility is imposed. We will not do this. Also, it is often assumed that k is algebraically closed. We will not do this either.

Here is a very handy consequence of separatedness.

6.8. *Proposition. — Suppose $X \to \operatorname{Spec} A$ is a separated morphism to an affine scheme, and U and V are affine open sets of X. Then $U \cap V$ is an affine open subset of X.*

Before proving this, we state a consequence that is otherwise nonobvious. If $X = \operatorname{Spec} A$, then the intersection of any two affine open sets is open (just take $A = \mathbb{Z}$ in the above proposition). This is certainly not an obvious fact! We know that the intersection of any two distinguished affine open sets is affine (from $D(f) \cap D(g) = D(fg)$), but we have very little handle on affine open sets in general.

Warning: this property does not characterize separatedness. For example, if $A = \operatorname{Spec} k$ and X is the line with doubled origin over k, then X also has this property.

Proof. By Proposition 6.6, $(U \times_A V) \cap \Delta = U \cap V$, where Δ is the diagonal. But $U \times_A V$ is affine (the fibered product is two affine schemes over an affine scheme is affine, Step 1 of our construction of fibered products, Theorem 1.1), and Δ is a closed subscheme of an affine scheme, and hence affine. $\square$

E-mail address: `vakil@math.stanford.edu`