FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 14

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CONTENTS

Today: projective schemes.

1. INTRODUCTION

At this point, we know that we can construct schemes by gluing affine schemes together. If a large number of affine schemes are involved, this can obviously be a laborious and tedious process. Our example of closed subschemes of projective space showed that we could piggyback on the construction of projective space to produce complicated and interesting schemes. In this chapter, we formalize this notion of *projective schemes*. Projective schemes over the complex numbers give good examples (in the classical topology) of compact complex varieties. In fact they are such good examples that it is quite hard to come up with an example of a compact complex variety that is provably not projective. (We will see examples later, although we won't concern ourselves with the relationship to the classical topology.) Similarly, it is quite hard to come up with an example of a complex variety that is provably not an open subset of a projective variety. In particular, most examples of complex varieties that come up in nature are of this form. More generally, projective schemes will be the key example of the algebro-geometric analogue of compactness (*properness*). Thus one advantage of the notion of projective scheme is that it encapsulates much of the algebraic geometry arising in nature.

In fact our example from last day already gives the notion of projective A-schemes in full generality. Recall that any collection of homogeneous elements of $A[x_0, \ldots, x_n]$ describes a closed subscheme of $\mathbb{P}^n_A$. Any closed subscheme of $\mathbb{P}^n_A$ cut out by a set of homogeneous polynomials will be called a *projective A-scheme*. (You may be initially most interested in the "classical" case where A is an algebraically closed field.) If I is the ideal

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in $A[x_0, \ldots, x_n]$ generated by these homogeneous polynomials, the scheme we have constructed will be called $\operatorname{Proj} A[x_0, \ldots, x_n]/I$. Then $x_0, \ldots, x_n$ are informally said to be *projective coordinates* on the scheme. Warning: they are not functions on the scheme. (We will later interpret them as sections of a line bundle.) This lecture will reinterpret this example in a more useful language. For example, just as there is a rough dictionary between rings and affine schemes, we will have an analogous dictionary between graded rings and projective schemes. Just as one can work with affine schemes by instead working with rings, one can work with projective schemes by instead working with graded rings.

1.1. A motivating picture from classical geometry.

We motivate a useful way of picturing projective schemes by recalling how one thinks of projective space "classically" (in the classical topology, over the real numbers). $\mathbb{P}^n$ can be interpreted as the lines through the origin in $\mathbb{R}^{n+1}$. Thus subsets of $\mathbb{P}^n$ correspond to unions of lines through the origin of $\mathbb{R}^{n+1}$, and closed subsets correspond to such unions which are closed. (The same is not true with "closed" replaced by "open"!)

One often pictures $\mathbb{P}^n$ as being the "points at infinite distance" in $\mathbb{R}^{n+1}$, where the points infinitely far in one direction are associated with the points infinitely far in the opposite direction. We can make this more precise using the decomposition

$$\text{(1)} \qquad\qquad \mathbb{P}^{n+1} = \mathbb{R}^{n+1} \coprod \mathbb{P}^n$$

by which we mean that there is an open subset in $\mathbb{P}^{n+1}$ identified with $\mathbb{R}^{n+1}$ (the points with last projective co-ordinate non-zero), and the complementary closed subset identified with $\mathbb{P}^n$ (the points with last projective co-ordinate zero).

Then for example any equation cutting out some set V of points in $\mathbb{P}^n$ will also cut out some set of points in $\mathbb{R}^n$ that will be a closed union of lines. We call this the *affine cone* of V. These equations will cut out some union of $\mathbb{P}^1$'s in $\mathbb{P}^{n+1}$, and we call this the *projective cone* of V. The projective cone is the disjoint union of the affine cone and V. For example, the affine cone over $x^2 + y^2 = z^2$ in $\mathbb{P}^2$ is just the "classical" picture of a cone in $\mathbb{R}^2$, see Figure 1.

We will make this analogy precise in our algebraic setting in §2.3.

2. THE Proj CONSTRUCTION

Let's abstract these notions, just as we abstracted the notion of the Spec of a ring with given generators and relations over k to the Spec of a ring in general.

In the examples we've seen, we have a graded ring $A[x_0, \ldots, x_n]/I$ where I is a **homogeneous ideal** (i.e. I is generated by homogeneous elements of $A[x_0, \ldots, x_n]$). Here we are taking the usual grading on $A[x_0, \ldots, x_n]$, where each x_i has weight 1. Then $A[x_0, \ldots, x_n]/I$ is also a graded ring $S_\bullet$, and we'll call its graded pieces S_0, S_1, etc. (The subscript $\bullet$ in $S_\bullet$ is intended to remind us of the indexing. In a graded ring, multiplication
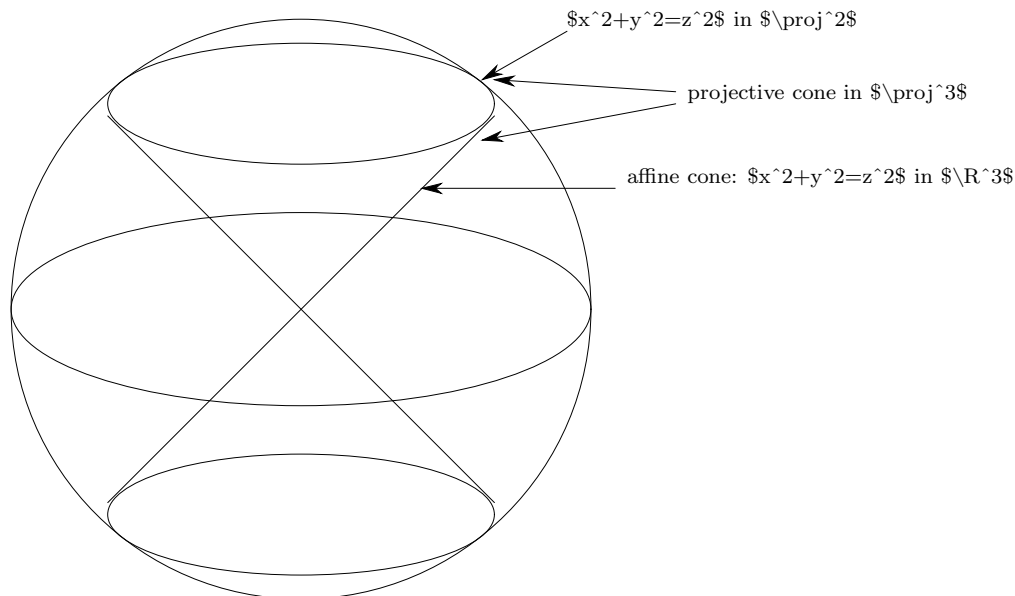
FIGURE 1. The affine and projective cone of $x^2 + y^2 = z^2$ in classical geometry

sends $S_m \times S_n$ to S_{m+n}. Note that S_0 is a subring, and S is a S_0-algebra.) In our examples that $S_0 = A$, and $S_\bullet$ is generated over S_0 by S_1.

2.1. Standing assumptions about graded rings. We make some standing assumptions on graded rings. Fix a ring A (the **base ring**). Our motivating example is $S_\bullet = A[x_0, x_1, x_2]$, with the usual grading. **Assume that** $S_\bullet$ **is graded by** $\mathbb{Z}^{\geq 0}$**, with** $S_0 = A$**.** Hence each S_n is an A-module. The subset $S_+ := \oplus_{i>0} S_i \subset S_\bullet$ is an ideal, called the **irrelevant ideal**. The reason for the name "irrelevant" will be clearer soon. **Assume that the irrelevant ideal** S_+ **is a finitely-generated ideal.**

2.A. EXERCISE. Show that $S_\bullet$ is a finitely-generated graded ring if and only if $S_\bullet$ is a finitely-generated graded A-algebra, i.e. generated over $A = S_0$ by a finite number of homogeneous elements of positive degree. (Hint for the forward implication: show that the generators of S_+ as an ideal are also generators of $S_\bullet$ as an algebra.)

If these assumptions hold, we say that $S_\bullet$ is a **finitely generated graded ring**.

We now define a scheme $\operatorname{Proj} S_\bullet$. You won't be surprised that we will define it as a *set*, with a *topology*, and a *structure sheaf*.

The set. The points of $\operatorname{Proj} S_\bullet$ are defined to be those homogeneous prime ideals *not containing the irrelevant ideal* S_+. The homogeneous primes containing the irrelevant ideal are irrelevant.

For example, if $S_\bullet = k[x, y, z]$ with the usual grading, then $(z^2 - x^2 - y^2)$ is a homogeneous prime ideal. We picture this as a subset of $\operatorname{Spec} S_\bullet$; it is a cone (see Figure 1). We picture $\mathbb{P}^2_k$ as the "plane at infinity". Thus we picture this equation as cutting out a conic "at infinity". We will make this intuition somewhat more precise in §2.3.

The topology. As with affine schemes, we define the Zariski topology by describing the closed subsets. They are of the form $V(I)$, where I is a homogeneous ideal. (Here $V(I)$ has essentially the same definition as before: those *homogeneous* prime ideals containing I.) Particularly important open sets will the **distinguished open sets** $D(f) = \operatorname{Proj} S_\bullet \setminus V(f)$, where $f \in S_+$ is homogeneous.

2.B. EASY EXERCISE. Verify that the distinguished open sets form a base of the topology. (The argument is essentially identical to the affine case.)

As with the affine case, if $D(f) \subset D(g)$, then $f^n \in (g)$ for some n, and vice versa. Clearly $D(f) \cap D(g) = D(fg)$, by the same immediate argument as in the affine case.

The structure sheaf. We define $\boxed{\mathcal{O}_{\operatorname{Proj} S_\bullet}(D(f)) = ((S_\bullet)_f)_0}$ where $((S_\bullet)_f)_0$ means the 0-graded piece of the graded ring $(S_\bullet)_f$. (The notation $((S_\bullet)_f)_0$ is admittedly unfortunate — the first and third subscripts refer to the grading, and the second refers to localization.) As in the affine case, we define restriction maps, and verify that this is well-defined (i.e. if $D(f) = D(f')$, then we are defining the same ring, and that the restriction maps are well-defined).

For example, if $S_\bullet = k[x_0, x_1, x_2]$ and $f = x_0$, we get $(k[x_0, x_1, x_2]_{x_0})_0 := k[x_{1/0}, x_{2/0}]$ (using our earlier language for projective patches).

We now check that this is a sheaf. We could show that this is a sheaf on the base, and the argument would be as in the affine case (which was not easy). Here instead is a sneakier argument. We first note that the topological space $D(f)$ and $\operatorname{Spec}((S_\bullet)_f)_0$ are canonically homeomorphic: they have matching distinguished bases. (To the distinguished open $D(g) \cap D(f)$ of $D(f)$, we associate $D(g^{\deg f}/f^{\deg g})$ in $\operatorname{Spec}(S_f)_0$. To $D(h)$ in $\operatorname{Spec}(S_f)_0$, we associate $D(f^n h) \subset D(f)$, where n is chosen large enough so that $f^n h \in S_\bullet$.) Second, we note that the sheaf of rings on the distinguished base of $D(f)$ can be associated (via this homeomorphism just described) with the sheaf of rings on the distinguished base of $\operatorname{Spec}((S_\bullet)_f)_0$: the sections match (the ring of sections $((S_\bullet)_{fg})_0$ over $D(g) \cap D(f) \subset D(f)$, those homogeneous degree 0 quotients of $S_\bullet$ with f's and g's in the denominator, is naturally identified with the ring of sections over the corresponding open set of $\operatorname{Spec}((S_\bullet)_f)_0$) and the restriction maps clearly match (think this through yourself!). Thus we have described an isomorphism of schemes

$$(D(f), \mathcal{O}_{\operatorname{Proj} S_\bullet}) \cong \operatorname{Spec}(S_f)_0.$$

2.C. EASY EXERCISE. Describe a natural "structure morphism" $\operatorname{Proj} S_\bullet \to \operatorname{Spec} A$.

2.2. Projective and quasiprojective schemes.

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We call a scheme of the form $\operatorname{Proj} S_\bullet$ (where $S_0 = A$) a **projective scheme over** A, or a **projective** A-**scheme**. A **quasiprojective** A-**scheme** is an open subscheme of a projective A-scheme. The "A" is omitted if it is clear from the context; often A is some field.

We now make a connection to classical terminology. A **projective variety (over** k**)**, or an **projective** k-**variety** is a *reduced* projective k-scheme. (Warning: in the literature, it is sometimes also required that the scheme be irreducible, or that k be algebraically closed.) A **quasiprojective** k-**variety** is an open subscheme of a projective k-variety. We defined affine varieties earlier, and you can check that affine open subsets of projective k-varieties are affine k-varieties. We will define varieties in general later.

The notion of quasiprojective k-scheme is a good one, covering most interesting cases which come to mind. We will see before long that the affine line with the doubled origin is not quasiprojective for somewhat silly reasons ("non-Hausdorffness"), but we'll call that kind of bad behavior "non-separated". Here is a surprisingly subtle question: *Are there quasicompact k-schemes that are not quasiprojective?* Translation: if we're gluing together a finite number of schemes each sitting in some $\mathbb{A}_k^n$, can we ever get something not quasiprojective? We will finally answer this question in the negative in the next quarter.

2.D. EASY EXERCISE. Show that all projective A-schemes are quasicompact. (Translation: show that any projective A-scheme is covered by a finite number of affine open sets.) Show that $\operatorname{Proj} S_\bullet$ is finite type over $A = S_0$. If S_0 is a Noetherian ring, show that $\operatorname{Proj} S_\bullet$ is a Noetherian scheme, and hence that $\operatorname{Proj} S_\bullet$ has a finite number of irreducible components. Show that any quasiprojective scheme is locally of finite type over A. If A is Noetherian, show that any quasiprojective A-scheme is quasicompact, and hence of finite type over A. Show this need not be true if A is not Noetherian. Better: give an example of a quasiprojective A-scheme that is not quasicompact (necessarily for some non-Noetherian A). (Hint: Flip ahead to silly example 3.2.)

2.3. Affine and projective cones.

If $S_\bullet$ is a finitely-generated graded ring, then the **affine cone** of $\operatorname{Proj} S_\bullet$ is $\operatorname{Spec} S_\bullet$. Note that this construction depends on $S_\bullet$, not just of $\operatorname{Proj} S_\bullet$. As motivation, consider the graded ring $S_\bullet = \mathbb{C}[x, y, z]/(z^2 - x^2 - y^2)$. Figure 2 is a sketch of $\operatorname{Spec} S_\bullet$. (Here we draw the "real picture" of $z^2 = x^2 + y^2$ in $\mathbb{R}^3$.) It is a cone in the most traditional sense; the origin $(0, 0, 0)$ is the "cone point".

This gives a useful way of picturing Proj (even over arbitrary rings than $\mathbb{C}$). Intuitively, you could imagine that if you discarded the origin, you would get something that would project onto $\operatorname{Proj} S_\bullet$. The following exercise makes that precise.

2.E. EXERCISE. If $S_\bullet$ is a projective scheme over a field k, Describe a natural morphism $\operatorname{Spec} S_\bullet \setminus \{0\} \to \operatorname{Proj} S_\bullet$.

This has the following generalization to A-schemes, which you might find geometrically reasonable. This again motivates the terminology "irrelevant".
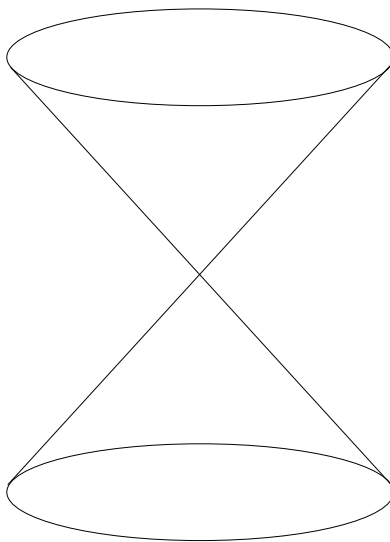
FIGURE 2. A sketch of the cone $\operatorname{Spec} k[x, y, z]/(z^2 - x^2 - y^2)$.

2.F. EXERCISE.   If $S_\bullet$ is a projective A-scheme, describe a natural morphism $\operatorname{Spec} S_\bullet \setminus V(S_+) \to \operatorname{Proj} S_\bullet$.

In fact, it can be made precise that $\operatorname{Proj} S_\bullet$ is the affine cone, minus the origin, modded out by multiplication by scalars.

The **projective cone** of $\operatorname{Proj} S_\bullet$ is $\operatorname{Proj} S_\bullet[T]$, where T is a new variable of degree 1. For example, the cone corresponding to the conic $\operatorname{Proj} k[x, y, z]/(z^2 - x^2 - y^2)$ is $\operatorname{Proj} k[x, y, z, T]/(z^2 - x^2 - y^2)$.

2.G. EXERCISE (CF. (1)).   Show that the projective cone of $\operatorname{Proj} S_\bullet[T]$ has a closed sub-scheme isomorphic to $\operatorname{Proj} S_\bullet$ (corresponding to $T = 0$), whose complement (the distinguished open set $D(T)$) is isomorphic to the affine cone $\operatorname{Spec} S_\bullet$.

You can also check that $\operatorname{Proj} S_\bullet$ is a locally principal closed subscheme, and is also locally not a zero-divisor (an *effective Cartier divisor*).

This construction can be usefully pictured as the affine cone union some points "at infinity", and the points at infinity form the Proj. The reader may which to ponder Figure 2, and try to visualize the conic curve "at infinity".

We have thus completely discribed the algebraic analog of the classical picture of 1.1.

3.1. *Example.* We (re)define projective space by $\mathbb{P}_A^n := \operatorname{Proj} A[x_0, \ldots, x_n]$. This definition involves no messy gluing, or choice of special patches.

3.A. EXERCISE. Check that this agrees with our earlier version of projective space.

3.2. *Silly example.* Note that $\mathbb{P}_A^0 = \operatorname{Proj} A[T] \cong \operatorname{Spec} A$. Thus "$\operatorname{Spec} A$ is a projective A-scheme".

Here is a useful generalization of this example that I forgot to say in class:

3.B. EXERCISE: FINITE MORPHISMS TO $\operatorname{Spec} A$ ARE PROJECTIVE. If B is a finitely generated A-algebra, define $S_\bullet$ by $S_0 = A$, and $S_n = B$ for $n > 0$ (with the obvious graded ring structure). Describe an isomorphism

$$\operatorname{Proj} S_\bullet \xleftarrow{\quad \sim \quad} \operatorname{Spec} B$$
$$\searrow \qquad \swarrow$$
$$\operatorname{Spec} A$$

3.C. EXERCISE. Show that $X = \mathbb{P}_k^2 \setminus \{x^2 + y^2 = z^2\}$ is an affine scheme. Show that $x = 0$ cuts out a locally principal closed subscheme that is not principal.

3.3. Example: $\mathbb{P}V$. We can make this definition of projective space even more choice-free as follows. Let V be an $(n+1)$-dimensional vector space over k. (Here k can be replaced by any ring A as usual.) Let $\operatorname{Sym}^\bullet V^\vee = k \oplus V^\vee \oplus \operatorname{Sym}^2 V^\vee \oplus \cdots$. (The reason for the dual is explained by the next exercise.) If for example V is the dual of the vector space with basis associated to $x_0, \ldots, x_n$, we would have $\operatorname{Sym}^\bullet V^\vee = k[x_0, \ldots, x_n]$. Then we can define $\mathbb{P}V := \operatorname{Proj} \operatorname{Sym}^\bullet V^\vee$. In this language, we have an interpretation for $x_0, \ldots, x_n$: they are the linear functionals on the underlying vector space V.

3.D. UNIMPORTANT EXERCISE. Suppose k is algebraically closed. Describe a natural bijection between one-dimensional subspaces of V and the points of $\mathbb{P}V$. Thus this construction canonically (in a basis-free manner) describes the one-dimensional subspaces of the vector space $\operatorname{Spec} V$.

On a related note: you can also describe a natural bijection between points of V and the points of $\operatorname{Spec} \operatorname{Sym}^\bullet V^\vee$. This construction respects the affine/projective cone picture of §2.3.

As maps of rings correspond to maps of affine schemes in the opposite direction, maps of graded rings sometimes give maps of projective schemes in the opposite direction. Before we make this precise, let's see an example to see what can go wrong. There isn't quite a map $\mathbb{P}^2_k \to \mathbb{P}^1_k$ given by $[x; y; z] \to [x; y]$, because this alleged map isn't defined only at the point $[0; 0; 1]$. What has gone wrong? The map $\mathbb{A}^3 = \operatorname{Spec} k[x, y, z] \to \mathbb{A}^2 = \operatorname{Spec} k[x, y]$ makes perfect sense. However, the z-axis in $\mathbb{A}^3$ maps to the origin in $\mathbb{A}^2$, so the point of $\mathbb{P}^2$ corresponding to the z-axis maps to the "cone point" of the affine cone, and hence not to the projective scheme. The image of this point of $\mathbb{A}^2$ contains the irrelevant ideal. If this problem doesn't occur, a map of rings gives a map of projective schemes in the opposite direction.

4.A. Important exercise. (a) Suppose that $f : S_\bullet \longrightarrow R_\bullet$ is a morphism of finitely-generated graded rings (i.e. a map of rings preserving the grading) over A. Suppose further that

(2) $$f(S_+) \supset R_+.$$

Show that this induces a morphism of schemes $\operatorname{Proj} R_\bullet \to \operatorname{Proj} S_\bullet$. (Warning: not every morphism arises in this way.)
(b) Suppose further that $S_\bullet \longrightarrow R_\bullet$ is a surjection of finitely-generated graded rings (so (2) is automatic). Show that the induced morphism $\operatorname{Proj} R_\bullet \to \operatorname{Proj} S_\bullet$ is a closed immersion. (Warning: not every closed immersion arises in this way!)

Hypothesis (2) can be replaced with the weaker hypothesis $\sqrt{f(S_+)} \supset R_+$, but in practice this hypothesis (2) suffices.

4.B. Exercise. Show that an injective linear map of k-vector spaces $V \hookrightarrow W$ induces a closed immersion $\mathbb{P}V \hookrightarrow \mathbb{P}W$.

This closed subscheme is called a **linear space**. Once we know about dimension, we will call this a linear space of dimension $\dim V - 1 = \dim \mathbb{P}V$. A linear space of dimension 1 (resp. 2, n, $\dim \mathbb{P}W - 1$) is called a **line** (resp. **plane**, n**-plane**, **hyperplane**). (If the linear map in the previous exercise is not injective, then the hypothesis (2) of Exercise 4.A fails.)

4.1. A particularly nice case: when $S_\bullet$ is generated in degree 1. If $S_\bullet$ is generated by S_1 as an A-algebra, we say that $S_\bullet$ is **generated in degree 1**.

4.C. Exercise. Suppose $S_\bullet$ is a finitely generated graded ring generated in degree 1. Show that S_1 is a finitely-generated module, and the irrelevant ideal S_+ is generated in degree 1.

4.D. EXERCISE. Show that if $S_\bullet$ is generated by S_1 (as an A-algebra) by $n+1$ elements $x_0, \ldots, x_n$, then $\operatorname{Proj} S_\bullet$ may be described as a closed subscheme of $\mathbb{P}^n_A$ as follows. Consider A^{n+1} as a free module with generators $t_0, \ldots, t_n$ associated to $x_0, \ldots, x_n$. The surjection of

$$\operatorname{Sym}^\bullet A^{n+1} = A[t_0, t_1, \ldots, t_n] \longrightarrow S_\bullet$$

implies $S_\bullet = A[t_0, t_1, \ldots t_n]/I$, where I is a homogeneous ideal.

This is completely analogous to the fact that if R is a finitely-generated A-algebra, then choosing n generators of R as an algebra is the same as describing $\operatorname{Spec} R$ as a closed subscheme of $\mathbb{A}^n_A$. In the affine case this is "choosing coordinates"; in the projective case this is "choosing projective coordinates".

For example, $\operatorname{Proj} k[x, y, z]/(z^2 - x^2 - y^2)$ is a closed subscheme of $\mathbb{P}^2_k$. (A picture is shown in Figure 2.)

Recall that we can interpret the closed points of $\mathbb{P}^n$ as the lines through the origin in $\mathbb{A}^{n+1}$. The following exercise states this more generally.

4.E. EXERCISE. Suppose $S_\bullet$ is a finitely-generated graded ring over an algebraically closed field k, generated in degree 1 by $x_0, \ldots, x_n$, inducing closed immersions $\operatorname{Proj} S_\bullet \hookrightarrow \mathbb{P}^n$ and $\operatorname{Spec} S_\bullet \hookrightarrow \mathbb{A}^n$. Describe a natural bijection between the closed points of $\operatorname{Proj} S_\bullet$ and the "lines through the origin" in $\operatorname{Spec} S_\bullet \subset \mathbb{A}^n$.

5. IMPORTANT EXERCISES

There are many fundamental properties that are best learned by working through problems.

5.1. Analogues of results on affine schemes.

5.A. EXERCISE.

(a) Suppose I is any homogeneous ideal, and f is a homogeneous element. Show that f vanishes on $V(I)$ if and only if $f^n \in I$ for some n. (Hint: Mimic the affine case; see an earlier exercise.)
(b) If $Z \subset \operatorname{Proj} S_\bullet$, define $I(\cdot)$. Show that it is a homogeneous ideal. For any two subsets, show that $I(Z_1 \cup Z_2) = I(Z_1) \cap I(Z_2)$.
(c) For any subset $Z \subset \operatorname{Proj} S_\bullet$, show that $V(I(Z)) = \overline{Z}$.

5.B. EXERCISE. Show that the following are equivalent. (This is more motivation for the S_+ being "irrelevant": any ideal whose radical contains it is "geometrically irrelevant".)

(a) $V(I) = \varnothing$

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(b) for any f_i (i in some index set) generating I, $\cup D(f_i) = \operatorname{Proj} S_\bullet$

(c) $\sqrt{I} \supset S_+$.

5.2. Scaling the grading, and the Veronese embedding.

Here is a useful construction. Define $S_{n\bullet} = \oplus_{j=0}^\infty S_{nj}$. (We could rescale our degree, so "old degree" n is "new degree" 1.)

5.C. EXERCISE. Show that $\operatorname{Proj} S_{n\bullet}$ is isomorphic to $\operatorname{Proj} S_\bullet$.

5.D. EXERCISE. Suppose $S_\bullet$ is generated over S_0 by $f_1, \ldots, f_n$. Find a d such that $S_{d\bullet}$ is generated in "new" degree 1 (= "old" degree d). This is handy, as it means that, using the previous Exercise 5.C, we can assume that any finitely-generated graded ring is generated in degree 1. In particular, we can place every Proj in some projective space via the construction of Exercise 4.D.

Example: Suppose $S_\bullet = k[x, y]$, so $\operatorname{Proj} S_\bullet = \mathbb{P}^1_k$. Then $S_{2\bullet} = k[x^2, xy, y^2] \subset k[x, y]$. We identify this subring as follows.

5.E. EXERCISE. Let $u = x^2, v = xy, w = y^2$. Show that $S_{2\bullet} = k[u, v, w]/(uw - v^2)$.

We have a graded ring generated by three elements in degree 1. Thus we think of it as sitting "in" $\mathbb{P}^2$, via the construction of §4.D. This can be interpreted as "$\mathbb{P}^1$ as a conic in $\mathbb{P}^2$".

Thus if k is algebraically closed of characteristic not 2, using the fact that we can diagonalize quadrics, the conics in $\mathbb{P}^2$, up to change of co-ordinates, come in only a few flavors: sums of 3 squares (e.g. our conic of the previous exercise), sums of 2 squares (e.g. $y^2 - x^2 = 0$, the union of 2 lines), a single square (e.g. $x^2 = 0$, which looks set-theoretically like a line, and is non-reduced), and 0 (not really a conic at all). Thus we have proved: any plane conic (over an algebraically closed field of characteristic not 2) that can be written as the sum of three squares is isomorphic to $\mathbb{P}^1$.

We now soup up this example.

5.F. EXERCISE. Show that $\operatorname{Proj} S_{3\bullet}$ is the *twisted cubic* "in" $\mathbb{P}^3$.

5.G. EXERCISE. Show that $\operatorname{Proj} S_{d\bullet}$ is given by the equations that

$$\begin{pmatrix} y_0 & y_1 & \cdots & y_{d-1} \\ y_1 & y_2 & \cdots & y_d \end{pmatrix}$$

is rank 1 (i.e. that all the 2×2 minors vanish). This is called the **degree d rational normal curve** "in" $\mathbb{P}^d$.
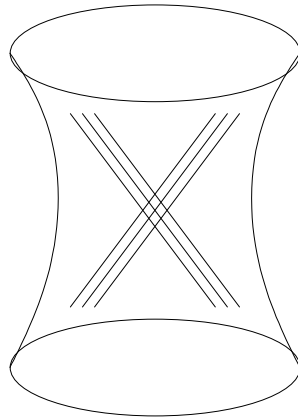
FIGURE 3. The two rulings on the quadric surface $V(wz - xy) \subset \mathbb{P}^3$. One ruling contains the line $V(w, x)$ and the other contains the line $V(w, y)$.

More generally, if $S_\bullet = k[x_0, \ldots, x_n]$, then $\operatorname{Proj} S_{d\bullet} \subset \mathbb{P}^{N-1}$ (where N is the number of degree d polynomials in $x_0, \ldots, x_n$) is called the d-**uple embedding** or d-**uple Veronese embedding**. It is enlightening to interpret this closed immersion as a map of graded rings.

5.H. COMBINATORIAL EXERCISE. Show that $N = \binom{n+d}{d}$.

5.I. UNIMPORTANT EXERCISE. Find five linearly independent quadric equations vanishing on the **Veronese surface** $\operatorname{Proj} S_{2\bullet}$ where $S_\bullet = k[x_0, x_1, x_2]$, which sits naturally in $\mathbb{P}^5$. (You needn't show that these equations generate all the equations cutting out the Veronese surface, although this is in fact true.)

5.3. Entertaining geometric exercises.

5.J. USEFUL GEOMETRIC EXERCISE. Describe all the lines on the quadric surface $wz - xy = 0$ in $\mathbb{P}^3_k$. (Hint: they come in two "families", called the **rulings** of the quadric surface.) This construction arises all over the place in nature.

Hence (by diagonalization of quadrics), if we are working over an algebraically closed field of characteristic not 2, we have shown that all rank 4 quadric surfaces have two rulings of lines.

5.K. EXERCISE. Show that $\mathbb{P}^n_k$ is normal. More generally, show that $\mathbb{P}^n_A$ is normal if A is a Unique Factorization Domain.

5.4. *Example.* If we put a non-standard weighting on the variables of $k[x_1, \ldots, x_n]$ — say we give x_i degree d_i — then $\operatorname{Proj} k[x_1, \ldots, x_n]$ is called **weighted projective space** $\mathbb{P}(d_1, d_2, \ldots, d_n)$.

5.L. EXERCISE. Show that $\mathbb{P}(m, n)$ is isomorphic to $\mathbb{P}^1$. Show that
$$\mathbb{P}(1, 1, 2) \cong \operatorname{Proj} k[u, v, w, z]/(uw - v^2).$$

Hint: do this by looking at the even-graded parts of $k[x_0, x_1, x_2]$, cf. Exercise 5.C.

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