FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASSES 11 AND 12

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CONTENTS

1. ASSOCIATED POINTS CONTINUED

Recall the four key facts to remember about associated points.

(1) *The generic points of the irreducible components are associated points.* The other associated points are called **embedded points**.

(2) *If X is reduced, then X has no embedded points.*

(3) If X is a locally Noetherian scheme, then for any $U \subset X$, the natural map

$$\Gamma(U, \mathcal{O}_X) \to \prod_{\text{associated } p \text{ in } U} \mathcal{O}_{X,p} \tag{1}$$

is an injection.

We define a **rational function** on a locally Noetherian scheme to be an element of the image of $\Gamma(U, \mathcal{O}_U)$ in (1) for some U containing all the associated points. The rational functions form a ring, called the **total fraction ring** of X, denoted $\mathrm{FF}(X)$. If $X = \operatorname{Spec} A$ is affine, then this ring is called the **total fraction ring** of A, $\mathrm{FF}(A)$.

(4) *A function on X is a zero divisor if and only if it vanishes at an associated point of X.*

Recall that an ideal $I \subset A$ in a ring is **primary** if $I \neq A$ and if $xy \in I$ implies either $x \in I$ or $y^n \in I$ for some $n > 0$. In other words, the quotient is not 0, and every zero-divisor is nilpotent. Hence the notion of "primary" should be seen as a condition on A/I, not on I.

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We know that if $\mathfrak{q}$ is primary, then $\sqrt{\mathfrak{q}}$ is prime, say $\mathfrak{p}$. We then say that $\mathfrak{q}$ is $\mathfrak{p}$-*primary*. We know that if $\mathfrak{q}$ and $\mathfrak{q}'$ are $\mathfrak{p}$-primary, then so is $\mathfrak{q} \cap \mathfrak{q}'$.

We also know that primary decompositions, and hence minimal primary decompositions, exist for any ideal of a Noetherian ring.

We proved:

1.1. *Theorem ("uniqueness" of primary decomposition). — Suppose* $I \subset A$ *has a minimal primary decomposition*

$$I = \cap_{i=1}^{n} \mathfrak{q}_i.$$

(For example, this is always true if A *is Noetherian.) Then the* $\sqrt{\mathfrak{q}_i}$ *are precisely the prime ideals that are of the form*

$$\sqrt{(I : x)}$$

for some $x \in A$. *Hence this list of primes is independent of the decomposition.*

These primes are called the **associated primes** of the ideal I. The **associated primes of** A are the associated primes of 0.

The proof involved the handy line

$$\sqrt{(I : x)} = \cap \sqrt{(\mathfrak{q}_i : x)} = \cap_{x \notin \mathfrak{q}_j} \mathfrak{p}_j. \tag{2}$$

So let's move forward!

1.A. EXERCISE (ASSOCIATED PRIMES BEHAVE WELL WITH RESPECT TO LOCALIZATION). Show that if A is a Noetherian ring, and S is a multiplicative subset (so there is an inclusion-preserving correspondence between the primes of $S^{-1}A$ and those primes of A not meeting S), then the associated primes of $S^{-1}A$ are just the associated primes of A not meeting S.

We then define the **associated points** of a locally Noetherian scheme X to be those points $p \in X$ such that, on any affine open set $\operatorname{Spec} A$ containing p, p corresponds to an associated prime of A. Note that this notion is well-defined: If p has two affine open neighborhoods $\operatorname{Spec} A$ and $\operatorname{Spec} B$ (say corresponding to primes $\mathfrak{p} \subset A$ and $\mathfrak{q} \subset B$ respectively), then p corresponds to an associated prime of A if and only if it corresponds to an associated prime of $A_{\mathfrak{p}} = \mathcal{O}_{X,p} = B_{\mathfrak{q}}$ if and only if it corresponds to an associated prime of B.

If furthermore X is quasicompact (i.e. X is a Noetherian scheme), then there are a finite number of associated points.

1.B. EXERCISE. (a) Show that the minimal primes of A are associated primes. We have now proved important fact **(1)**. (Hint: suppose $\mathfrak{p} \supset \cap_{i=1}^{n} \mathfrak{q}_i$. Then $\mathfrak{p} = \sqrt{\mathfrak{p}} \supset \sqrt{\cap_{i=1}^{n} \mathfrak{q}_i} = \cap_{i=1}^{n} \sqrt{\mathfrak{q}_i} = \cap_{i=1}^{n} \mathfrak{p}_i$, so by a previous exercise, $\mathfrak{p} \supset \mathfrak{p}_i$ for some i. If $\mathfrak{p}$ is minimal, then as $\mathfrak{p} \supset \mathfrak{p}_i \subset (0)$, we must have $\mathfrak{p} = \mathfrak{p}_i$.)

(b) Show that there can be other associated primes that are not minimal. (Hint: we've seen an example...) Your argument will show more generally that the minimal primes of I are associated primes of I.

1.C. EXERCISE. Show that if A is reduced, then the only associated primes are the minimal primes. (This establishes **(2)**.)

The $\mathfrak{q}_i$ corresponding to minimal primes are unique, but the $\mathfrak{q}_i$ corresponding to other associated primes are not unique. We will not need this fact, and hence won't prove it.

1.2. *Proposition.* — *The set of zero-divisors is the union of the associated primes.*

This establishes **(4)**: a function is a zero-divisor if and only if it vanishes at an associated point. Thus (for a Noetherian scheme) a function is a zero divisor if and only if its zero locus contains one of a finite set of points.

You may wish to try this out on the example of the affine line with fuzz at the origin.

Proof. If $\mathfrak{p}_i$ is an associated prime, then $\mathfrak{p}_i = \sqrt{(0 : x)}$ from the proof of Theorem 1.1, so $\cup \mathfrak{p}_i$ is certainly contained in the set Z of zero-divisors.

For the converse:

1.D. EXERCISE. Show that

$$Z = \cup_{x \neq 0} (0 : x) \subseteq \cup_{x \neq 0} \sqrt{(0 : x)} \subseteq Z.$$

Hence

$$Z = \cup_{x \neq 0} \sqrt{(0 : x)} = \cup_x \left(\cap_{x \notin \mathfrak{q}_j} \mathfrak{p}_j \right) \subseteq \cup \mathfrak{p}_j$$

using (2). □

1.E. UNIMPORTANT EXERCISE (RABINOFF'S THEOREM). Here is an interesting variation on **(4)**: show that $a \in A$ is nilpotent if and only if it vanishes at the associated points of Spec A. Algebraically: we know that the nilpotents are the intersection of *all* prime ideals; now show that in the Noetherian case, the nilpotents are in fact the intersection of the (finite number of) associated prime ideals.

1.3. *Proposition.* — *The natural map* $A \to \prod_{\textit{associated } \mathfrak{p}} A_{\mathfrak{p}}$ *is an inclusion.*

Proof. Suppose $r \neq 0$ maps to 0 under this map. Then there are $s_i \in A - \mathfrak{p}$ with $s_i r = 0$. Then $I := (s_1, \ldots, s_n)$ is an ideal consisting only of zero-divisors. Hence $I \subseteq \cup \mathfrak{p}_i$. Then $I \subset \mathfrak{p}_i$ for some i by an exercise from last week, contradicting $s_i \notin \mathfrak{p}_i$. □

1.F. EASIER AND LESS IMPORTANT EXERCISE. Prove fact **(3)**. (The previous Proposition establishes it for affine open sets.)

2. INTRODUCTION TO MORPHISMS OF SCHEMES

Whenever you learn about a new type of object in mathematics, you should naturally be curious about maps between them, which means understanding how they form a category. In order to satisfy this curiosity, we'll introduce the notion of morphism of schemes now, and at the same time we may as well define some easy-to-state properties of morphisms. However, we'll leave more subtle properties of morphisms for next quarter.

Recall that a scheme is (i) a set, (ii) with a topology, (iii) and a (structure) sheaf of rings, and that it is sometimes helpful to think of the definition as having three steps. In the same way, the notion of morphism of schemes $X \to Y$ may be defined (i) as a map of sets, (ii) that is continuous, and (iii) with some further information involving the sheaves of functions. In the case of affine schemes, we have already seen the map as sets, and later saw that this map is continuous.

Here are two motivations for how morphisms should behave. The first is algebraic, and the second is geometric.

(a) We'll want morphisms of affine schemes $\operatorname{Spec} B \to \operatorname{Spec} A$ to be precisely the ring maps $A \to B$. We have already seen that ring maps $A \to B$ induce maps of topological spaces in the opposite direction; the main new ingredient will be to see how to add the structure sheaf of functions into the mix. Then a morphism of schemes should be something that "on the level of affines, looks like this".

(b) We are also motivated by the theory of differentiable manifolds. Notice that if $\pi : X \to Y$ is a map of differentiable manifolds, then a differentiable function on Y pulls back to a differentiable function on X. More precisely, given an open subset $U \subset Y$, there is a natural map $\Gamma(U, \mathcal{O}_Y) \to \Gamma(\pi^{-1}(U), \mathcal{O}_X)$. This behaves well with respect to restriction (restricting a function to a smaller open set and pulling back yields the same result as pulling back and then restricting), so in fact we have a map of sheaves on Y: $\mathcal{O}_Y \to \pi_* \mathcal{O}_X$. Similarly a morphism of schemes $X \to Y$ should induce a map $\mathcal{O}_Y \to \pi_* \mathcal{O}_X$. But in fact in the category of differentiable manifolds a continuous map $X \to Y$ is a map of differentiable manifolds precisely when differentiable functions on Y pull back to differentiable functions on X (i.e. the pullback map from differentiable functions on Y to *functions* on X in fact lies in the subset of *differentiable functions*, i.e. the continuous map $X \to Y$ induces a pullback of differential functions $\mathcal{O}_Y \to \mathcal{O}_X$), so this map of sheaves *characterizes* morphisms in the differentiable category. So we could use this as the *definition* of morphism in the differentiable category.

But how do we apply this to the category of schemes? In the category of differentiable manifolds, a continuous map $X \to Y$ *induces* a pullback of (the sheaf of) functions, and we can ask when this induces a pullback of *differentiable* functions. However, functions are odder on schemes, and we can't recover the pullback map just from the map of topological spaces. A reasonable patch is to hardwire this into the definition of morphism, i.e. to have a continuous map $f : X \to Y$, along with a pullback map $f^\# : \mathcal{O}_Y \to f_* \mathcal{O}_X$. This leads to the definition of the category of ringed spaces.

One might hope to define morphisms of schemes as morphisms of ringed spaces. This isn't quite right, as then motivation (a) isn't satisfied: as desired, to each morphism $A \to B$ there is a morphism $\operatorname{Spec} B \to \operatorname{Spec} A$, but there can be additional morphisms of ringed spaces $\operatorname{Spec} B \to \operatorname{Spec} A$ not arising in this way (Exercise 3.C). A revised definition as morphisms of ringed spaces that locally looks of this form will work, but this is awkward to work with, and we take a different tack. However, we will check that our eventual definition actually is equivalent to this.

We'll begin by discussing morphisms of ringed spaces.

Before we do, we take this opportunity to use motivation (a) to motivate the definition of *equivalence of categories*. We wish to say that the category of rings and the category of affine schemes are opposite categories, i.e. that the "opposite category of affine schemes" (where all the arrows are reversed) is "essentially the same" as the category of rings. We indeed have a functor from rings to affine schemes (sending A to $\operatorname{Spec} A$), and a functor from affine schemes to rings (sending X to $\Gamma(X, \mathcal{O}_X)$). But if you think about it, you'll realize their composition isn't exactly the identity. (It all boils down to the meaning of "is" or "same", and this can get confusing.) Rather than trying to set things up so the composition *is* the identity, we just don't let this bother us, and make precise the notion that the composition is "essentially" the identity.

Suppose F and G are two functors from $\mathcal{A}$ to $\mathcal{B}$. A **natural transformation of functors** $F \to G$ is the data of a morphism $m_a : F(a) \to G(a)$ for each $a \in \mathcal{A}$ such that for each $f : a \to a'$ in $\mathcal{A}$, the diagram

$$\begin{array}{ccc} F(a) & \xrightarrow{F(f)} & F(a') \\ {\scriptstyle m_a}\downarrow & & \downarrow{\scriptstyle m_{a'}} \\ G(a) & \xrightarrow[G(f)]{} & G(a') \end{array}$$

A **natural isomorphism** of functors is a natural transformation such that each m_a is an isomorphism. The data of functors $F : \mathcal{A} \to \mathcal{B}$ and $F' : \mathcal{B} \to \mathcal{A}$ such that $F \circ F'$ is naturally isomorphic to the identity $I_{\mathcal{B}}$ on $\mathcal{B}$ and $F' \circ F$ is naturally isomorphic to $I_{\mathcal{A}}$ is said to be an **equivalence of categories**. This is the "right" notion of isomorphism of categories.

Two examples might make this strange concept more comprehensible. The double dual of a finite-dimensional vector space V is *not* V, but we learn early to say that it is canonically isomorphic to V. We make can that precise as follows. Let **f.d.** $\mathbf{Vec}_k$ be the category of finite-dimensional vector spaces over k. Note that this category contains oodles of vector spaces of each dimension.

2.A. EXERCISE. Let $\vee\vee$: **f.d.** $\mathbf{Vec}_k \to$ **f.d.** $\mathbf{Vec}_k$ be the double dual functor from the category of vector spaces over k to itself. Show that $\vee\vee$ is naturally isomorphic to the identity. (Without the finite-dimensional hypothesis, we only get a natural transformation of functors from id to $\vee\vee$.)

Let $\mathcal{V}$ be the category whose objects are k^n for each n (there is one vector space for each n), and whose morphisms are linear transformations. This latter space can be thought of as vector spaces with bases, and the morphisms are honest matrices. There is an obvious functor $\mathcal{V} \to \mathbf{f.d.Vec}_k$, as each k^n *is* a finite-dimensional vector space.

2.B. EXERCISE. Show that $\mathcal{V} \to \mathbf{f.d.Vec}_k$ gives an equivalence of categories, by describing an "inverse" functor. (You'll need the axiom of choice, as you'll simultaneously choose bases for each vector space in $\mathbf{f.d.Vec}_k$!)

Once you have come to terms with the notion of equivalence of categories, you will quickly see that rings and affine schemes are basically the same thing, with the arrows reversed:

2.C. EXERCISE. Assuming that morphisms of schemes are defined so that Motivation (a) holds, show that the category of rings and the opposite category of affine schemes are equivalent.

3. MORPHISMS OF RINGED SPACES

3.1. Definition. A **morphism** $\pi : X \to Y$ **of ringed spaces** is a continuous map of topological spaces (which we unfortunately also call π) along with a "pullback map" $\mathcal{O}_Y \to \pi_* \mathcal{O}_X$. By adjointness, this is the same as a map $\pi^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. There is an obvious notion of composition of morphisms; hence there is a category of ringed spaces. Hence we have notion of automorphisms and isomorphisms. You can easily verify that an isomorphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a homeomorphism $f : X \to Y$ along with an isomorphism $\mathcal{O}_Y \to f_* \mathcal{O}_X$ (or equivalently $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$).

If $U \subset Y$ is an open subset, then there is a natural morphism of ringed spaces $(U, \mathcal{O}_Y|_U) \to (Y, \mathcal{O}_Y)$. (Check this! The f^{-1} interpretation is cleaner to use here.) This is our model for an open immersion. More precisely, if $U \to Y$ is an isomorphism of U with an open subset V of Y, and we are given an isomorphism $(U, \mathcal{O}_U) \cong (V, \mathcal{O}_V)$ (via the isomorphism $U \cong V$), then the resulting map of ringed spaces is called an **open immersion** of ringed spaces.

3.A. EXERCISE (MORPHISMS OF RINGED SPACES GLUE). Suppose $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are ringed spaces, $X = \cup_i U_i$ is an open cover of X, and we have morphisms of ringed spaces $f_i : U_i \to Y$ that "agree on the overlaps", i.e. $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$. Show that there is a unique morphism of ringed spaces $f : X \to Y$ such that $f|_{U_i} = f_i$. (An earlier exercise essentially showed this for topological spaces.)

3.B. EASY IMPORTANT EXERCISE. Given a morphism of ringed spaces $f : X \to Y$ with $f(p) = q$, show that there is a map of stalks $(\mathcal{O}_Y)_q \to (\mathcal{O}_X)_p$.

3.2. Key Exercise. Suppose $f^\# : B \to A$ is a morphism of rings. Define a morphism of ringed spaces $f : \operatorname{Spec} A \to \operatorname{Spec} B$ as follows. The map of topological spaces was given earlier. To describe a morphism of sheaves $\mathcal{O}_B \to f_* \mathcal{O}_A$ on $\operatorname{Spec} B$, it suffices to describe a morphism of sheaves on the distinguished base of $\operatorname{Spec} B$. On $D(g) \subset \operatorname{Spec} B$, we define

$$\mathcal{O}_B(D(g)) \to \mathcal{O}_A(f^{-1}D(g)) = \mathcal{O}_A(D(f^\# g))$$

by $B_g \to A_{f^\# g}$. Verify that this makes sense (e.g. is independent of g), and that this describes a morphism of sheaves on the distinguished base. (This is the third in a series of exercises. We showed that a morphism of rings induces a map of sets first, a map of topological spaces later, and now a map of ringed spaces here.)

This will soon be an example of morphism of schemes! In fact we could make that definition right now.

3.3. Definition we won't start with. A morphism of schemes $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces that "locally looks like" the maps of affine schemes described in Key Exercise 3.2. Precisely, for each choice of affine opens $\operatorname{Spec} A \subset X$, $\operatorname{Spec} B \subset Y$, such that $f(\operatorname{Spec} A) \subset \operatorname{Spec} B$, the induced map of ringed spaces should be of the form shown in Key Exercise 3.2.

We would like this definition to be checkable on an affine cover, and we might hope to use the affine communication lemma to develop the theory in this way. This works, but it will be more convenient to use a clever trick: in the next section, we will use the notion of locally ringed spaces, and then once we have used it, we will discard it like yesterday's garbage.

The map of ringed spaces of Key Exercise 3.2 is really not complicated. Here is an example. Consider the ring map $\mathbb{C}[x] \to \mathbb{C}[y]$ given by $x \mapsto y^2$. We are mapping the affine line with co-ordinate y to the affine line with co-ordinate x. The map is (on closed points) $a \mapsto a^2$. For example, where does $[(y-3)]$ go to? Answer: $[(x-9)]$, i.e. $3 \mapsto 9$. What is the preimage of $[(x-4)]$? Answer: those prime ideals in $\mathbb{C}[y]$ containing $[(y^2-4)]$, i.e. $[(y-2)]$ and $[(y+2)]$, so the preimage of 4 is indeed ± 2. This is just about the map of sets, which is old news, so let's now think about functions pulling back. What is the pullback of the function $3/(x-4)$ on $D([(x-4)]) = \mathbb{A}^1 - \{4\}$? Of course it is $3/(y^2-4)$ on $\mathbb{A}^1 - \{-2, 2\}$.

We conclude with an example showing that not every morphism of ringed spaces between affine schemes is of the form of Key Exercise 3.2.

3.C. UNIMPORTANT EXERCISE. Recall that $\operatorname{Spec} k[x]_{(x)}$ has two points, corresponding to (0) and (x), where the second point is closed, and the first is not. Consider the map *of ringed spaces* $\operatorname{Spec} k(x) \to \operatorname{Spec} k[x]_{(x)}$ sending the point of $\operatorname{Spec} k(x)$ to $[(x)]$, and the pullback map $f^\# \mathcal{O}_{\operatorname{Spec} k(x)} \to \mathcal{O}_{\operatorname{Spec} k[x]_{(x)}}$ is induced by $k \hookrightarrow k(x)$. Show that this map of ringed spaces is not of the form described in Key Exercise 3.2.

In order to prove that morphisms behave in a way we hope, we will introduce the notion of a *locally ringed space*. It will not be used later, although it is useful elsewhere in geometry. The notion of locally ringed spaces is inspired by what we know about manifolds. If $\pi : X \to Y$ is a morphism of manifolds, with $\pi(p) = q$, and f is a function on Y vanishing at q, then the pulled back function $\pi^{\#} f$ on X should vanish on p. Put differently: germs of functions (at $q \in Y$) vanishing at q should pull back to germs of functions (at $p \in X$) vanishing at p.

A **locally ringed space** is a ringed space $(X, \mathcal{O}_X)$ such that the stalks $\mathcal{O}_{X,x}$ are all local rings. A **morphism of locally ringed spaces** $f : X \to Y$ is a morphism of ringed spaces such that the induced map of stalks $\mathcal{O}_{Y,q} \to \mathcal{O}_{X,p}$ (Exercise 3.B) sends the maximal ideal of the former into the maximal ideal of the latter (a "local morphism of local rings"). This means something rather concrete and intuitive: "if $p \mapsto q$, and g is a function vanishing at q, then it will pull back to a function vanishing at p." Note that locally ringed spaces form a category.

4.A. EXERCISE. Show that morphisms of locally ringed spaces glue (cf. Exercise 3.A). (Hint: Basically, the proof of Exercise 3.A works.)

4.B. EASY IMPORTANT EXERCISE. (a) Show that $\operatorname{Spec} A$ is a locally ringed space. (b) The morphism of ringed spaces $f : \operatorname{Spec} A \to \operatorname{Spec} B$ defined by a ring morphism $f^{\#} : B \to A$ is a morphism of locally ringed spaces.

4.1. *Key Proposition. — If $f : \operatorname{Spec} A \to \operatorname{Spec} B$ is a morphism of locally ringed spaces then it is the morphism of locally ringed spaces induced by the map $f^{\#} : B = \Gamma(\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}) \to \Gamma(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) = A$ as in Exercise 4.B(b).*

Proof. Suppose $f : \operatorname{Spec} A \to \operatorname{Spec} B$ is a morphism of locally ringed spaces. Then we wish to show that $f^{\#} : \mathcal{O}_{\operatorname{Spec} B} \to f_* \mathcal{O}_{\operatorname{Spec} A}$ is the morphism of sheaves given by Exercise 3.2 (cf. Exercise 4.B(b)). It suffices to checked this on the distinguished base.

Note that if $b \in B$, $f^{-1}(D(b)) = D(f^{\#} b)$; this is where we use the hypothesis that f is a morphism of locally ringed spaces.

The commutative diagram

$$\begin{array}{ccc} \Gamma(\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}) & \xrightarrow{f^{\#}_{\operatorname{Spec} B}} & \Gamma(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) \\ \downarrow & & \downarrow \otimes_B B_b \\ \Gamma(D(b), \mathcal{O}_{\operatorname{Spec} B}) & \xrightarrow{f^{\#}_{D(b)}} & \Gamma(D(f^{\#} b), \mathcal{O}_{\operatorname{Spec} A}) \end{array}$$

may be written as

$$\begin{array}{ccc} B & \xrightarrow{f^\#_{\operatorname{Spec} B}} & A \\ \downarrow & & \downarrow \otimes_B B_b \\ B_b & \xrightarrow{f^\#_{D(b)}} & A_{f^\# b}. \end{array}$$

We want that $f^\#_{D(b)} = (f^\#_{\operatorname{Spec} B})_b$. This is clear from the commutativity of that last diagram.
□

We are ready for our definition.

4.2. Definition. If X and Y are schemes, then a morphism of locally ringed spaces is called a **morphism of schemes**. We have thus defined a *category* of schemes. (We then have notions of **isomorphism** — just the same as before — and **automorphism**.)

The definition in terms of locally ringed spaces easily implies tentative definition 3.3:

4.C. IMPORTANT EXERCISE. Show that a morphism of schemes $f : X \to Y$ is a morphism of ringed spaces that looks locally like morphisms of affines. Precisely, if $\operatorname{Spec} A$ is an affine open subset of X and $\operatorname{Spec} B$ is an affine open subset of Y, and $f(\operatorname{Spec} A) \subset \operatorname{Spec} B$, then the induced morphism of ringed spaces is a morphism of affine schemes. Show that it suffices to check on a set $(\operatorname{Spec} A_i, \operatorname{Spec} B_i)$ where the $\operatorname{Spec} A_i$ form an open cover X.

In practice, we will use the fact the affine cover interpretation, and forget completely about locally ringed spaces.

It is also clear (from the corresponding facts about locally ringed spaces) that morphisms glue (Exercise 4.A), and the composition of two morphisms is a morphism. Isomorphisms in this category are precise what we defined them to be earlier (homeomorphism of topological spaces with isomorphisms of structure sheaves).

4.3. The category of schemes (or k-schemes, or A-schemes, or $\mathbb{Z}$-schemes). It is often convenient to consider subcategories. For example, the category of k-schemes (where k is a field) is defined as follows. The objects are morphisms of the form

$$\begin{array}{c} X \\ \downarrow \text{structure morphism} \\ \operatorname{Spec} k \end{array}$$

(This is the same definition as earlier, but in a more satisfactory form.) The morphisms in the category of k-schemes are commutative diagrams

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \xrightarrow{=} & \operatorname{Spec} k \end{array}$$

which is more conveniently written as a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & Y \\ & \searrow \quad \swarrow & \\ & \operatorname{Spec} k. & \end{array}$$

For example, complex geometers may consider the category of $\mathbb{C}$-schemes.

When there is no confusion, simply the top row of the diagram is given. More generally, if A is a ring, the category of A-schemes is defined in the same way, with A replacing k. And if Z is a scheme, the category of Z-schemes is defined in the same way, with Z replacing $\operatorname{Spec} k$.

4.4. Examples.

4.D. IMPORTANT EXERCISE. (This exercise will give you some practice with understanding morphisms of schemes by cutting up into affine open sets.) Make sense of the following sentence: "$\mathbb{A}^{n+1} \setminus \{\vec{0}\} \to \mathbb{P}^n$ given by

$$(x_0, x_1, \ldots, x_{n+1}) \mapsto [x_0; x_1; \ldots; x_n]$$

is a morphism of schemes." Caution: you can't just say where points go; you have to say where functions go. So you'll have to divide these up into affines, and describe the maps, and check that they glue.

4.E. IMPORTANT EXERCISE. Show that morphisms $X \to \operatorname{Spec} A$ are in natural bijection with ring morphisms $A \to \Gamma(X, \mathcal{O}_X)$. (Hint: Show that this is true when X is affine. Use the fact that morphisms glue.)

In particular, there is a canonical morphism from a scheme to Spec of its space of global sections. (Warning: Even if X is a finite-type k-scheme, the ring of global sections might be nasty! In particular, it might not be finitely generated.)

4.5. *Side fact for experts: Γ and* Spec *are adjoints.* We have a functor Spec from rings to locally ringed spaces, and a functor Γ from locally ringed spaces to rings. Exercise 4.E implies $(\Gamma, \operatorname{Spec})$ is an adjoint pair! Thus we could have defined Spec by requiring it to be adjoint to Γ.

4.F. EXERCISE. Show that $\operatorname{Spec} \mathbb{Z}$ is the final object in the category of schemes. In other words, if X is any scheme, there exists a unique morphism to $\operatorname{Spec} \mathbb{Z}$. (Hence the category of schemes is isomorphic to the category of $\mathbb{Z}$-schemes.)

4.G. EXERCISE. Show that morphisms $X \to \operatorname{Spec} \mathbb{Z}[t]$ correspond to global sections of the structure sheaf.

4.6. ⋆ **Representable functors.** This is one of our first explicit examples of an important idea, that of representable functors. This is a very useful idea, whose utility isn't immediately apparent. We have a contravariant functor from schemes to sets, taking a scheme to its set of global sections. We have another contravariant functor from schemes to sets, taking X to $\mathrm{Hom}(X, \mathrm{Spec}\,\mathbb{Z}[t])$. This is describing a (natural) isomorphism of the functors. More precisely, we are describing an isomorphism $\Gamma(X, \mathcal{O}_X) \cong \mathrm{Hom}(X, \mathrm{Spec}\,\mathbb{Z}[t])$ that behaves well with respect to morphisms of schemes: given any morphism $f : X \to Y$, the diagram

$$\begin{array}{ccc} \Gamma(Y, \mathcal{O}_Y) & \xrightarrow{\sim} & \mathrm{Hom}(Y, \mathrm{Spec}\,\mathbb{Z}[t]) \\ \downarrow f^* & & \downarrow f\circ \\ \Gamma(X, \mathcal{O}_X) & \xrightarrow{\sim} & \mathrm{Hom}(X, \mathrm{Spec}\,\mathbb{Z}[t]) \end{array}$$

commutes. *Given a contravariant functor from schemes to sets, by the usual universal property argument, there is only one possible scheme Y, up to isomorphism, such that there is a natural isomorphism between this functor and* $\mathrm{Hom}(\cdot, Y)$. If there is such a Y, we say that the functor is **representable**.

Here are a couple of examples of something you've seen to put it in context. (i) The contravariant functor $h^Y = \mathrm{Hom}(\cdot, Y)$ (i.e. $X \mapsto \mathrm{Hom}(X, Y)$) is representable by Y. (ii) Suppose we have morphisms $X, Y \to Z$. The contravariant functor $\mathrm{Hom}(\cdot, X) \times_{\mathrm{Hom}(\cdot, Z)} \mathrm{Hom}(\cdot, Y)$ is representable if and only if the fibered product $X \times_Z Y$ exists (and indeed then the contravariant functor is represented by $\mathrm{Hom}(\cdot, X \times_Z Y)$). This is purely a translation of the definition of the fibered product — you should verify this yourself.

Remark for experts: The global sections form something better than a set — they form a scheme. You can define the notion of ring scheme, and show that if a contravariant functor from schemes to rings is representable (as a contravariant functor from schemes to sets) by a scheme Y, then Y is guaranteed to be a ring scheme. The same is true for group schemes.

4.H. RELATED EXERCISE. Show that global sections of $\mathcal{O}_X^*$ correspond naturally to maps to $\mathrm{Spec}\,\mathbb{Z}[t, t^{-1}]$. ($\mathrm{Spec}\,\mathbb{Z}[t, t^{-1}]$ is a *group scheme*.)

5. SOME TYPES OF MORPHISMS

(This section has been moved forward to class 13.)

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