FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASSES 7 AND 8

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CONTENTS

1. DECOMPOSITION INTO IRREDUCIBLE COMPONENTS, AND NOETHERIAN INDUCTION

At the end of last day, we defined *irreducible component*: If X is a topological space, and Z is an irreducible closed subset not contained in any larger irreducible closed subset, Z is said to be an **irreducible component** of X. We think of these as the "pieces of X" (see Figure 1).

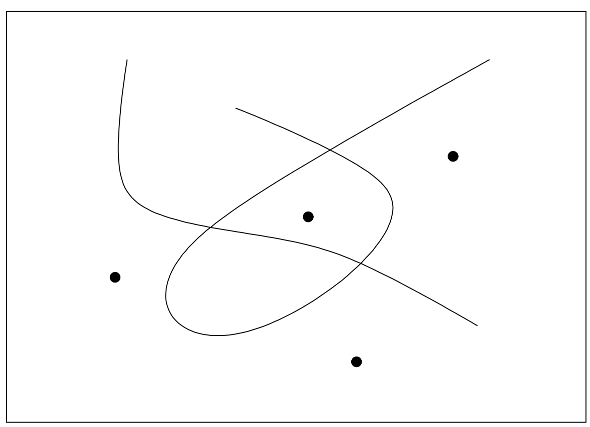

FIGURE 1. This closed subset of $\mathbb{A}^2$ has six irreducible components

We saw the exercise: If A is any ring, show that the irreducible components of $\operatorname{Spec} A$ are in bijection with the minimal primes of A.

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For example, the only minimal prime of $k[x,y]$ is (0). What are the minimal primes of $k[x,y]/(xy)$?

1.1. *Proposition. — Suppose* X *is a Noetherian topological space. Then every non-empty closed subset* Z *can be expressed uniquely as a finite union* $Z = Z_1 \cup \cdots \cup Z_n$ *of irreducible closed subsets, none contained in any other.*

Translation: any non-empty closed subset Z has of a finite number of pieces.

As a corollary, this implies that a Noetherian ring A has only finitely many minimal primes.

Proof. The following technique is often called *Noetherian induction*, for reasons that will become clear. Justin prefers the phrase "Noetherian descent".

Consider the collection of closed subsets of X that *cannot* be expressed as a finite union of irreducible closed subsets. We will show that it is empty. Otherwise, let Y_1 be one such. If it properly contains another such, then choose one, and call it Y_2. If this one contains another such, then choose one, and call it Y_3, and so on. By the descending chain condition, this must eventually stop, and we must have some Y_r that cannot be written as a finite union of irreducible closed subsets, but every closed subset contained in it can be so written. But then Y_r is not itself irreducible, so we can write $Y_r = Y' \cup Y''$ where Y' and Y'' are both proper closed subsets. Both of these by hypothesis can be written as the union of a finite number of irreducible subsets, and hence so can Y_r, yielding a contradiction. Thus each closed subset can be written as a finite union of irreducible closed subsets. We can assume that none of these irreducible closed subsets contain any others, by discarding some of them.

We now show uniqueness. Suppose

$$Z = Z_1 \cup Z_2 \cup \cdots \cup Z_r = Z_1' \cup Z_2' \cup \cdots \cup Z_s'$$

are two such representations. Then $Z_1' \subset Z_1 \cup Z_2 \cup \cdots \cup Z_r$, so $Z_1' = (Z_1 \cap Z_1') \cup \cdots \cup (Z_r \cap Z_1')$. Now Z_1' is irreducible, so one of these is Z_1' itself, say (without loss of generality) $Z_1 \cap Z_1'$. Thus $Z_1' \subset Z_1$. Similarly, $Z_1 \subset Z_a'$ for some a; but because $Z_1' \subset Z_1 \subset Z_a'$, and Z_1' is contained in no other Z_i', we must have $a = 1$, and $Z_1' = Z_1$. Thus each element of the list of Z's is in the list of Z''s, and vice versa, so they must be the same list. $\square$

2. The function $I(\cdot)$, taking subsets of $\operatorname{Spec} A$ to ideals of A

We now introduce a notion that is in some sense "inverse" to the vanishing set function $V(\cdot)$. Given a subset $S \subset \operatorname{Spec} A$, $I(S)$ is the set of functions vanishing on S.

We make three quick observations:

- $I(S)$ is clearly an ideal.
- $I(\overline{S}) = I(S)$.

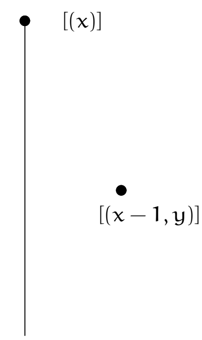

FIGURE 2. The set S of Exercise/example 2.A, pictured as a subset of $\mathbb{A}^2$

- $I(\cdot)$ is inclusion-reversing: if $S_1 \subset S_2$, then $I(S_2) \subset I(S_1)$.

2.A. EXERCISE/EXAMPLE. Let $A = k[x, y]$. If $S = \{[(x)], [(x-1, y)]\}$ (see Figure 2), then $I(S)$ consists of those polynomials vanishing on the y axis, and at the point $(1, 0)$. Give generators for this ideal.

2.B. TRICKY EXERCISE. Suppose $X \subset \mathbb{A}^3$ is the union of the three axes. (The x-axis is defined by $y = z = 0$, and the y-axis and z-axis are deined analogously.) Give generators for the ideal $I(X)$. Be sure to prove it! Hint: We will see later that this ideal is not generated by less than three elements.

2.C. EXERCISE. Show that $V(I(S)) = \overline{S}$. Hence $V(I(S)) = S$ for a closed set S. (Compare this to Exercise 2.D below.)

Note that $I(S)$ is always a radical ideal — if $f \in \sqrt{I(S)}$, then f^n vanishes on S for some $n > 0$, so then f vanishes on S, so $f \in I(S)$.

2.D. EXERCISE. Prove that if $I \subset A$ is an ideal, then $I(V(I)) = \sqrt{I}$.

This exercise and Exercise 2.C suggest that V and I are "almost" inverse. More precisely:

2.1. *Theorem. —* $V(\cdot)$ *and* $I(\cdot)$ *give a bijection between closed subsets of* $\operatorname{Spec} A$ *and radical ideals of* A *(where a closed subset gives a radical ideal by* $I(\cdot)$*, and a radical ideal gives a closed subset by* $V(\cdot)$*).*

2.E. IMPORTANT EXERCISE. Show that $V(\cdot)$ and $I(\cdot)$ give a bijection between *irreducible closed subsets* of $\operatorname{Spec} A$ and *prime* ideals of A. From this conclude that in $\operatorname{Spec} A$ there is a bijection between points of $\operatorname{Spec} A$ and irreducible closed subsets of $\operatorname{Spec} A$ (where a point determines an irreducible closed subset by taking the closure). Hence *each irreducible closed subset of* $\operatorname{Spec} A$ *has precisely one generic point* — any irreducible closed subset Z can be written uniquely as $\overline{\{z\}}$.

3. DISTINGUISHED OPEN SETS

If $f \in A$, define the **distinguished open set** $D(f) = \{[\mathfrak{p}] \in \operatorname{Spec} A : f \notin \mathfrak{p}\}$. It is the locus where f doesn't vanish. (I often privately write this as $D(f \neq 0)$ to remind myself of this. I also privately call this a "**D**oesn't-vanish set" in analogy with $V(f)$ being the **V**anishing set.) We have already seen this set when discussing $\operatorname{Spec} A_f$ as a subset of $\operatorname{Spec} A$. For example, we have observed that the Zariski-topology on the distinguished open set $D(f) \subset \operatorname{Spec} A$ coincides with the Zariski topology on $\operatorname{Spec} A_f$.

Here are some important but not difficult exercises to give you a feel for these important open sets.

3.A. EXERCISE. Show that the distinguished open sets form a base for the Zariski topology. (Hint: Given an ideal I, show that the complement of $V(I)$ is $\cup_{f \in I} D(f)$.)

3.B. EXERCISE. Suppose $f_i \in A$ as i runs over some index set J. Show that $\cup_{i \in J} D(f_i) = \operatorname{Spec} A$ if and only if $(f_i) = A$. (One of the directions will use the fact that any proper ideal of A is contained in some maximal ideal.)

3.C. EXERCISE. Show that if $\operatorname{Spec} A$ is an infinite union $\cup_{i \in J} D(f_i)$, then in fact it is a union of a finite number of these. (Hint: use the previous exercise 3.B.) Show that $\operatorname{Spec} A$ is quasicompact.

3.D. EXERCISE. Show that $D(f) \cap D(g) = D(fg)$.

3.E. EXERCISE. Show that if $D(f) \subset D(g)$, if and only if $f^n \in (g)$ for some n if and only if g is a unit in A_f. (Hint for the first equivalence: $f \in I(V((g)))$. We will use this shortly.

3.F. EXERCISE. Show that $D(f) = \varnothing$ if and only if $f \in \mathfrak{N}$.

4. THE STRUCTURE SHEAF

The final ingredient in the definition of an affine scheme is the **structure sheaf** $\mathcal{O}_{\operatorname{Spec} A}$, which we think of as the "sheaf of algebraic functions". As motivation, in $\mathbb{A}^2$, we expect that on the open set $D(xy)$ (away from the two axes), $(3x^4 + y + 4)/x^7$ should be an algebraic function.

These functions will have values at points, but won't be determined by their values at points. But like all sheaves, they will indeed be determined by their germs. This is discussed in Section 4.4.

It suffices to describe the structure sheaf as a sheaf (of rings) on the base of distinguished open sets. Our strategy is as follows. We will define the sections on the base by

$$\mathcal{O}_{\operatorname{Spec} A}(D(f)) = A_f \tag{1}$$

We need to make sure that this is well-defined, i.e. that we have a natural isomorphism $A_f \to A_g$ if $D(f) = D(g)$. We will define the restriction maps $\operatorname{res}_{D(g),D(f)}$ as follows. If $D(f) \subset D(g)$, then we have shown that $D(fg) = D(f)$. There is a natural map $A_g \to A_{fg}$ given by $r/g^m \mapsto (rf^m)/(fg)^m$, and we will define

$$\operatorname{res}_{D(g),D(fg)=D(f)} : \mathcal{O}_{\operatorname{Spec} A}(D(g)) \to \mathcal{O}_{\operatorname{Spec} A}(D(fg))$$

to be this map. But it will be cleaner to state things a little differently.

If $D(f) \subset D(g)$, then by Exercise 3.E, g is a unit in A_f Thus by the universal property of localization, there is a natural map $A_g \to A_f$ which we temporarily denote $\operatorname{res}_{g,f}$, but which we secretly think of as $\operatorname{res}_{D(g),D(f)}$. If $D(f) \subset D(g) \subset D(h)$, then these restriction maps commute:

$$\begin{tikzcd} A_h \arrow[rr, "\operatorname{res}_{h,g}"] \arrow[dr, "\operatorname{res}_{h,f}"'] & & A_g \arrow[dl, "\operatorname{res}_{g,f}"] \\ & A_f & \end{tikzcd} \tag{2}$$

commutes. (The map $A_h \to A_f$ is defined by universal property, and the composition $\operatorname{res}_{g,f} \circ \operatorname{res}_{h,g}$ satisfies this universal property.)

In particular, if $D(f) = D(g)$, then $\operatorname{res}_{g,f} \circ \operatorname{res}_{f,g}$ is the identity on A_f, (take $h = f$ in the above diagram (2)), and similarly $\operatorname{res}_{f,g} \circ \operatorname{res}_{g,f} = \operatorname{id}_{A_g}$. Thus we can define $\mathcal{O}_{\operatorname{Spec} A}(D(f)) = A_f$, and this is well-defined (independent of the choice of f).

By (2), we have defined a presheaf on the distinguished base.

We now come to a key theorem.

4.1. *Theorem. — The data just described gives a sheaf on the distinguished base, and hence determines a sheaf on the topological space* $\operatorname{Spec} A$.

This sheaf is called the **structure sheaf**, and will be denoted $\mathcal{O}_{\operatorname{Spec} A}$, or sometimes $\mathcal{O}$ if the scheme in question is clear from the context. Such a topological space, with sheaf, will be called an **affine scheme**. The notation $\operatorname{Spec} A$ will hereafter denote the data of a topological space with a structure sheaf.

Proof. We first check identity on the base. We deal with the case of a cover of the entire space A, and let you verify that essentially the same argument holds for a cover of some A_f. Suppose that $\operatorname{Spec} A = \cup_{i \in I} D(f_i)$ where i runs over some index set I. Then there is some finite subset of I, which we name $\{1, \ldots, n\}$, such that $\operatorname{Spec} A = \cup_{i=1}^n D(f_i)$, i.e. $(f_1, \ldots, f_n) = A$ (quasicompactness of $\operatorname{Spec} A$, Exercise 3.C). Suppose we are given $s \in A$ such that $\operatorname{res}_{\operatorname{Spec} A, D(f_i)} s = 0$ in A_{f_i} for all i. (We wish to show that $s = 0$.) Hence there is some m such that for each $i \in \{1, \ldots, n\}$, $f_i^m s = 0$. Now $(f_1^m, \ldots, f_n^m) = A$ ($\operatorname{Spec} A = \cup D(f_i) = \cup D(f_i^m)$), so there are $r_i \in A$ with $\sum_{i=1}^n r_i f_i^m = 1$ in A, from which

$$s = \left(\sum r_i f_i^m\right) s = \sum r_i (f_i^m s) = 0.$$

Thus we have checked the "base identity" axiom for $\operatorname{Spec} A$. (Serre has described this as a "partition of unity" argument, and if you look at it in the right way, his insight is very enlightening.)

4.A. EXERCISE. Make the tiny changes to the above argument to show base identity for any distinguished open $D(f)$. (Possible strategy: show that the argument is the same as the previous argument for $\operatorname{Spec} A_f$.)

We next show base gluability. As with base identity, we deal with the case where we wish to glue sections to produce a section over $\operatorname{Spec} A$. As before, we leave the general case where we wish to glue sections to produce a section over $D(f)$ to the reader (Exercise 4.B).

Suppose $\cup_{i \in I} D(f_i) = \operatorname{Spec} A$, where I is a index set (possibly horribly uncountably infinite). Suppose we are given elements in each A_{f_i} that agree on the overlaps $A_{f_i f_j}$. (Note that intersections of distinguished opens are also distinguished opens.)

Aside: experts might realize that we are trying to show exactness of

$$0 \to A \to \prod_i A_{f_i} \to \prod_{i \neq j} A_{f_i f_j}.$$

(What is the right-hand map?) Base identity corresponds to injectivity at A. The composition of the right two morphisms is trivially zero, and gluability is verifying exactness at $\prod_i A_{f_i}$.

Choose a finite subset $\{1, \ldots, n\} \subset I$ with $(f_1, \ldots, f_n) = A$ (i.e. use quasicompactness of $\operatorname{Spec} A$ to choose a finite subcover by $D(f_i)$). We have elements $a_i / f_i^{l_i} \in A_{f_i}$ agreeing on overlaps $A_{f_i f_j}$. Letting $g_i = f_i^{l_i}$, using $D(f_i) = D(g_i)$, we can simplify notation by considering our elements as of the form $a_i / g_i \in A_{g_i}$.

The fact that a_i/g_i and a_j/g_j "agree on the overlap" (i.e. in $A_{g_i g_j}$) means that for some m_{ij},

$$(g_i g_j)^{m_{ij}} (g_j a_i - g_i a_j) = 0$$

in A. By taking $m = \max m_{ij}$ (here we use the finiteness of I), we can simplify notation:

$$(g_i g_j)^m (g_j a_i - g_i a_j) = 0$$

for all i, j. Let $b_i = a_i g_i^m$ for all i, and $h_i = g_i^{m+1}$ (so $D(h_i) = D(g_i)$). Then we can simplify notation even more: on each $D(h_i)$, we have a function b_i / h_i, and the overlap condition is $h_j b_i - h_i b_j = 0$

Now $\cup_i D(h_i) = A$, implying that $1 = \sum_{i=1}^n r_i h_i$ for some $r_i \in A$. Define $r = \sum r_i b_i$. This will be the element of A that restricts to each b_j/h_j. Indeed,

$$rh_j - b_j = \sum_i r_i b_i h_j - \sum_i b_j r_i h_i = \sum_i r_i (b_i h_j - b_j h_i) = 0.$$

We are not quite done! We are supposed to have something that restricts to $a_i/f_i^{l_i}$ for *all* $i \in I$, not just $i = 1, \ldots, n$. But a short trick takes care of this. We now show that for any $\alpha \in I - \{1, \ldots, n\}$, r restricts to the desired element $a_\alpha\, A_{f_\alpha}$. Repeat the entire process above with $\{1, \ldots, n, \alpha\}$ in place of $\{1, \ldots, n\}$, to obtain $r' \in A$ which restricts to α_α for $i \in \{1, \ldots, n, \alpha\}$. Then by base identity, $r' = r$. (Note that we use base identity to *prove* base gluability. This is an example of how base identity is "prior" to base gluability.) Hence r restricts to $a_\alpha/f_\alpha^{l_\alpha}$ as desired.

4.B. EXERCISE. Alter this argument appropriately to show base gluability for any distinguished open $D(f)$.

We have now completed the proof of Theorem 4.1.

□

The proof of Theorem 4.1 immediately generalizes, as the following exercise shows. This exercise will be essential for the definition of a quasicoherent sheaf later on [say where].

4.C. IMPORTANT EXERCISE/DEFINITION. Suppose M is an A-module. Show that the following construction describes a sheaf $\tilde{M}$ on the distinguished base. To $D(f)$ we associate $M_f = M \otimes_A A_f$; the restriction map is the "obvious" one. This is an $\mathcal{O}_{\operatorname{Spec} A}$-module! This sort of sheaf $\tilde{M}$ will be very important soon; it is an example of a *quasicoherent sheaf*.

Here is a useful fact for later: As a consequence, note that if $(f_1, \ldots, f_r) = A$, we have identified M with a specific submodule of $M_{f_1} \times \cdots \times M_{f_r}$. Even though $M \to M_{f_i}$ may not be an inclusion for any f_i, $M \to M_{f_1} \times \cdots \times M_{f_r}$ *is* an inclusion. We don't care yet, but we'll care about this later, and I'll invoke this fact. (Reason: we'll want to show that if M has some nice property, then M_f does too, which will be easy. We'll also want to show that if $(f_1, \ldots, f_n) = R$, then if M_{f_i} have this property, then M does too.)

4.2. Definition. We can now define *scheme* in general. First, define an **isomorphism of ringed spaces** $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ as (i) a homeomorphism $f : X \to Y$, and (ii) an isomorphism of sheaves $\mathcal{O}_X$ and $\mathcal{O}_Y$, considered to be on the same space via f. (Condition (ii), more precisely: an isomorphism $\mathcal{O}_X \to f^{-1}\mathcal{O}_Y$ of sheaves on X, or $f_*\mathcal{O}_X \to \mathcal{O}_Y$ of sheaves on Y.) In other words, we have a correspondence of sets, topologies, and structure sheaves. An **affine scheme** is a ringed space that is isomorphic to $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$. A **scheme** $(X, \mathcal{O}_X)$ is a ringed space such that any point $x \in X$ has a neighborhood U such that $(U, \mathcal{O}_X|_U)$ is an affine scheme. The scheme can be denoted $(X, \mathcal{O}_X)$, although it is often denoted X, with the structure sheaf implicit.

An **isomorphism of two schemes** $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ is an isomorphism as ringed spaces.

4.3. *Remark.* From this definition of the structure sheaf on an affine scheme, several things are clear. First of all, if we are told that $(X, \mathcal{O}_X)$ is an affine scheme, we may recover its ring (i.e. find the ring A such that $\operatorname{Spec} A = X$) by taking the ring of global sections, as $X = D(1)$, so:

$$\begin{aligned} \Gamma(X, \mathcal{O}_X) &= \Gamma(D(1), \mathcal{O}_{\operatorname{Spec} A}) \quad \text{as } D(1) = \operatorname{Spec} A \\ &= A_1 \quad \text{(i.e. allow 1's in the denominator) by definition} \\ &= A. \end{aligned}$$

(You can verify that we get more, and can "recognize X as the scheme $\operatorname{Spec} A$": we get a natural isomorphism $f : (\operatorname{Spec} \Gamma(X, \mathcal{O}_X), \mathcal{O}_{\operatorname{Spec} \Gamma(X, \mathcal{O}_X)}) \to (X, \mathcal{O}_X)$. For example, if $\mathfrak{m}$ is a maximal ideal of $\Gamma(X, \mathcal{O}_X)$, $f([\mathfrak{m}]) = V(\mathfrak{m})$.) More generally, given $f \in A$, $\Gamma(D(f), \mathcal{O}_{\operatorname{Spec} A}) \cong A_f$. Thus under the natural inclusion of sets $\operatorname{Spec} A_f \hookrightarrow \operatorname{Spec} A$, the Zariski topology on $\operatorname{Spec} A$ restricts to give the Zariski topology on $\operatorname{Spec} A_f$ (as we've seen in an earlier Exercise), and the structure sheaf of $\operatorname{Spec} A$ restricts to the structure sheaf of $\operatorname{Spec} A_f$, as the next exercise shows.

4.D. IMPORTANT BUT EASY EXERCISE. Suppose $f \in A$. Show that under the identification of $D(f)$ in $\operatorname{Spec} A$ with $\operatorname{Spec} A_f$, there is a natural isomorphism of sheaves $(D(f), \mathcal{O}_{\operatorname{Spec} A}|_{D(f)}) \cong (\operatorname{Spec} A_f, \mathcal{O}_{\operatorname{Spec} A_f})$.

4.E. EXERCISE. Show that if X is a scheme, then the affine open sets form a base for the Zariski topology.

4.F. EXERCISE. If X is a scheme, and U is *any* open subset, prove that $(U, \mathcal{O}_X|_U)$ is also a scheme.

$(U, \mathcal{O}_X|_U)$ is called an *open subscheme of* U. If U is also an affine scheme, we often say U is an *affine open subset*, or an *affine open subscheme*, or sometimes informally just an *affine open*. For an example, $D(f)$ is an affine open subscheme of $\operatorname{Spec} A$.

4.4. Stalks of the structure sheaf: germs, and values at a point. Like every sheaf, the structure sheaf has stalks, and we shouldn't be surprised if they are interesting from an algebraic point of view. In fact, we have seen them before.

4.G. IMPORTANT EXERCISE. Show that the stalk of $\mathcal{O}_{\operatorname{Spec} A}$ at the point $[\mathfrak{p}]$ is the ring $A_{\mathfrak{p}}$.

Essentially the same argument will show that the stalk of the sheaf $\tilde{M}$, defined in Exercise 4.C at $[\mathfrak{p}]$ is $M_{\mathfrak{p}}$. Here is an interesting consequence, or if you prefer, a geometric interpretation of an algebraic fact. A section is determined by it stalks (an earlier Exercise), meaning that $M \to \prod_{\mathfrak{p}} M_{\mathfrak{p}}$ is an inclusion. So for example an A-module is zero if and only if all its localizations at primes are zero.

The **residue field of a scheme at a point** is the local ring modulo its maximal ideal.

So now we can make some of our vague discussion earlier precise. Suppose $[\mathfrak{p}]$ is a point in some open set U of $\operatorname{Spec} A$. For example, say $A = k[x, y]$, $\mathfrak{p} = [(x)]$ [draw picture], and $U = \mathbb{A}^2 - (0, 0)$.

Then a function on U, i.e. a section of $\mathcal{O}_{\operatorname{Spec} A}$ over U, has a *germ near* $[\mathfrak{p}]$, which is an element of $A_{\mathfrak{p}}$. This stalk $A_{\mathfrak{p}}$ is a local ring, with maximal ideal $\mathfrak{p}A_{\mathfrak{p}}$. In our example, consider the function $(3x^4 + x^2 + xy + y^2)/(3x^2 + xy + y^2 + 1)$, which is defined on the open set $D(3x^2 + xy + y^2 + 1)$. Because there are no factors of x in the denominator, it is indeed in $A_{\mathfrak{p}}$.

A germ has a *value* at $[\mathfrak{p}]$, which is an element of $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$. This is isomorphic to $FF(A/\mathfrak{p})$, the fraction field of the quotient domain. It is useful to note that localization at $\mathfrak{p}$ and taking quotient by $\mathfrak{p}$ "commute", i.e. the following diagram commutes.

$$\begin{tikzcd}
& A_{\mathfrak{p}} \arrow[dr] & \\
A \arrow[ur] \arrow[dr] & & A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} = FF(A/\mathfrak{p}) \\
& A/\mathfrak{p} \arrow[ur, "FF(\cdot)"'] &
\end{tikzcd}$$

So the value of a function at a point always takes values in a field. In our example, to see the value of our germ at $x = 0$, we simply set $x = 0$. So we get the value $y^2/(y^2 + 1)$, which is certainly in $FF(k[y])$. (If you think you care only about complex schemes, and hence only about algebraically closed fields, let this be a first warning: $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ won't be algebraically closed in general, even if A is a finitely generated $\mathbb{C}$-algebra!)

We say that the germ *vanishes* at $\mathfrak{p}$ if the value is zero. In our example, the germ doesn't vanish at $\mathfrak{p}$.

If anything makes you nervous, you should make up an example to assuage your nervousness. (Example: 27/4 is a regular function on $\operatorname{Spec} \mathbb{Z} - \{[(2)], [(7)]\}$. What is its value at $[(5)]$? Answer: $2/(-1) \equiv -2 \pmod 5$. What is its value at the generic point $[(0)]$? Answer: 27/4. Where does it vanish? At $[(3)]$.)

We now give three extended examples. Our short term goal is to see that we can really work with this sheaf, and can compute the ring of sections of interesting open sets that aren't just distinguished open sets of affine schemes. Our long-term goal is to see interesting examples that will come up repeatedly in the future. All three examples are non-affine schemes, so these examples are genuinely new to us.

4.5. Example: The plane minus the origin. I now want to work through an example with you, to show that this distinguished base is indeed something that you can work with. Let $A = k[x, y]$, so $\operatorname{Spec} A = \mathbb{A}_k^2$. If you want, you can let k be $\mathbb{C}$, but that won't be relevant. Let's work out the space of functions on the open set $U = \mathbb{A}^2 - (0, 0)$.

It is a non-obvious fact that you can't cut out this set with a single equation, so this isn't a distinguished open set. We'll see why fairly soon [where?]. But in any case, even if we're not sure that this is a distinguished open set, we can describe it as the union of two things which *are* distinguished open sets. If I throw out the x axis, i.e. the set $y = 0$, I get the distinguished open set $D(y)$. If I throw out the y axis, i.e. $x = 0$, I get the distinguished open set $D(x)$. So $U = D(x) \cup D(y)$. (Remark: $U = \mathbb{A}^2 - V(x, y)$ and $U = D(x) \cup D(y)$. Coincidence? I think not!) We will find the functions on U by gluing together functions on $D(x)$ and $D(y)$.

What are the functions on $D(x)$? They are, by definition, $A_x = k[x, y, 1/x]$. In other words, they are things like this: $3x^2 + xy + 3y/x + 14/x^4$. What are the functions on $D(y)$? They are, by definition, $A_y = k[x, y, 1/y]$. Note that $A \hookrightarrow A_x, A_y$. This is because x and y are not zero-divisors. (A is an integral domain — it has no zero-divisors, besides 0 — so localization is always an inclusion.) So we are looking for functions on $D(x)$ and $D(y)$ that agree on $D(x) \cap D(y) = D(xy)$, i.e. they are just the same Laurent polynomial. Which things of this first form are also of the second form? Just old-fashioned polynomials —

$$\Gamma(U, \mathcal{O}_{\mathbb{A}^2}) \equiv k[x, y]. \tag{3}$$

In other words, we get no extra functions by throwing out this point. Notice how easy that was to calculate!

4.6. (Aside: Notice that any function on $\mathbb{A}^2 - (0, 0)$ extends over all of $\mathbb{A}^2$. This is an analog of *Hartogs' Lemma* in complex geometry: you can extend a holomorphic function defined on the complement of a set of codimension at least two on a complex manifold over the missing set. This will work more generally in the algebraic setting: you can extend over points in codimension at least 2 not only if they are smooth, but also if they are mildly singular — what we will call *normal*. We will make this precise later. This fact will be very useful for us.)

We can now verify an interesting fact: $(U, \mathcal{O}_{\mathbb{A}^2}|_U)$ is a scheme, but it is not an affine scheme. Here's why: otherwise, if $(U, \mathcal{O}_{\mathbb{A}^2}|_U) = (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$, then we can recover A by taking global sections:

$$A = \Gamma(U, \mathcal{O}_{\mathbb{A}^2}|_U),$$

which we have already identified in (3) as $k[x, y]$. So if U is affine, then $U = \mathbb{A}_k^2$. But we get more: we can recover the points of $\operatorname{Spec} A$ by taking the primes of A. In particular, the prime ideal (x, y) of A should cut out a point of $\operatorname{Spec} A$. But on U, $V(x) \cap V(y) = \varnothing$. Conclusion: U is *not* an affine scheme. (If you are ever looking for a counterexample to something, and you are expecting one involving a non-affine scheme, keep this example in mind!)

You've seen two examples of non-affine schemes: an infinite disjoint union of non-empty schemes (Exercise 4.M), and now $\mathbb{A}^2 - (0,0)$. I want to give you two more important examples. They are important because they are the first examples of fundamental behavior, the first pathological, and the second central.

First, I need to tell you how to glue two schemes together. And before that, you should review how to glue topological spaces together along isomorphic open sets. Given two topological spaces X and Y, and open subsets $U \subset X$ and $V \subset Y$ along with a homeomorphism $U \cong V$, we can create a new topological space W, that we think of as gluing X and Y together along $U \cong V$. It is the quotient of the disjoint union $X \coprod Y$ by the equivalence relation $U \cong V$, where the quotient is given the quotient topology. Then X and Y are naturally (identified with) open subsets of W, and indeed cover W. Can you restate this with an arbitrary number of topological spaces glued together?

Now that we have discussed gluing topological spaces, let's glue schemes together. Suppose you have two schemes $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$, and open subsets $U \subset X$ and $V \subset Y$, along with a homeomorphism $f : U \xrightarrow{\sim} V$, and an isomorphism of structure sheaves $\mathcal{O}_X \cong f^*\mathcal{O}_Y$ (i.e. an isomorphism *of schemes* $(U, \mathcal{O}_X|_U) \cong (V, \mathcal{O}_Y|_V)$). Then we can glue these together to get a single scheme. Reason: let W be X and Y glued together using the isomorphism $U \cong V$. Then an earlier exercise on gluing sheaves shows that the structure sheaves can be glued together to get a sheaf of rings. Note that this is indeed a scheme: any point has a neighborhood that is an affine scheme. (Do you see why?)

4.H. EXERCISE. For later reference, show that you can glue together an arbitrary number of schemes together. Suppose we are given:

- schemes X_i (as i runs over some index set I, not necessarily finite),
- open subschemes $X_{ij} \subset X_i$,
- isomorphisms $f_{ij} : X_{ij} \to X_{ji}$
- such that the isomorphisms "agree along triple intersections", i.e. $f_{ik}|_{X_{ij} \cap X_{ik}} = f_{jk}|_{U_{ji} \cap U_{jk}} \circ f_{ij}|_{X_{ij} \cap X_{ik}}$.

Show that there is a unique scheme X (up to unique isomorphism) along with open subset isomorphic to X_i respecting this gluing data in the obvious sense.

I'll now give you two non-affine schemes. In both cases, I will glue together two copies of the affine line $\mathbb{A}^1_k$. Again, if it makes you feel better, let $k = \mathbb{C}$, but it really doesn't matter.

Let $X = \operatorname{Spec} k[t]$, and $Y = \operatorname{Spec} k[u]$. Let $U = D(t) = \operatorname{Spec} k[t, 1/t] \subset X$ and $V = D(u) = \operatorname{Spec} k[u, 1/u] \subset Y$. We will get both examples by gluing X and Y together along U and V. The difference will be in how we glue.

4.7. Extended example: the affine line with the doubled origin. Consider the isomorphism $U \cong V$ via the isomorphism $k[t, 1/t] \cong k[u, 1/u]$ given by $t \leftrightarrow u$. Let the resulting scheme be X. This is called the *affine line with doubled origin*. Figure 3 is a picture of it.

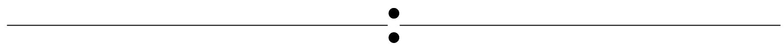

FIGURE 3. The affine line with doubled origin

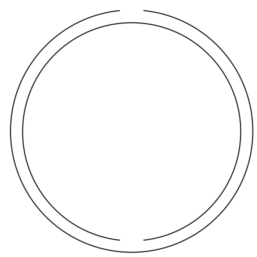

FIGURE 4. Gluing two affine lines together to get $\mathbb{P}^1$

As the picture suggests, intuitively this is an analogue of a failure of Hausdorffness. $\mathbb{A}^1$ itself is not Hausdorff, so we can't say that it is a failure of Hausdorffness. We see this as weird and bad, so we'll want to make up some definition that will prevent this from happening. This will be the notion of *separatedness*. This will answer other of our prayers as well. For example, on a separated scheme, the "affine base of the Zariski topology" is nice — the intersection of two affine open sets will be affine.

4.I. EXERCISE. Show that X is not affine. Hint: calculate the ring of global sections, and look back at the argument for $\mathbb{A}^2 - (0,0)$.

4.J. EXERCISE. Do the same construction with $\mathbb{A}^1$ replaced by $\mathbb{A}^2$. You'll have defined the *affine plane with doubled origin*. Use this example to show that the affine base of the Zariski topology isn't a nice base, by describing two affine open sets whose intersection is not affine.

4.8. Example 2: the projective line. Consider the isomorphism $U \cong V$ via the isomorphism $k[t, 1/t] \cong k[u, 1/u]$ given by $t \leftrightarrow 1/u$. Figure 4 is a suggestive picture of this gluing. Call the resulting scheme the **projective line over the field** k, $\mathbb{P}^1_k$.

Notice how the points glue. Let me assume that k is algebraically closed for convenience. (You can think about how this changes otherwise.) On the first affine line, we have the closed (= "old-fashioned") points $[(t-a)]$, which we think of as "a on the t-line", and we have the generic point. On the second affine line, we have closed points that are "b on the u-line", and the generic point. Then a on the t-line is glued to $1/a$ on the u-line (if $a \neq 0$ of course), and the generic point is glued to the generic point (the ideal (0) of $k[t]$ becomes the ideal (0) of $k[t, 1/t]$ upon localization, and the ideal (0) of

$k[u]$ becomes the ideal (0) of $k[u, 1/u]$. And (0) in $k[t, 1/t]$ is (0) in $k[u, 1/u]$ under the isomorphism $t \leftrightarrow 1/u$).

We can interpret the closed ("old-fashioned") points of $\mathbb{P}^1$ in the following way, which may make this sound closer to the way you have seen projective space defined earlier. The points are of the form $[a; b]$, where a and b are not both zero, and $[a; b]$ is identified with $[ac; bc]$ where $c \in k^*$. Then if $b \neq 0$, this is identified with a/b on the t-line, and if $a \neq 0$, this is identified with b/a on the u-line.

4.9. *Proposition. —* $\mathbb{P}^1_k$ *is not affine.*

Proof. We do this by calculating the ring of global sections.

The global sections correspond to sections over X and sections over Y that agree on the overlap. A section on X is a polynomial $f(t)$. A section on Y is a polynomial $g(u)$. If I restrict $f(t)$ to the overlap, I get something I can still call $f(t)$; and ditto for $g(u)$. Now we want them to be equal: $f(t) = g(1/t)$. How many polynomials in t are at the same time polynomials in $1/t$? Not very many! Answer: only the constants k. Thus $\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = k$. If $\mathbb{P}^1$ were affine, then it would be $\operatorname{Spec} \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = \operatorname{Spec} k$, i.e. one point. But it isn't — it has lots of points. □

Note that we have proved an analog of a theorem: the only holomorphic functions on $\mathbb{CP}^1$ are the constants!

4.K. IMPORTANT EXERCISE. Figure out how to define projective n-space $\mathbb{P}^n_k$. Glue together $n + 1$ opens each isomorphic to $\mathbb{A}^n_k$. Show that the only global sections of the structure sheaf are the constants, and hence that $\mathbb{P}^n_k$ is not affine if $n > 0$. (Hint: you might fear that you will need some delicate interplay among all of your affine opens, but you will only need two of your opens to see this. There is even some geometric intuition behind this: the complement of the union of two opens has codimension 2. But "Hartogs' Theorem" (to be stated rigorously later) says that any function defined on this union extends to be a function on all of projective space. Because we're expecting to see only constants as functions on all of projective space, we should already see this for this union of our two affine open sets.)

4.L. EXERCISE. Show that if k is algebraically closed, the closed points of $\mathbb{P}^n_k$ may be interpreted in the same way as we interpreted the points of $\mathbb{P}^1_k$. (The points are of the form $[a_0; \ldots; a_n]$, where the a_i are not all zero, and $[a_0; \ldots; a_n]$ is identified with $[ca_0; \ldots; ca_n]$ where $c \in k^*$.)

We will later give another definition of projective space . Your definition (from Exercise 4.K) will be handy for computing things. But there is something unnatural about it — projective space is highly symmetric, and that isn't clear from your point of view.

Note that your definition will give a definition of $\mathbb{P}^n_A$ for any *ring* A. This will be useful later.

4.10. Fun aside: The Chinese Remainder Theorem is a *geometric* fact. I want to show you that the Chinese Remainder theorem is embedded in what we've done, which shouldn't be obvious to you. I'll show this by example. The Chinese Remainder Theorem says that knowing an integer modulo 60 is the same as knowing an integer modulo 3, 4, and 5. Here's how to see this in the language of schemes. What is $\operatorname{Spec} \mathbb{Z}/(60)$? What are the primes of this ring? Answer: those prime ideals containing (60), i.e. those primes dividing 60, i.e. (2), (3), and (5). So here is my picture of the scheme [picture of 3 dots]. They are all closed points, as these are all maximal ideals, so the topology is the discrete topology. What are the stalks? You can check that they are $\mathbb{Z}/4$, $\mathbb{Z}/3$, and $\mathbb{Z}/5$. My picture is actually like this [draw a bit of one-dimensional fuzz on (2)]: the scheme has nilpotents here ($2^2 \equiv 0 \pmod 4$). I indicate nilpotents with "fuzz". So what are global sections on this scheme? They are sections on this open set (2), this other open set (3), and this third open set (5). In other words, we have a natural isomorphism of rings

$$\mathbb{Z}/60 \to \mathbb{Z}/4 \times \mathbb{Z}/3 \times \mathbb{Z}/5.$$

On a related note:

4.M. EXERCISE. (a) Show that the disjoint union of a *finite* number of affine schemes is also an affine scheme. (Hint: say what the ring is.)
(b) Show that an infinite disjoint union of (non-empty) affine schemes is not an affine scheme.

4.11. $\star$ *Example.* Here is an example of a function on an open subset of a scheme that is a bit surprising. On $X = \operatorname{Spec} k[w, x, y, z]/(wx - yz)$, consider the open subset $D(y) \cup D(w)$. Show that the function x/y on $D(y)$ agrees with z/w on $D(w)$ on their overlap $D(y) \cap D(w)$. Hence they glue together to give a section. You may have seen this before when thinking about analytic continuation in complex geometry — we have a "holomorphic" function which has the description x/y on an open set, and this description breaks down elsewhere, but you can still "analytically continue" it by giving the function a different definition on different parts of the space.

Follow-up for curious experts: This function has no "single description" as a well-defined expression in terms of w, x, y, z! There is lots of interesting geometry here. This example will be a constant source of interesting examples for us. We will later recognize it as the cone over the quadric surface. Here is a glimpse, in terms of words we have not yet defined. $\operatorname{Spec} k[w, x, y, z]$ is $\mathbb{A}^4$, and is, not surprisingly, 4-dimensional. We are looking at the set X, which is a hypersurface, and is 3-dimensional. It is a cone over a smooth quadric surface in $\mathbb{P}^3$. $D(y)$ is X minus some hypersurface, so we are throwing away a codimension 1 locus. $D(z)$ involves throwing another codimension 1 locus. You might think that their intersection is then codimension 2, and that maybe failure of extending this weird function to a global polynomial comes because of a failure of our Hartogs'-type theorem, which will be a failure of normality. But that's not true — $V(y) \cap V(z)$ is in fact codimension 1 — so no Hartogs-type theorem holds. Here is what is actually going on. $V(y)$ involves throwing away the (cone over the) union of two lines l and m_1, one in each "ruling" of the surface, and $V(z)$ also involves throwing away the (cone over the) union of two lines l and m_2. The intersection is the (cone over the) line l, which is a codimension 1 set. Neat fact: despite being "pure codimension 1", it is not cut out even set-theoretically by a single equation. (It is hard to get an example of this behavior. This example is the simplest example I know.) This means that any expression $f(w,x,y,z)/g(w,x,y,z)$ for our function cannot correctly describe our function on $D(y) \cup D(z)$ — at some point of $D(y) \cup D(z)$ it must be $0/0$. Here's why. Our function can't be defined on $V(y) \cap V(z)$, so g must vanish here. But g can't vanish just on the cone over l — it must vanish elsewhere too. (For the experts among the experts: here is why the cone over l is not cut out set-theoretically by a single equation. If $l = V(f)$, then $D(f)$ is affine. Let l' be another line in the same ruling as l, and let $C(l)$ (resp. l') be the cone over l (resp. l'). Then $C(l')$ can be given the structure of a closed subscheme of $\operatorname{Spec} k[w,x,y,z]$, and can be given the structure of $\mathbb{A}^2$. Then $C(l') \cap V(f)$ is a closed subscheme of $D(f)$. Any closed subscheme of an affine scheme is affine. But $l \cap l' = \varnothing$, so the cone over l intersects the cone over l' in a point, so $C(l') \cap V(f)$ is $\mathbb{A}^2$ minus a point, which we've seen is not affine, so we have a contradiction.)

We concluded with some initial discussion of properties of schemes, including irreducible, closed point, specialization, generization, generic point, connected component, and irreducible component.

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