FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 4

RAVI VAKIL

CONTENTS

Last day: abelian categories: kernels, cokernels, and all that jazz. Definition of (pre)sheaves.

A quick comment on last day's material:

When you see a left-exact functor, you should always dream that you are seeing the end of a long exact sequence. If

$$0 \to M' \to M \to M'' \to 0$$

is an exact sequence in abelian category $\mathcal{A}$, and $F : \mathcal{A} \to \mathcal{B}$ is a left-exact functor, then

$$0 \to FM' \to FM \to FM''$$

is exact, and you should always dream that it should continue in some natural way. For example, the next term should depend only on M', call it R^1FM', and if it is zero, then $FM \to FM''$ is an epimorphism. This remark holds true for left-exact and contravariant functors too. In good cases, such a continuation exists, and is incredibly useful. We'll see this when we come to cohomology.

1. MORPHISMS OF PRESHEAVES AND SHEAVES

Whenever one defines a new mathematical *object*, category theory has taught us to try to understand maps between them. We now define morphisms of presheaves, and similarly for sheaves. In other words, we will descibe the *category of presheaves* (of abelian groups, etc.) and the *category of sheaves*.

A morphism of presheaves of sets (or indeed with values in any category) $f : \mathcal{F} \to \mathcal{G}$ is the data of maps $f(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ for all U behaving well with respect to restriction:

Date: Wednesday, October 3, 2007. Updated October 26.

if $U \hookrightarrow V$ then

$$\begin{array}{ccc} \mathcal{F}(V) & \xrightarrow{f(V)} & \mathcal{G}(V) \\ \downarrow \mathrm{res}_{V,U} & & \downarrow \mathrm{res}_{V,U} \\ \mathcal{F}(U) & \xrightarrow{f(U)} & \mathcal{G}(U) \end{array}$$

commutes. (Notice: the underlying space remains X.)

A morphism of sheaves is defined in the same way: the morphisms from a sheaf $\mathcal{F}$ to a sheaf $\mathcal{G}$ are precisely the morphisms from $\mathcal{F}$ to $\mathcal{G}$ as presheaves. (Translation: The category of sheaves on X is a full subcategory of the category of presheaves on X.)

An example of a morphism of sheaves is the map from the sheaf of differentiable functions on $\mathbb{R}$ to the sheaf of continuous functions. This is a "forgetful map": we are forgetting that these functions are differentiable, and remembering only that they are continuous.

1.1. *Side-remarks for category-lovers.* If you interpret a presheaf on X as a contravariant functor (from the category of open sets), a morphism of presheaves on X is a natural transformation of functors. We haven't defined natural transformation of functors, but you might be able to guess the definition from this remark.

1.A. EXERCISE. Suppose $f : X \to Y$ is a continuous map of topological spaces (i.e. a morphism in the category of topological spaces). Show that pushforward gives a functor from $\{$ sheaves of sets on $X \}$ to $\{$ sheaves of sets on $Y \}$. Here "sets" can be replaced by any category. (Watch out for some possible confusion: a presheaf is a functor, and presheaves form a category. It may be best to forget that presheaves form a functor for the time being.)

1.B. IMPORTANT EXERCISE AND DEFINITION: "SHEAF Hom". Suppose $\mathcal{F}$ and $\mathcal{G}$ are two sheaves of abelian groups on X. (In fact, it will suffice that $\mathcal{F}$ is a presheaf.) Let $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})$ be the collection of data

$$\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})(U) := \mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U).$$

(Recall the notation $\mathcal{F}|_U$, the restriction of the sheaf to the open set U, see last day's notes.) Show that this is a sheaf. This is called the "sheaf $\underline{\mathrm{Hom}}$". Show that if $\mathcal{G}$ is a sheaf of abelian groups, then $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})$ is a sheaf of abelian groups. (The same construction will obviously work for sheaves with values in any category.)

1.2. Presheaves of abelian groups or $\mathcal{O}_X$-modules form an abelian category.

We can make module-like constructions using presheaves of abelian groups on a topological space X. (In this section, all (pre)sheaves are of abelian groups.) For example, we can clearly add maps of presheaves and get another map of presheaves: if $f, g : \mathcal{F} \to \mathcal{G}$, then we define the map $f + g$ by $(f + g)(V) = f(V) + g(V)$. (There is something small to check here: that the result is indeed a map of presheaves.) In this way, presheaves of abelian groups form an additive category (recall: the morphisms between any two presheaves of abelian groups form an abelian group; there is a 0-morphism; and one can take finite products.) For exactly the same reasons, sheaves of abelian groups also form an additive category.

If $f : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves, define the presheaf kernel $\ker_{\text{pre}} f$ by $(\ker_{\text{pre}} f)(U) = \ker f(U)$.

1.C. EXERCISE. Show that $\ker_{\text{pre}} f$ is a presheaf. (Hint: if $U \hookrightarrow V$, there is a natural map $\text{res}_{V,U} : \mathcal{G}(V)/f(V)(\mathcal{F}(V)) \to \mathcal{G}(U)/f(U)(\mathcal{F}(U))$ by chasing the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \ker_{\text{pre}} f(V) & \longrightarrow & \mathcal{F}(V) & \longrightarrow & \mathcal{G}(V) \\ & & \Big\downarrow \exists! & & \Big\downarrow \text{res}_{V,U} & & \Big\downarrow \text{res}_{V,U} \\ 0 & \longrightarrow & \ker_{\text{pre}} f(U) & \longrightarrow & \mathcal{F}(U) & \longrightarrow & \mathcal{G}(U) \end{array}$$

You should check that the restriction maps compose as desired.)

Define the presheaf cokernel $\text{coker}_{\text{pre}} f$ similarly. It is a presheaf by essentially the same argument.

1.D. EXERCISE: THE COKERNEL DESERVES ITS NAME. Show that the presheaf cokernel satisfies the universal property of cokernels in the category of presheaves.

Similarly, $\ker_{\text{pre}} f \to \mathcal{F}$ satisfies the unversal property for kernels in the category of presheaves.

It is not too tedious to verify that presheaves of abelian groups form an abelian category, and the reader is free to do so. (The key idea is that all abelian-categorical notions may be defined and verified open set by open set.) Hence we can define terms such as *subpresheaf*, *image presheaf*, *quotient presheaf*, *cokernel presheaf*, and they behave the way one expect. You construct kernels, quotients, cokernels, and images open set by open set. Homological algebra (exact sequences etc.) works, and also "works open set by open set". In particular:

1.E. EXERCISE. If $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \cdots \to \mathcal{F}_n \to 0$ is an exact sequence of presheaves of abelian groups, then $0 \to \mathcal{F}_1(U) \to \mathcal{F}_2(U) \to \cdots \to \mathcal{F}_n(U) \to 0$ is also an exact sequence for all U, and vice versa.

The above discussion carries over without any change to presheaves with values in any abelian category.

However, we are interested in more geometric objects, sheaves, where things are can be understood in terms of their local behavior, thanks to the identity and gluing axioms. We will soon see that sheaves of abelian groups also form an abelian category, but a complication will arise that will force the notion of *sheafification* on us. Sheafification will be the answer to many of our prayers. We just don't realize it yet.

Kernels work just as with presheaves:

1.F. IMPORTANT EXERCISE. Suppose $f : \mathcal{F} \to \mathcal{G}$ is a morphism of *sheaves*. Show that the presheaf kernel $\ker_{\text{pre}} f$ is in fact a sheaf. Show that it satisfies the universal property of kernels. (Hint: the second question follows immediately from the fact that $\ker_{\text{pre}} f$ satisfies the universal property in the category of *presheaves*.)

Thus if f is a morphism of sheaves, we define

$$\ker f := \ker_{\text{pre}} f.$$

The problem arises with the cokernel.

1.G. IMPORTANT EXERCISE. Let X be $\mathbb{C}$ with the classical topology, let $\underline{\mathbb{Z}}$ be the locally constant sheaf on X with group $\mathbb{Z}$, $\mathcal{O}_X$ the sheaf of holomorphic functions, and $\mathcal{F}$ the *presheaf* of functions admitting a holomorphic logarithm. (Why is $\mathcal{F}$ not a sheaf?) Consider

$$0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow \mathcal{O}_X \xrightarrow{f \mapsto \exp 2\pi i f} \mathcal{F} \longrightarrow 0$$

where $\underline{\mathbb{Z}} \to \mathcal{O}_X$ is the natural inclusion. Show that this is an exact sequence *of presheaves*. Show that $\mathcal{F}$ is *not* a sheaf. (Hint: $\mathcal{F}$ does not satisfy the gluability axiom. The problem is that there are functions that don't have a logarithm that locally have a logarithm.) This will come up again in Example 2.8.

We will have to put our hopes for understanding cokernels of sheaves on hold for a while. We will first take a look at how to understand sheaves using stalks.

2. PROPERTIES DETERMINED AT THE LEVEL OF STALKS

In this section, we'll see that lots of facts about sheaves can be checked "at the level of stalks". This isn't true for presheaves, and reflects the local nature of sheaves. We will flag each case of a property determined by stalks.

2.A. IMPORTANT EXERCISE **(sections are determined by stalks)**. Prove that a section of a sheaf is determined by its germs, i.e. the natural map

$$\mathcal{F}(U) \to \prod_{x \in U} \mathcal{F}_x \tag{1}$$

is injective. (Hint # 1: you won't use the gluability axiom, so this is true for separated presheaves. Hint # 2: it is false for presheaves in general, see Exercise 2.F, so you *will* use the identity axiom.)

This exercise suggests an important question: which elements of the right side of (1) are in the image of the left side?

2.1. Important definition. We say that an element $\prod_{x \in U} s_x$ of the right side $\prod_{x \in U} \mathcal{F}_x$ of (1) consists of *compatible germs* if for all $x \in U$, there is some representative $(U_x, s'_x \in \Gamma(U_x, \mathcal{F}))$ for s_x (where $x \in U_x \subset U$) such that the germ of s'_x at all $y \in U_x$ is s_y. You'll have to think about this a little. Clearly any section s of $\mathcal{F}$ over U gives a choice of compatible germs for U — take $(U_x, s'_x) = (U, s)$.

2.B. IMPORTANT EXERCISE. Prove that any choice of compatible germs for $\mathcal{F}$ over U is the image of a section of $\mathcal{F}$ over U. (Hint: you will use gluability.)

We have thus completely described the image of (1), in a way that we will find useful.

2.2. *Remark.* This perspective is part of the motivation for the agricultural terminology "sheaf": it is the data of a bunch of stalks, bundled together appropriately.

Now we throw morphisms into the mix.

2.C. EXERCISE. Show a morphism of (pre)sheaves (of sets, or rings, or abelian groups, or $\mathcal{O}_X$-modules) induces a morphism of stalks. More precisely, if $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of (pre)sheaves on X, and $x \in X$, describe a natural map $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$.

2.D. EXERCISE **(morphisms are determined by stalks)**. Show that morphisms of sheaves are determined by morphisms of stalks. Hint: consider the following diagram.

$$\begin{array}{ccc} \mathcal{F}(U) & \longrightarrow & \mathcal{G}(U) \\ \downarrow & & \downarrow \\ \prod_{x \in U} \mathcal{F}_x & \longrightarrow & \prod_{x \in U} \mathcal{G}_x \end{array} \tag{2}$$

2.E. TRICKY EXERCISE **(isomorphisms are determined by stalks)**. Show that a morphism of sheaves is an isomorphism if and only if it induces an isomorphism of all stalks. (Hint: Use (2). Injectivity uses the previous exercise 2.D. Surjectivity will use gluability, and is more subtle.)

2.F. EXERCISE. (a) Show that Exercise 2.A is false for general presheaves.
(b) Show that Exercise 2.D is false for general presheaves.
(c) Show that Exercise 2.E is false for general presheaves.
(General hint for finding counterexamples of this sort: consider a 2-point space with the discrete topology, i.e. every subset is open.)

2.3. Sheafification.

Every sheaf is a presheaf (and indeed by definition sheaves on X form a full subcategory of the category of presheaves on X). Just as groupification gives a group that best approximates a semigroup, sheafification gives the sheaf that best approximates a presheaf, with an analogous universal property.

2.4. *Definition.* If $\mathcal{F}$ is a presheaf on X, then a morphism of presheaves $sh : \mathcal{F} \to \mathcal{F}^{sh}$ on X is a *sheafification of* $\mathcal{F}$ if $\mathcal{F}^{sh}$ is a sheaf, and for any other sheaf $\mathcal{G}$, and any presheaf morphism $g : \mathcal{F} \to \mathcal{G}$, there *exists* a *unique* morphism of sheaves $f : \mathcal{F}^{sh} \to \mathcal{G}$ making the diagram

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{sh} & \mathcal{F}^{sh} \\ & {\scriptstyle g}\searrow & \downarrow{\scriptstyle f} \\ & & \mathcal{G} \end{array}$$

commute.

2.G. EXERCISE. Show that sheafification is unique up to unique isomorphism. Show that if $\mathcal{F}$ is a sheaf, then the sheafification is $\mathcal{F} \xrightarrow{id} \mathcal{F}$. (This should be second nature by now.)

2.5. Construction. We next show that any presheaf has a sheafification. Suppose $\mathcal{F}$ is a *presheaf*. Define $\mathcal{F}^{sh}$ by defining $\mathcal{F}^{sh}(U)$ as the set of compatible germs of the presheaf $\mathcal{F}$ over U. Explicitly:

$$\mathcal{F}^{sh}(U) := \{(f_x \in \mathcal{F}_x)_{x \in U} : \forall x \in U, \exists x \in V \subset U, s \in \mathcal{F}(V) : s_y = f_y \forall y \in V\}.$$

(Those who want to worry about the empty set are welcome to.)

2.H. EASY EXERCISE. Show that $\mathcal{F}^{sh}$ (using the tautological restriction maps) forms a sheaf.

2.I. EASY EXERCISE. Describe a natural map $sh : \mathcal{F} \to \mathcal{F}^{sh}$.

2.J. EXERCISE. Show that the map sh satisfies the universal property 2.4 of sheafification.

2.K. EXERCISE. Use the universal property to show that for any morphism of presheaves $\phi : \mathcal{F} \to \mathcal{G}$, we get a natural induced morphism of sheaves $\phi^{sh} : \mathcal{F}^{sh} \to \mathcal{G}^{sh}$. Show that sheafification is a functor from presheaves to sheaves.

2.L. USEFUL EXERCISE FOR CATEGORY-LOVERS. Show that the sheafification functor is left-adjoint to the forgetful functor from sheaves on X to presheaves on X.

2.M. EXERCISE. Show $\mathcal{F} \to \mathcal{F}^{sh}$ induces an isomorphism of stalks. (Possible hint: Use the concrete description of the stalks. Another possibility: judicious use of adjoints.)

2.6. *Unimportant remark.* Sheafification can be defined in a topological way, via the "espace étalé" construction, see Hartshorne II.1.13, and likely Serre's totemic *FAC*. This is essentially the same construction as the one given here. Another construction is described in Eisenbud-Harris.

2.7. Subsheaves and quotient sheaves.

2.N. EXERCISE. Suppose $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves (of sets) on at topological space X. Show that the following are equivalent.

(a) ϕ is a monomorphism in the category of sheaves.
(b) ϕ is injective on the level of stalks: $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ injective for all $x \in X$.
(c) ϕ is injective on the level of open sets: $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for all open $U \subset X$.

(Possible hints: for (b) implies (a), recall that morphisms are determined by stalks, Exercise 2.D. For (a) implies (b), judiciously choose a skyscraper sheaf. For (a) implies (c), judiciously the "indicator sheaf" with one section over every open set contained in U, and no section over any other open set.)

If these conditions hold, we say that $\mathcal{F}$ is a *subsheaf* of $\mathcal{G}$ (where the "inclusion" ϕ is sometimes left implicit).

2.O. EXERCISE. Continuing the notation of the previous exercise, show that the following are equivalent.

(a) ϕ is a epimorphism in the category of sheaves.
(b) ϕ is surjective on the level of stalks: $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ surjective for all $x \in X$.

If these conditions hold, we say that $\mathcal{G}$ is a *quotient sheaf* of $\mathcal{F}$.

Thus **monomorphisms and epimorphisms — subsheafiness and quotient sheafiness — can be checked at the level of stalks.**

Both exercises generalize immediately to sheaves with values in any category, where "injective" is replaced by "monomorphism" and "surjective" is replaced by "epimorphism".

Notice that there was no part (c) to the previous exercise, and here is an example showing why.

2.8. *Example.* Let $X = \mathbb{C}$ with the usual (analytic) topology, and define $\mathcal{O}_X$ to be the sheaf of holomorphic functions, and $\mathcal{O}_X^*$ to be the sheaf of invertible (nowhere zero) holomorphic functions. This is a sheaf of abelian groups under multiplication. We have maps of sheaves

$$0 \longrightarrow \underline{\mathbb{Z}} \xrightarrow{\times 2\pi i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \longrightarrow 1 \tag{3}$$

where $\underline{\mathbb{Z}}$ is the locally constant sheaf associated to $\mathbb{Z}$. (You can figure out what the sheaves 0 and 1 mean; they are isomorphic, and are written in this way for reasons that may be clear). We will soon interpret this as an exact sequence of sheaves of abelian groups (the *exponential exact sequence*), although we don't yet have the language to do so.

2.P. EXERCISE. Show that $\mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^*$ describes $\mathcal{O}_X^*$ as a quotient sheaf of $\mathcal{O}_X$. Show that it is not surjective on all open sets.

This is a great example to get a sense of what "surjectivity" means for sheaves. Nonzero holomorphic functions locally have logarithms, but they need not globally.

3. SHEAVES OF ABELIAN GROUPS, AND $\mathcal{O}_X$-MODULES, FORM ABELIAN CATEGORIES

We are now ready to see that sheaves of abelian groups, and their cousins, $\mathcal{O}_X$-modules, form abelian categories. In other words, we may treat them in the same way we treat vector spaces, and modules over a ring. In the process of doing this, we will see that this is much stronger than an analogy; kernels, cokernels, exactness, etc. can be understood at the level of germs (which are just abelian groups), and the compatibility of the germs will come for free.

The category of sheaves of abelian groups is clearly an additive category. In order to show that it is an abelian category, we must show that any morphism $\phi : \mathcal{F} \to \mathcal{G}$ has a kernel and a cokernel. We have already seen that ϕ has a kernel (Exercise 1.F): the presheaf kernel is a sheaf, and is a kernel.

3.A. EXERCISE. Show that the stalk of the kernel is the kernel of the stalks: there is a natural isomorphism

$$(\ker(\mathcal{F} \to \mathcal{G}))_x \cong \ker(\mathcal{F}_x \to \mathcal{G}_x).$$

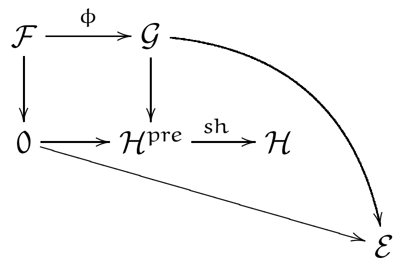
So we next address the issue of the cokernel. Now $\phi : \mathcal{F} \to \mathcal{G}$ has a cokernel in the category of presheaves; call it $\mathcal{H}^{\text{pre}}$ (where the superscript is meant to remind us that this is a presheaf). Let $\mathcal{H}^{\text{pre}} \xrightarrow{\text{sh}} \mathcal{H}$ be its sheafification. Recall that the cokernel is defined using a universal property: it is the colimit of the diagram

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\phi} & \mathcal{G} \\ \downarrow & & \\ 0 & & \end{array}$$

in the category of presheaves. We claim that $\mathcal{H}$ is the cokernel of ϕ in the category of sheaves, and show this by proving the universal property. Given any sheaf $\mathcal{E}$ and a commutative diagram

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\phi} & \mathcal{G} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{E} \end{array}$$

We construct



We show that there is a unique morphism $\mathcal{H} \to \mathcal{E}$ making the diagram commute. As $\mathcal{H}^{\text{pre}}$ is the cokernel in the category of presheaves, there is a unique morphism of presheaves $\mathcal{H}^{\text{pre}} \to \mathcal{E}$ making the diagram commute. But then by the universal property of sheafification (Defn. 2.4), there is a unique morphism of *sheaves* $\mathcal{H} \to \mathcal{E}$ making the diagram commute.

3.B. EXERCISE. Show that the stalk of the cokernel is naturally isomorphic to the cokernel of the stalk.

We have now defined the notions of kernel and cokernel, and verified that they may be checked at the level of stalks. We have also verified that the qualities of a morphism being monic or epi are also determined at the level of stalks (Exercises 2.N and 2.O). Hence sheaves of abelian groups on X form an abelian category.

We see more: all structures coming from the abelian nature of this category may be checked at the level of stalks. For example, **exactness of a sequence of sheaves may be checked at the level of stalks**. A fancy-sounding consequence: taking stalks is an exact functor from sheaves of abelian groups on X to abelian groups.

3.C. EXERCISE (LEFT-EXACTNESS OF THE GLOBAL SECTION FUNCTOR). Suppose $U \subset X$ is an open set, and $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H}$ is an exact sequence of sheaves of abelian groups. Show that

$$0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U)$$

is exact. Give an example to show that the global section functor is not exact. (Hint: the exponential exact sequence (3).)

3.D. EXERCISE: LEFT-EXACTNESS OF PUSHFORWARD. Suppose $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H}$ is an exact sequence of sheaves of abelian groups on X. If $f : X \to Y$ is a continuous map, show that

$$0 \to f_*\mathcal{F} \to f_*\mathcal{G} \to f_*\mathcal{H}$$

is exact. (The previous exercise, dealing with the left-exactness of the global section functor can be interpreted as a special case of this, in the case where Y is a point.)

3.E. EXERCISE. Suppose $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves of abelian groups. Show that the image sheaf $\operatorname{im} \phi$ is the sheafification of the image presheaf. (You must use the definition of image in an abelian category. In fact, this gives the accepted definition of image sheaf for a morphism of sheaves of sets.)

3.F. EXERCISE. Show that if $(X, \mathcal{O}_X)$ is a ringed space, then $\mathcal{O}_X$-modules form an abelian category. (There isn't much more to check!)

We end with a useful construction using some of the ideas in this section.

3.G. IMPORTANT EXERCISE: TENSOR PRODUCTS OF $\mathcal{O}_X$-MODULES. (a) Suppose $\mathcal{O}_X$ is a sheaf of rings on X. Define (categorically) what we should mean by tensor product of two $\mathcal{O}_X$-modules. Give an explicit construction, and show that it satisfies your categorical definition. *Hint:* take the "presheaf tensor product" — which needs to be defined — and sheafify. Note: $\otimes_{\mathcal{O}_X}$ is often written $\otimes$ when the subscript is clear from the context.
(b) Show that the tensor product of stalks is the stalk of tensor product.

I then said a very little about where we are going. The last two things we'll discuss about sheaves in particular are the *inverse image sheaf* and *sheaves on a base* of a topology.

E-mail address: vakil@math.stanford.edu