FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 1

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CONTENTS

1. WELCOME

Welcome! This is Math 216A, Foundations of Algebraic Geometry, the first of a three-quarter sequence on the topic. I'd like to tell you a little about what I intend with this course.

Algebraic geometry is a subject that somehow connects and unifies several parts of mathematics, including obviously algebra and geometry, but also number theory, and depending on your point of view many other things, including topology, string theory, etc. As a result, it can be a handy thing to know if you are in a variety of subjects, notably number theory, symplectic geometry, and certain kinds of topology. The power of the field arises from a point of view that was developed in the 1960's in Paris, by the group led by Alexandre Grothendieck. The power comes from rather heavy formal and technical machinery, in which it is easy to lose sight of the intuitive nature of the objects under consideration. This is one reason why it used to strike fear into the hearts of the uninitiated.

The rough edges have been softened over the ensuing decades, but there is an inescapable need to understand the subject on its own terms.

This class is the second version of an experiment. I hope to try several things, which are mutually incompatible. Over the year, I want to cover the foundations of the subject quite completely: the varieties and schemes, the morphisms between them, their properties, cohomology theories, and more. I would like to do this rigorously, while trying hard to keep track of the geometric intuition behind it. I'm going to try to do this without working from a text, so I'll occasionally talk myself into a corner, and then realize I'll have to go backwards and fix something earlier.

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Beginning algebraic geometry traditionally requires a lot of background. I'm going to try to assume as little as possible, ideally just commutative ring theory, and some comfort with things like prime ideals and localization. The more you know, the better, of course. But if I say things that you don't understand, please slow me down in class, and also talk to me after class. Given the amount of material that there is in the foundations of the subject, I'm afraid I'm going to move faster than I would like, which means that for you it will be like drinking from a firehose. If it helps, I'm very happy to do my part to make it easier for you, and I'm happy to talk about things outside of class. I also intend to post notes for as many classes as I can. They will usually appear before the next class, but not always.

In particular, this will not be the type of class where you can sit back and hope to pick up things casually. The only way to avoid losing yourself in a sea of definitions is to become comfortable with the ideas by playing with examples.

To this end, I intend to give problem sets, to be handed in. They aren't intended to be onerous, and if they become so, please tell me. But they *are* intended to force you to become familiar with the ideas we'll be using.

Okay, I think I've said enough to scare most of you away from coming back, so I want to emphasize that I'd like to do everything in my power to make it better, short of covering less material. The best way to get comfortable with the material is to talk to me on a regular basis about it.

Office hours: I haven't decided if it will be useful to have formal office hours rather than being available to talk after class, and also on many days by appointment.

Grader/TA: Jarod Alper, jarod@math.

Texts: Here are some books to have handy. Hartshorne's *Algebraic Geometry* has most of the material that I'll be discussing. It isn't a book that you should sit down and read, but you might find it handy to flip through for certain results. It may be at the bookstore, and is on 2-day reserve at the library. Mumford's *Red Book of Varieties and Schemes* has a good deal of the material I'll be discussing, and with a lot of motivation too. That is also on 2-day reserve in the library. The second edition is strictly worse than the 1st, because someone at Springer retyped it without understanding the math, introducing an irritating number of errors. If you would like something gentler, I would suggest Shafarevich's books on algebraic geometry. Another excellent foundational reference is Eisenbud and Harris' book *The Geometry of Schemes*, and Harris' earlier book *Algebraic Geometry* is a beautiful tour of the subject.

For background, it will be handy to have your favorite commutative algebra book around. Good examples are Eisenbud's *Commutative Algebra with a View to Algebraic Geometry*, or Atiyah and Macdonald's *Commutative Algebra*. If you'd like something with homological algebra, category theory, and abstract nonsense, I'd suggest Weibel's book *Introduction to Homological Algebra*.

Assumptions. All my rings are commutative, and with unit. I currently don't require $0 \neq 1$, so the 0-ring, with one ring, counts as a ring for me. I may regret this later. I believe in the axiom of choice, and in particular that every proper ideal in a ring is contained in a maximal ideal.

2. Why algebraic geometry?

It is hard to define algebraic geometry in its vast generality in a couple of sentences. So I'll talk around it a bit.

As a motivation, consider the study of manifolds. Real manifolds are things that locally look like bits of real n-space, and they are glued together to make interesting shapes. There is already some subtlety here — when you glue things together, you have to specify what kind of gluing is allowed. For example, if the transition functions are required to be differentiable, then you get the notion of a differentiable manifold.

A great example of a manifold is a submanifold of $\mathbb{R}^n$ (consider a picture of a torus). In fact, any compact manifold can be described in such a way. You could even make this your definition, and not worry about gluing. This is a good way to think about manifolds, but not the best way. There is something arbitrary and inessential about defining manifolds in this way. Much cleaner is the notion of an *abstract manifold*, which is the current definition used by the mathematical community.

There is an even more sophisticated way of thinking about manifolds. A differentiable manifold is obviously a topological space, but it is a little bit more. There is a very clever way of summarizing what additional information is there, basically by declaring what functions on this topological space are differentiable. The right notion is that of a sheaf, which is a simple idea, that I'll soon define for you. It is true, but non-obvious, that this ring of functions that we are declaring to be differentiable determines the differentiable manifold structure.

Very roughly, algebraic geometry, at least in its geometric guise, is the kind of geometry you can describe with polynomials. So you are allowed to talk about things like $y^2 = x^3 + x$, but not $y = \sin x$. So some of the fundamental geometric objects under consideration are things in n-space cut out by polynomials. Depending on how you define them, they are called *affine varieties* or *affine schemes*. They are the analogues of the patches on a manifold. Then you can glue these things together, using things that you can describe with polynomials, to obtain more general varieties and schemes. So then we'll have these algebraic objects, that we call varieties or schemes, and we can talk about maps between them, and things like that.

In comparison with manifold theory, we've really restricted ourselves by only letting ourselves use polynomials. But on the other hand, we have gained a huge amount too. First of all, we can now talk about things that aren't smooth (that are *singular*), and we can work with these things. Algebraic geometry provides particularly powerful tools for dealing with singular objects. (One thing we'll have to do is to define what we mean by smooth and singular!) Also, we needn't work over the real or complex numbers, so we can talk about arithmetic questions, such as: what are the rational points on $y^2 = x^3 + x^2$? (Here, we work over the field $\mathbb{Q}$.) More generally, the recipe by which we make geometric objects out of things to do with polynomials can generalize drastically, and we can make a geometric object out of rings. This ends up being surprisingly useful — all sorts of old facts in algebra can be interpreted geometrically, and indeed progress in the field of commutative algebra these days usually requires a strong geometric background.

Let me give you some examples that will show you some surprising links between geometry and number theory. To the ring of integers $\mathbb{Z}$, we will associate a smooth curve $\operatorname{Spec} \mathbb{Z}$. In fact, to the ring of integers in a number field, there is always a smooth curve, and to its orders (subrings), we have singular = non-smooth curves.

An old flavor of Diophantine question is something like this. Given an equation in two variables, $y^2 = x^3 + x^2$, how many rational solutions are there? So we're looking to solve this equation over the field $\mathbb{Q}$. Instead, let's look at the equation over the field $\mathbb{C}$. It turns out that we get a complex surface, perhaps singular, and certainly non-compact. So let me separate all the singular points, and compactify, by adding in points. The resulting thing turns out to be a compact oriented surface, so (assuming it is connected) it has a genus g, which is the number of holes it has. For example, $y^2 = x^3 + x^2$ turns out to have genus 0. Then Mordell conjectured that if the genus is at least 2, then there are at most a finite number of rational solutions. The set of complex solutions somehow tells you about the number of rational solutions! Mordell's conjecture was proved by Faltings, and earned him a Fields Medal in 1986. As an application, consider Fermat's Last Theorem. We're looking for integer solutions to $x^n + y^n = z^n$. If you think about it, we are basically looking for rational solutions to $X^n + Y^n = 1$. Well, it turns out that this has genus $\binom{n-1}{2}$ — we'll verify something close to this at some point in the future. Thus if n is at least 4, there are only a finite number of solutions. Thus Falting's Theorem implies that for each $n \geq 4$, there are only a finite number of counterexamples to Fermat's last theorem. Of course, we now know that Fermat is true — but Falting's theorem applies much more widely — for example, in more variables. The equations $x^3 + y^2 + z^{14} + xy + 17 = 0$ and $3x^{14} + x^{34}y + \cdots = 0$, assuming their complex solutions form a surface of genus at least 2, which they probably do, have only a finite number of solutions.

So here is where we are going. Algebraic geometry involves a new kind of "space", which will allow both singularities, and arithmetic interpretations. We are going to define these spaces, and define maps between them, and other geometric constructions such as vector bundles and sheaves, and before long, cohomology groups.

Motivating example: Varieties. This course will deal with the geometric notion of a *scheme*, which generalizes the earlier notion of a variety. Ideally I'd like to give you a semester's worth of a pre-course, dealing with varieties.

3. A LITTLE BIT OF CATEGORY THEORY

That which does not kill me, makes me stronger. — Nietzsche

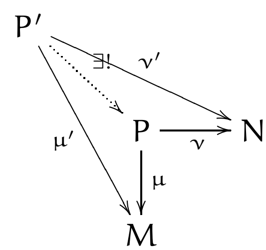
Before we get to any interesting geometry, we need to develop the language to discuss things cleanly and effectively. This is best done in the language of categories. If algebraic geometry tends to strike fear into peoples' hearts, category theory tends to induce sleep and boredom, as abstract meaningless concepts are introduced and symbols are pushed around. If I use the word topoi, you can shoot me. Here's how you should think about category theory for our purposes. There is not much to know about categories to get started; it is just a very useful language. Like all mathematical languages, category theory comes with an embedded logic, which allows us to abstract intuitions in settings we know well to far more general situations.

Our motivation is as follows. We will be creating some new mathematical objects (such as schemes, and families of sheaves), and we expect them to act like objects we have seen before. We could try to nail down precisely what we mean by "act like", and what minimal set of things we have to check in order to verify that they act the way we expect. Fortunately, we don't have to — other people have done this before us, by defining key notions, such as *abelian categories*, which behave like modules over a ring.

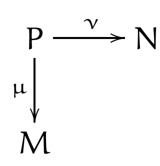
For example, we will define the notion of *product* of the geometric spaces (schemes). We could just give a definition of product, but then you should want to know why this precise definition deserves the name of "product". As a motivation, we revisit the notion of product in a situation we know well: (the category of) sets. One way to define the product of sets U and V is as the set of ordered pairs $\{(u, v) : u \in U, v \in V\}$. But someone from a different mathematical culture might reasonably define it as the set of symbols $\{[v, u] : u \in U, v \in V\}$. These notions are "obviously the same". Better: there is "an obvious bijection between the two".

This can be made precise by giving a better definition of product, in terms of a *universal property*. Given two sets M and N, a product is a set P, along with maps $\mu : P \to M$ and $\nu : P \to N$, such that for *any other set* P' *with maps* $\mu' : P' \to M$ *and* $\nu' : P' \to N$, these maps must factor *uniquely* through P:

(1)

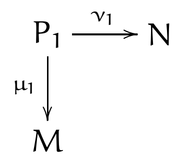


Thus a product is a *diagram*

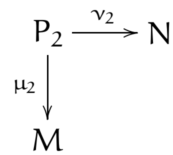


and not just a set P, although the maps μ and ν are often left implicit.

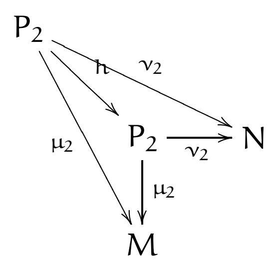
This definition agrees with the usual definition, with one twist: there isn't just a single product; but any two products come with a *canonical* isomorphism between them. In other words, the product is unique up to unique isomorphism. Here is why: if you have a product



and I have a product



then by the universal property of my product (letting (P_2, μ_2, ν_2) play the role of (P, μ, ν), and (P_1, μ_1, ν_1) play the role of (P', μ', ν') in (1)), there is a unique map $f : P_1 \to P_2$ making the appropriate diagram commute (i.e. $\mu_1 = \mu_2 \circ f$ and $\nu_1 = \nu_2 \circ f$). Similarly by the universal property of your product, there is a unique map $g : P_2 \to P_1$ making the appropriate diagram commute. Now consider the universal property of my product, this time letting (P_2, μ_2, ν_2) play the role of both (P, μ, ν) and (P', μ', ν') in (1). There is a unique map $h : P_2 \to P_2$ such that



commutes. However, I can name two such maps: the identity map id_{P_2}, and $g \circ f$. Thus $g \circ f = \mathrm{id}_{P_2}$. Similarly, $f \circ g = \mathrm{id}_{P_1}$. Thus the maps f and g arising from the universal property are bijections. In short, there is a unique bijection between P_1 and P_2 preserving the "product structure" (the maps to M and N). This gives us the right to name any such product $M \times N$, since any two such products are canonically identified.

This definition has the advantage that it works in many circumstances, and once we define category, we will soon see that the above argument applies verbatim in any category to show that products, if they exist, are unique up to unique isomorphism. Even if you haven't seen the definition of category before, you can verify that this agrees with your notion of product in some category that you have seen before (such as the category of vector spaces, where the maps are taken to be linear maps; or the category of real manifolds, where the maps are taken to be submersions).

This is handy even in cases that you understand. For example, one way of defining the product of two manifolds M and N is to cut them both up into charts, then take products of charts, then glue them together. But if I cut up the manifolds in one way, and you cut them up in another, how do we know our resulting manifolds are the "same"? We could wave our hands, or make an annoying argument about refining covers, but instead, we should just show that they are indeed products, and hence the "same" (i.e. isomorphic).

This argument is essentialy Yoneda's lemma, which we will formalize shortly in Section 5.

4. Categories and functors

Let's now get our hands dirty. We begin with an informal definition of categories and functors.

4.1. Categories.

A category $\mathcal{C}$ consists of a collection of *objects*, and for each pair of objects, a set of maps, or *morphisms* (or *arrows*), between them. The collection of objects of a category $\mathcal{C}$ are often denoted $\mathrm{obj}(\mathcal{C})$, but we will usually denote the collection $\mathcal{C}$ also by $\mathcal{C}$. If $A, B \in \mathcal{C}$, then the morphisms from A to B are denoted $\mathrm{Mor}(A, B)$. A morphism is often written $f : A \to B$, and A is said to be the *source* of f, and B the *target* of f. Morphisms compose as expected: there is a composition $\mathrm{Mor}(A, B) \times \mathrm{Mor}(B, C) \to \mathrm{Mor}(A, C)$, and if $f \in \mathrm{Mor}(A, B)$ and $g \in \mathrm{Mor}(B, C)$, then their composition is denoted $g \circ f$. Composition is associative: if $f \in \mathrm{Mor}(A, B)$, $g \in \mathrm{Mor}(B, C)$, and $h \in \mathrm{Mor}(C, D)$, then $h \circ (g \circ f) = (h \circ g) \circ f$. For each object $A \in \mathcal{C}$, there is always an *identity morphism* $\mathrm{id}_A : A \to A$, such that when you (left- or right-)compose a morphism with the identity, you get the same morphism. More precisely, if $f : A \to B$ is a morphism, then $f \circ \mathrm{id}_A = f = \mathrm{id}_B \circ f$.

If we have a category, then we have a notion of *isomorphism* between two objects (if we have two morphisms $f : A \to B$ and $g : B \to A$, *both* of whose compositions are the identity on the appropriate object), and a notion of *automorphism* of an object (an isomorphism of the object with itself).

4.2. *Example.* The prototypical example to keep in mind is the category of sets, denoted **Sets**. The objects are sets, and the morphisms are maps of sets.

4.3. *Example.* Another good example is the category $\mathbf{Vec}_k$ of vector spaces over a given field k. The objects are k-vector spaces, and the morphisms are linear transformations.

4.A. Unimportant exercise. A category in which each morphism is an isomorphism is called a *groupoid*. (This notion is not important in this class. The point of this exercise is to give you some practice with categories, by relating them to an object you know well.)
(a) A perverse definition of a group is: a groupoid with one element. Make sense of this.
(b) Describe a groupoid that is not a group.
(For readers with a topological background: if X is a topological space, then the fundamental groupoid is the category where the objects are points of x, and the morphisms from $x \to y$ are paths from x to y, up to homotopy. Then the automorphism group of x_0 is the (pointed) fundamental group $\pi_1(X, x_0)$. In the case where X is connected, and the $\pi_1(X)$ is not abelian, this illustrates the fact that for a connected groupoid — whose definition you can guess — the automorphism groups of the objects are all isomorphic, but not canonically isomorphic.)

4.B. EXERCISE. If A is an object in a category $\mathcal{C}$, show that the isomorphisms of A with itself $\mathrm{Isom}(A, A)$ form a group (called the *automorphism group of* A, denoted $\mathrm{Aut}(A)$). What are the automorphism groups of the objects in Examples 4.2 and 4.3? Show that two isomorphic objects have isomorphic automorphism groups.

4.4. *Example: abelian groups.* The abelian groups, along with group homomorphisms, form a category **Ab**.

4.5. *Example: modules over a ring.* If A is a ring, then the A-modules form a category $\mathbf{Mod}_A$. (This category has additional structure; it will be the prototypical example of an *abelian category*, which we'll define next day.) Taking $A = k$, we obtain Example 4.3; taking $A = \mathbb{Z}$, we obtain Example 4.4.

4.6. *Example: rings.* There is a category **Rings**, where the objects are rings, and the morphisms are morphisms of rings (which I'll assume send 1 to 1).

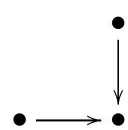
4.7. *Example: topological spaces.* The topological spaces, along with continuous maps, form a category **Top**. The isomorphisms are homeomorphisms.

4.8. *Example: partially ordered sets.* A *partially ordered set*, or *poset*, is a set $(S, \geq)$ along with a binary relation $\geq$ satisfying:

(i) $x \geq x$,
(ii) $x \geq y$ and $y \geq z$ imply $x \geq z$ (transitivity), and
(iii) if $x \geq y$ and $y \geq x$ then $x = y$.

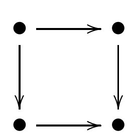
A partially ordered set $(S, \geq)$ can be interpreted as a category whose objects are the elements of S, and with a single morphism from x to y if and only if $x \geq y$ (and no morphism otherwise).

A trivial example is $(S, \geq)$ where $x \geq y$ if and only if $x = y$. Another example is



Here there are three objects. The identity morphisms are omitted for convenience, and the three non-identity morphisms are depicted. A third example is

(2)



Here the "obvious" morphisms are again omitted: the identity morphisms, and the morphism from the upper left to the lower right. Similarly,

$$\cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$$

depicts a partially ordered set, where again, only the "generating morphisms" are depicted.

4.9. *Example: the category of subsets of a set, and the category of open sets in a topological space.* If X is a set, then the subsets form a partially ordered set, where the order is given by inclusion. Similarly, if X is a topological space, then the *open* sets form a partially ordered set, where the order is given by inclusion. (What is the initial object? What is the final object?)

4.10. Functors.

A *covariant functor* F from a category $\mathcal{A}$ to a category $\mathcal{B}$, denoted $F : \mathcal{A} \to \mathcal{B}$, is the following data. It is a map of objects $F : \mathrm{obj}(\mathcal{A}) \to \mathrm{obj}(\mathcal{B})$, and for each $a_1, a_2 \in \mathcal{A}$ a morphism $m : a_1 \to a_2$, $F(m)$ is a morphism from $F(A_1) \to F(A_2)$ in $\mathcal{B}$. F preserves identity morphisms: for $A \in \mathcal{A}$, $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$. F preserves composition: $F(m_1 \circ m_2) = F(m_1) \circ F(m_2)$.

If $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$, then we may define a functor $G \circ F : \mathcal{A} \to \mathcal{C}$ in the obvious way. Composition of functors is associative.

4.11. *Example: a forgetful functor..* Consider the functor from the category of complex vector spaces $\mathbf{Vec}_k$ to **Sets**, that associates to each vector space its underlying set. The functor sends a linear transformation to its underlying map of sets. This is an example of a *forgetful functor*, where some additional structure is forgotten. Another example of a forgetful functor is $\mathbf{Mod}_A \to \mathbf{Ab}$ from A-modules to abelian groups, remembering only the abelian group structure of the A-module.

4.12. *Topological examples.* Examples of covariant functors include the fundamental group functor π_1, which sends a topological space with X choice of a point $x_0 \in X$ to a group $\pi_1(X, x_0)$, and the ith homology functor $\mathbf{Top} \to \mathbf{Ab}$, which sends a topological space X to its ith homology group $H_i(X, \mathbb{Z})$. The covariance corresponds to the fact that a (continuous) morphism of pointed topological spaces $f : X \to Y$ with $f(x_0) = y_0$ induces a map of fundamental groups $\pi_1(X, x_0) \to \pi_1(Y, y_0)$, and similarly for homology groups.

4.13. *Example.* Suppose A is an element of a category $\mathcal{C}$. Then there is a functor $h_A : \mathcal{C} \to$ **Sets** sending $B \in \mathcal{C}$ to $\mathrm{Mor}(A, B)$, and sending $f : B_1 \to B_2$ to $\mathrm{Mor}(A, B_1) \to \mathrm{Mor}(A, B_2)$ described by

$$[g : A \to B_1] \mapsto [f \circ g : A \to B_1 \to B_2].$$

4.14. *Example: partially ordered sets as index categories.* Partially ordered sets will often turn up as index categories. As a first example, if $\square$ is the category of (2), and $\mathcal{A}$ is a category, then a functor $\square \to \mathcal{A}$ is precisely the information of a commuting square in $\mathcal{A}$.

4.15. Definition. A *contravariant functor* is defined in the same way as a covariant functor, except the arrows switch directions: in the above language, $F(A_1 \to A_2)$ is now an arrow from $F(A_2)$ to $F(A_1)$.

It is wise to always state whether a functor is covariant or contravariant. If it is not stated, the functor is often assumed to be covariant.

4.16. *Topological example (cf. Example 4.12).* The the ith cohomology functor $H^i(\cdot, \mathbb{Z})$: **Top** $\to$ **Ab** is a contravariant functor.

4.17. *Example.* If $\mathbf{Vec}_k$ is the category of complex k-vector spaces, then taking duals gives a contravariant functor $^\vee : \mathbf{Vec}_k \to \mathbf{Vec}_k$. Indeed, to each linear transformation $f : V \to W$, we have a dual transformation $f^\vee : W^\vee \to V^\vee$, and $(f \circ g)^\vee = g^\vee \circ f^\vee$.

4.18. *Example.* There is a contravariant functor **Top** $\to$ **Rings** taking a topological space X to the continuous functions on X. A morphism of topological spaces $X \to Y$ (a continuous map) induces the pullback map from functions on Y to maps on X.

4.19. *Example (cf. 4.13).* Suppose A is an element of a category $\mathcal{C}$. Then there is a contravariant functor $h^A : \mathcal{C} \to$ **Sets** sending $B \in \mathcal{C}$ to $\mathrm{Mor}(B, A)$, and sending $f : B_1 \to B_2$ to $\mathrm{Mor}(B_2, A) \to \mathrm{Mor}(B_1, A)$ described by

$$[g : B_2 \to A] \mapsto [g \circ f : B_2 \to B_1 \to A].$$

This example initially looks weird and different, but the previous two examples are just special cases of this; do you see how? What is A in each case?

5. Universal properties

Given some category that we come up with, we often will have ways of producing new objects from old. In good circumstances, such a definition can be made using the notion of a *universal property*. Informally, we wish that there is an object with some property. We first show that if it exists, then it is essentially unique, or more precisely, is unique up to unique isomorphism. Then we go about constructing an example of such an object to show existence.

With a little practice, universal properties are useful in proving things quickly slickly. However, explicit constructions are often intuitively easier to work with, and sometimes also lead to short proofs.

We have seen one important example of a universal property argument already in our discussion of products. You should go back and verify that our discussion there gives a notion of product in category, and shows that products, *if they exist*, are unique up to canonical isomorphism.

5.1. Another good example of a universal property construction is the notion of a tensor product of A-modules

$$\otimes_A : \qquad \mathrm{obj}(\mathbf{Mod}_A) \times \mathrm{obj}(\mathbf{Mod}_A) \longrightarrow \mathrm{obj}(\mathbf{Mod}_A)$$

$$M \times N \longmapsto M \otimes_A N$$

The subscript A is often suppressed when it is clear from context. Tensor product is often defined as follows. Suppose you have two A-modules M and N. Then elements of the tensor product $M \otimes_A N$ are of the form $m \otimes n$ ($m \in M$, $n \in N$), subject to relations $(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n$, $m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2$, $a(m \otimes n) = (am) \otimes n = m \otimes (an)$ (where $a \in A$).

If A is a field k, we get the tensor product of vector spaces.

5.A. EXERCISE (IF YOU HAVEN'T SEEN TENSOR PRODUCTS BEFORE). Calculate $\mathbb{Z}/10 \otimes_{\mathbb{Z}} \mathbb{Z}/12$. (This exercise is intended to give some hands-on practice with tensor products.)

5.B. EXERCISE: RIGHT-EXACTNESS OF $\cdot \otimes_A N$. Show that $\cdot \otimes_A N$ gives a covariant functor $\mathbf{Mod}_A \to \mathbf{Mod}_A$. Show that $\cdot \otimes_A N$ is a *right-exact functor*, i.e. if

$$M' \to M \to M'' \to 0$$

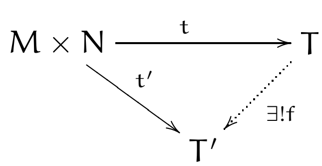
is an exact sequence of A-modules, then the induced sequence

$$M' \otimes_A N \to M \otimes_A N \to M'' \otimes_A N \to 0$$

is also exact. (For experts: is there a universal property proof?)

This is a weird definition, and really the "wrong" definition. To motivate a better one: notice that there is a natural A-bilinear map $M \times N \to M \otimes_A N$. *Any* A-bilinear map $M \times N \to C$ factors through the tensor product uniquely: $M \times N \to M \otimes_A N \to C$. (Think this through!)

We can take this as the *definition* of the tensor product as follows. It is an A-module T along with an A-bilinear map $t : M \times N \to T$, such any other such map factors through t that given any other $t' : M \times N \to T'$, there is a unique map $f : T \to T'$ such that $t' = f \circ t$.



5.C. EXERCISE. Show that $(T, t : M \times N \to T)$ is unique up to unique isomorphism. Hint: first figure out what "unique up to unique isomorphism" means for such pairs. Then follow the analogous argument for the product. (This exercise will prime you for Yoneda's Lemma.)

In short: there is an A-bilinear map $t : M \times N \to M \otimes_A N$, unique up to unique isomorphism, defined by the following universal property: for any A-bilinear map $t' : M \times N \to T'$ there is a unique $f : M \otimes_A N \to T'$ such that $t' = f \circ t$.

Note that this argument shows uniqueness *assuming existence*. We need to still show the existence of such a tensor product. This forces us to do something constructive.

5.D. EXERCISE. Show that the construction of §5.1 satisfies the universal property of tensor product.

The uniqueness of tensor product is our second example of the proof of uniqueness (up to unique isomorphism) by a *universal property*. If you have never seen this sort of argument before, then you might think you get it, but you don't, so you should think over it some more. We will be using such arguments repeatedly in the future. We'll soon formalize this way of thinking in Yoneda's Lemma.

Before getting to it, we'll give another exercise that involves universal properties.

5.2. Definition. An object of a category $\mathcal{C}$ is an *initial object* if it has precisely one map to every other object. It is a *final object* if it has precisely one map from every other object. It is a *zero-object* if it is both an initial object and a final object.

5.E. EXERCISE. Show that any two initial objects are canonically isomorphic. Show that any two final objects are canonically isomorphic.

This (partially) justifies the phrase "*the* initial object" rather than "*an* initial object", and similarly for "*the* final object" and "*the* zero object".

5.F. EXERCISE. State what the initial and final objects are in **Sets**, **Rings**, and **Top** (if they exist).

Next day: Yoneda's lemma. Limits. Maybe even some sheaves.

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