FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 1

RAVI VAKIL

This set is due Monday, October 17. It covers classes 1 and 2. Hand in five of these problems. If you are pressed for time, try more A problems. If you are ambitious, try more B problems.

I intend there to be weekly problem sets, to be given out each Monday and handed in the following Monday (although this set is an exception). If you are taking this course for a grade, you'll have to hand in all but one of the sets. These problems are not intended to be (too) onerous, but they are intended to help you get practical experience with ideas that may be new to you. Even if you are not taking the course for a grade, I strongly encourage you to try these problems, and if you are handing in problems, I encourage you to try more than the minimum number. Choose problems that stretch your knowledge, and not problems that you already know how to do. Feedback on the problems would be appreciated.

You are encouraged to talk to each other about the problems. (Write up your solutions individually.) You are also encouraged to talk to me about them. Ideally, you should find out who did problems that you didn't do.

I will be away Wednesday, October 5 until Thursday, October 13. The next class after Monday, October 3 will be Friday, October 14. *The week after we will meet Monday, Wednesday, and Friday (Oct. 17, 19, 21). Then we will be only one class behind.*

A1. A category in which each morphism is an isomorphism is called a *groupoid*. A perverse definition of a group is: a groupoid with one element. Make sense of this. (The notion of "groupoid" isn't important for this course. The point of this exercise is to give you some practice with categories, by relating them to an object you know well.)

A2 (if you haven't seen tensor products before). Calculate $\mathbb{Z}/10 \otimes_{\mathbb{Z}} \mathbb{Z}/12$. (The point of this exercise is to give you a very little hands-on practice with tensor products.)

A3. Interpret fibered product in the category of sets: If we are given maps from sets X and Y to the set Z, interpret $X \times_Z Y$. (This will help you build intuition about this concept.)

A4. A morphism $f : X \to Y$ is said to be a a **monomorphism** if any two morphisms $g_1, g_2 : Z \to X$ such that $f \circ g_1 = f \circ g_2$ must satisfy $g_1 = g_2$. This is the generalization of an injection of sets. Suppose $X \to Y$ is a monomorphism, and $W, Z \to X$ are two morphisms. Show that $W \times_X Z$ and $W \times_Y Z$ are canonically isomorphic. (We will use this later when talking about fibered products.)

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A5. Given $X \to Y \to Z$, show that there is a natural morphism $X \times_Y X \to X \times_Z X$, assuming these fibered products exist. (This is trivial once you figure out what it is saying. The point of this exercise is to see why it is trivial.)

A6. Define coproduct in a category by reversing all the arrows in the definition of product. Show that coproduct for sets is disjoint union.

A7. If Z is the final object in a category $\mathcal{C}$, and $X, Y \in \mathcal{C}$, then "$X \times_Z Y = X \times Y$" ("the" fibered product over Z is canonically isomorphic to "the" product). (This is an exercise about unwinding the definition.)

A8 ("A presheaf is the same as a contravariant functor"). Given any topological space X, we can get a category, which I will call the "category of open sets". The objects are the open sets. The morphisms are the inclusions $U \hookrightarrow V$. (What is the initial object? What is the final object?) Verify that the data of a presheaf is precisely the data of a contravariant functor from the category of open sets of X to the category of sets, plus the final object axiom, that there is one section over $\emptyset$. (This exercise is intended for people wanting practice with categories.)

A9. (a) Let X be a topological space, and S a set with more than one element, and define $\mathcal{F}(U) = S$ for all open sets U. Show that this forms a presheaf (with the obvious restriction maps), and even satisfies the identity axiom. Show that this needn't form a sheaf. (Here we need the axiom that $\mathcal{F}(\emptyset)$ must be the final object, not S. Without this patch, the constant presheaf *is* a sheaf.) This is called the *constant presheaf with values in* S. We will denote this presheaf $\underline{S}^{\text{pre}}$.
(b) Now let $\mathcal{F}(U)$ be the maps to S that are *locally constant*, i.e. for any point x in U, there is a neighborhood of x where the function is constant. (Better description is this: endow S with the discrete topology, and let $\mathcal{F}(U)$ be the continuous maps $U \to S$.) Show that this is a *sheaf*. (Here we need $\mathcal{F}(\emptyset)$ to be the final object again.) We will try to call this the *locally constant sheaf*. (In the real world, this is called the *constant sheaf*. I don't understand why.) We will denote this sheaf $\underline{S}$.

B1 (Yoneda's lemma). Pick an object in your favorite category $A \in \mathcal{C}$. For any object $C \in \mathcal{C}$, we have a set of morphisms $\mathrm{Mor}(C, A)$. If we have a morphism $f : B \to C$, we get a map of sets

$$\mathrm{Mor}(C, A) \to \mathrm{Mor}(B, A), \tag{1}$$

just by composition: given a map from C to A, we immediately get a map from B to A by precomposing with f. Yoneda's lemma, or at least part of it, says that this functor determines A up to unique isomorphism. Translation: If we have two objects A and A', and isomorphisms

$$i_C : \mathrm{Mor}(C, A) \to \mathrm{Mor}(C, A') \tag{2}$$

that commute with the maps (1), then the i_C must be induced from a unique isomorphism $A \to A'$. Prove this.

B2. Prove that a morphism is a monomorphism if and only if the natural morphism $X \to X \times_Y X$ is an isomorphism. (We may then take this as the definition of monomorphism.)

(Monomorphisms aren't very central to future discussions, although they will come up again. This exercise is just good practice.)

B3 (tensor product). (This will be important later!) Suppose $T \to R, S$ are two ring morphisms. Let I be an ideal of R. Let I^e be the extension of I to $R \otimes_T S$. These are the elements $\sum_j i_j \otimes s_j$ where $i_j \in I$, $s_j \in S$. Show that there is a natural isomorphism

$$R/I \otimes_T S \cong (R \otimes_T S)/I^e.$$

Hence the natural morphism $R \otimes_T S \to R/I \otimes_T S$ is a surjection. As an application, we can compute tensor products of finitely generated k algebras over k. For example,

$$k[x_1, x_2]/(x_1^2 - x_2) \otimes_k k[y_1, y_2]/(y_1^3 + y_2^3) \cong k[x_1, x_2, y_1, y_2]/(x_1^2 - x_2, y_1^3 + y_2^3).$$

B4 (direct limits). We say a partially ordered set I is a *directed set* if for $i, j \in I$, there is some $k \in I$ with $i, j \leq k$. In this exercise, you will show that the direct limit of any system of A-modules indexed by I exists, by constructing it. Say the system is given by M_i ($i \in I$), and $f_{ij} : M_i \to M_j$ ($i \leq j$ in I). Let $M = \oplus_i M_i$, where each M_i is identified with its image in M, and let R be the submodule generated by all elements of the form $m_i - f_{ij}(m_i)$ where $m_i \in M_i$ and $i \leq j$. Show that M/R (with the projection maps from the M_i) is $\lim_{\to} M_i$. You will notice that the same argument works in other interesting categories, such as: sets; groups; and abelian groups. (This example came up in interpreting/defining stalks as direct limits.)

B5 (practice with universal properties). The purpose of this exercise is to give you some practice with "adjoints of forgetful functors", the means by which we get groups from semigroups, and sheaves from presheaves. Suppose R is a ring, and S is a multiplicative subset. Then $S^{-1}R$-modules are a fully faithful subcategory of the category of R-modules (meaning: the objects of the first category are a subset of the objects of the second; and the morphisms between any two objects of the second that are secretly objects of the first are just the morphisms from the first). Then $M \to S^{-1}M$ satisfies a universal property. Figure out what the universal property is, and check that it holds. In other words, describe the universal property enjoyed by $M \to S^{-1}M$, and prove that it holds.

(Here is the larger story. Let $S^{-1}R$-Mod be the category of $S^{-1}R$-modules, and R-Mod be the category of R-modules. Every $S^{-1}R$-module is an R-module, so we have a (covariant) forgetful functor $F : S^{-1}R\text{-Mod} \to R\text{-Mod}$. In fact this is a fully faithful functor: it is injective on objects, and the morphisms between any two $S^{-1}R$-modules *as R-modules* are just the same when they are considered as $S^{-1}R$-modules. Then there is a functor $G : R$-Mod $\to S^{-1}R$-Mod, which might reasonably be called "localization with respect to S", which is left-adjoint to the forgetful functor. Translation: If A is an R-module, and B is an $S^{-1}R$-module, then $\mathrm{Hom}(GA, B)$ (morphisms as $S^{-1}R$-modules, which is incidentally the same as morphisms as R-modules) are in natural bijection with $\mathrm{Hom}(A, FB)$ (morphisms as R-modules).)

B6 (good examples of sheaves). Suppose Y is a topological space. Show that "continuous maps to Y" form a sheaf of sets on X. More precisely, to each open set U of X, we associate the set of continuous maps to Y. Show that this forms a sheaf.

(b) Suppose we are given a continuous map $f : Y \to X$. Show that "sections of f" form a sheaf. More precisely, to each open set U of X, associate the set of continuous maps s to Y such that $f \circ s = \mathrm{id}|_U$. Show that this forms a sheaf. (A classical construction of sheaves in general is to interpret them in precisely this way. See Serre's revolutionary article *Faisceaux Algébriques Cohérents*.)

B7 (an important construction, the pushforward sheaf). (a) Suppose $f : X \to Y$ is a continuous map, and $\mathcal{F}$ is a sheaf on X. Then define $f_*\mathcal{F}$ by $f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$, where V is an open subset of Y. Show that $f_*\mathcal{F}$ is a sheaf. This is called a *pushforward sheaf*. More precisely, $f_*\mathcal{F}$ is called the *pushforward of* $\mathcal{F}$ *by* f.
(b) Assume $\mathcal{F}$ is a sheaf of sets (or rings or R-modules), so stalks exist. If $f(x) = y$, describe the natural morphism of stalks $(f_*\mathcal{F})_y \to \mathcal{F}_x$.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 2

This set is due Monday, October 24. It covers classes 3 and 4. Read all of these problems, and hand in six solutions. The problems are arranged roughly in "chronological order", not by difficulty. Try to solve problems on a range of topics. If you are pressed for time, try more straightforward problems. If you are ambitious, push the envelope a bit. You are encouraged to talk to each other about the problems. (Write up your solutions individually.) You are also encouraged to talk to me about them. Ideally, you should find out who did problems that you didn't do.

1. Suppose

$$0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} A^n \xrightarrow{d^n} 0$$

is a complex of vector spaces (often called $A^\bullet$ for short), i.e. $d^i \circ d^{i-1} = 0$. Show that $\sum (-1)^i \dim A^i = \sum (-1)^i h^i(A^\bullet)$. (Recall that $h^i(A^\bullet) = \dim \ker(d^i)/\operatorname{im}(d^{i-1})$.) In particular, if $A^\bullet$ is exact, then $\sum(-1)^i \dim A^i = 0$. (If you haven't dealt much with cohomology, this will give you some practice. If you have, you shouldn't do this problem.)

Problems on presheaves and sheaves

2. Suppose $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves of abelian groups or $\mathcal{O}_X$-modules. If $\mathcal{H}$ is defined by the collection of data $\mathcal{H}(U) = \mathcal{G}(U)/\phi(\mathcal{F}(U))$ for all open U, show that $\mathcal{H}$ is a presheaf, and show that it is a cokernel in the category of presheaves. (I stated this as a fact in class, but you aren't allowed to appeal to authority.)

3. Suppose that $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \cdots \to \mathcal{F}_n \to 0$ is an *exact sequence of presheaves* of groups or $\mathcal{O}_X$-modules. Show that $0 \to \mathcal{F}_1(U) \to \mathcal{F}_2(U) \to \cdots \to \mathcal{F}_n(U) \to 0$ is also an exact sequence for all U.

4. (This problem sounds more confusing than it is!) Show that the presheaf kernel of a morphism of sheaves (of abelian groups, or $\mathcal{O}_X$-modules) is also sheaf. Show that it is the sheaf kernel (a kernel in the category of sheaves) as well. (This is one reason that kernels are easier than cokernels.)

5. The presheaf cokernel was defined in problem 2. Show that the sheafification of the presheaf cokernel is in fact the sheaf cokernel, by verifying that it satisfies the universal property.

6. Suppose $f : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves of abelian groups or $\mathcal{O}_X$-modules. Let $\operatorname{im} f$ be the sheafification of the "presheaf image". Show that there are natural isomorphisms $\operatorname{im} f \cong \mathcal{F}/\ker f$ and $\operatorname{coker} f \cong \mathcal{G}/\operatorname{im} f$. (This problem shows that this construction deserves to be called the "image".)

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7. Suppose $\mathcal{O}_X$ is a sheaf of rings on X. Define (categorically) what we should mean by tensor product of two presheaves or sheaves of $\mathcal{O}_X$-modules. Give an explicit construction, and show that it satisfies your categorical definition. *Hint:* take the "presheaf tensor product" — which needs to be defined — and sheafify. (This is admittedly a vague problem. If it is confusing, just ask. But it is good practice to turn your rough intuition into precise statements.)

8. Suppose $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H}$ is an exact sequence of sheaves (of abelian groups) on X. If $f : X \to Y$ is a continuous map, show that

$$0 \to f_*\mathcal{F} \to f_*\mathcal{G} \to f_*\mathcal{H}$$

is exact. Translation: pushforward is a left-exact functor. (The case of left-exactness of the global section functor can be interpreted as a special case of this, in the case where Y is a point.) Show that it needn't be exact on the right, i.e. that $f_*\mathcal{G} \to f_*\mathcal{H}$ needn't be surjective (= an epimorphism). (Hint: see the previous parenthetical comment, and think of your favorite short exact sequence of sheaves.)

The next three problems present some new concepts: gluing sheaves, sheaf homomorphisms, and flasque sheaves. I will feel comfortable using these concepts in class.

9. Suppose $X = \cup U_i$ is an open cover of X, and we have sheaves $\mathcal{F}_i$ on U_i along with isomorphisms $\phi_{ij} : \mathcal{F}_i|_{U_i \cap U_j} \to \mathcal{F}_j|_{U_i \cap U_j}$ that agree on triple overlaps (i.e. $\phi_{ij} \circ \phi_{jk} = \phi_{ij}$ on $U_i \cap U_j \cap U_k$). Show that these sheaves can be glued together into a unique sheaf $\mathcal{F}$ on X, such that $\mathcal{F}_i = \mathcal{F}|_{U_i}$, and the isomorphisms over $U_i \cap U_j$ are the obvious ones. (Thus we can "glue sheaves together", using limited patching information.)

10. Suppose $\mathcal{F}$ and $\mathcal{G}$ are two sheaves on X. Let $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})$ be the collection of data

$$\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})(U) := \mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U).$$

Show that this is a sheaf. (This is called the "sheaf $\underline{\mathrm{Hom}}$". If $\mathcal{F}$ and $\mathcal{G}$ are sheaves of sets, $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})$ is a sheaf of sets. If $\mathcal{G}$ is a sheaf of abelian groups, then $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})$ is a sheaf of abelian groups.) I've decided to call this $\underline{\mathrm{Hom}}$ rather than $\mathcal{H}\mathrm{om}$ because of the convention that "underlining often denotes sheaf". (Of course, the calligraphic font also often denotes sheaf.)

11. A sheaf $\mathcal{F}$ is said to be *flasque* if for every $U \subset V$, the restriction map $\mathrm{res}_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$ is surjective. In other words, every section over U extends to a section over V. This is a very strong condition, but it comes up surprisingly often.
(a) Show that $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is exact, and $\mathcal{F}'$ and $\mathcal{F}''$ are flasque, then so is $\mathcal{F}$.
(b) Suppose $f : X \to Y$ is a continuous map, and $\mathcal{F}$ is a flasque sheaf on X. Show that $f_*\mathcal{F}$ is a flasque sheaf on Y.
(If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is exact, and $\mathcal{F}'$ is flasque, then $0 \to \mathcal{F}'(U) \to \mathcal{F}(U) \to \mathcal{F}''(U) \to 0$ is exact, i.e. the global section functor is exact here, even on the right. Similarly, for any continuous map $f : X \to Y$, $0 \to f_*\mathcal{F}' \to f_*\mathcal{F} \to f_*\mathcal{F}'' \to 0$ is exact. I haven't thought about how hard this is yet, so I haven't made this part of the exercise. But it is good to know, and gives a reason to like flasque sheaves.)

Understanding sheaves via stalks

12. Prove that a section of a sheaf is determined by its germs, i.e.

$$\mathcal{F}(U) \to \coprod_{x \in U} \mathcal{F}_x$$

is injective. Hint: you won't use the gluability axiom. So this is true of morphisms of "separated presheaves". (This exercise is important, as you've seen!) Corollary: If a sheaf has all stalks 0, then it is the 0-sheaf.

13. Show that a morphism of sheaves on a topological space X induces a morphism of stalks. More precisely, if $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves on X, describe a natural map $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$.

14. Show that morphisms of sheaves are determined by morphisms of stalks. Hint # 1: you won't use the gluability axiom. Hint # 2: study the following diagram.

(1)
$$\begin{array}{ccc} \mathcal{F}(U) & \longrightarrow & \mathcal{G}(U) \\ \big\downarrow & & \big\downarrow \\ \prod_{x \in U} \mathcal{F}_x & \longrightarrow & \prod_{x \in U} \mathcal{G}_x \end{array}$$

15. Show that a morphism of sheaves is an isomorphism if and only if it induces an isomorphism of all stalks. Hint: Use (1). Injectivity of $\mathcal{F}(U) \to \mathcal{G}(U)$ uses the previous exercise. Surjectivity requires gluability. (I largely did this in class, so you should try this mainly if you want to make sure you are clear on the concept.)

16. Show that problems 12, 14, and 15 are false for presheaves in general. (Hint: take a 2-point space with the discrete topology, i.e. every subset is open.)

17. Show that for any morphism of presheaves $\phi : \mathcal{F} \to \mathcal{G}$, we get a natural induced morphism of sheaves $\phi^{sh} : \mathcal{F}^{sh} \to \mathcal{G}^{sh}$.

18. Show that the stalks of $\mathcal{F}^{sh}$ are the same as ("are naturally isomorphic to") the stalks of $\mathcal{F}$. Hint: Use the concrete description of the stalks.

Sheaves on a nice base

19. Suppose $\{B_i\}$ is a "nice base" for the topology of X.
(a) Verify that a morphism of sheaves is determined by the induced morphism of sheaves on the base.
(b) Show that a morphism of sheaves on the base (i.e. such that the diagram

$$\begin{array}{ccc} \Gamma(B_i, \mathcal{F}) & \longrightarrow & \Gamma(B_i, \mathcal{G}) \\ \big\downarrow & & \big\downarrow \\ \Gamma(B_j, \mathcal{F}) & \longrightarrow & \Gamma(B_j, \mathcal{G}) \end{array}$$

commutes for all $B_j \hookrightarrow B_i$) gives a morphism of the induced sheaves.

The inverse image sheaf

Suppose we have a continuous map $f : X \to Y$. If $\mathcal{F}$ is a sheaf on X, we have defined the pushforward $f_*\mathcal{F}$, which is a sheaf on Y. There is also a notion of inverse image. If $\mathcal{G}$ is a sheaf on Y, then there is a sheaf on X, denoted $f^{-1}\mathcal{G}$. This gives a covariant functor from sheaves on Y to sheaves on X. For example, if we have a morphism of sheaves on Y, we'll get an induced morphism of their inverse image sheaves on X.

Here is a concrete but unmotivated (and frankly unpleasant) definition: temporarily define $f^{-1}\mathcal{G}^{\mathrm{pre}}(U) = \lim_{\to V \supset f(U)} \mathcal{G}(V)$. (Recall explicit description of direct limit: sections are sections on open sets containing $f(U)$, with an equivalence relation.)

20. Show that this defines a presheaf on X.

Now define the *inverse image sheaf* $f^{-1}\mathcal{G} := (f^{-1}\mathcal{G}^{\mathrm{pre}})^{\mathrm{sh}}$.

21. Show that the stalks of $f^{-1}\mathcal{G}$ are the same as the stalks of $\mathcal{G}$. More precisely, if $f(x) = y$, describe a natural isomorphism $\mathcal{G}_y \cong (f^{-1}\mathcal{G})_x$. (Hint: use the concrete description of the stalk, as a direct limit.)

22. Show that f^{-1} is an exact functor from sheaves of abelian groups on Y to sheaves of abelian groups on X. (Hint: exactness can be checked on stalks, and by the previous exercise, stalks are the same.) The identical argument will show that f^{-1} is an exact functor from sheaves of $\mathcal{O}_Y$-modules on Y to sheaves of $f^{-1}\mathcal{O}_Y$-modules on X, but don't bother writing that down.

Here is a categorical definition of inverse image: it is left-adjoint to f_*. More precisely, suppose $f : X \to Y$ is a continuous map (= morphism) of topological spaces, and $\mathcal{F}$ is a sheaf of sets on X, and $\mathcal{G}$ is a sheaf of sets on Y. There is a natural bijection between $\mathrm{Hom}(f^{-1}(\mathcal{G}), \mathcal{F})$ and $\mathrm{Hom}(\mathcal{G}, f_*\mathcal{F})$. (The same argument will apply for sheaves of abelian groups etc.)

23. Show that the explicit definition of inverse image satisfies this universal property. (Just describe the bijection. One should also check that this bijection is natural, i.e. that for any $\mathcal{F}_1 \to \mathcal{F}_2$, the diagram

$$\begin{array}{ccc} \mathrm{Hom}(f^{-1}(\mathcal{G}), \mathcal{F}_2) & \longrightarrow & \mathrm{Hom}(\mathcal{G}, f_*\mathcal{F}_2) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(f^{-1}(\mathcal{G}), \mathcal{F}_1) & \longrightarrow & \mathrm{Hom}(\mathcal{G}, f_*\mathcal{F}_1) \end{array}$$

commutes, and something similar for the "left argument", but don't worry about that.) This problem requires some elbow grease.

A small exercise on a small affine scheme

24. Describe the set $\mathrm{Spec}\, k[x]/x^2$. This seems like a very boring example, but it will grow up to be very important indeed! (This is not hard.)

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FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 3

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This set is due Monday, October 31. It covers classes 5, 6, and 7. Read all of these problems, and hand in six solutions. Try to solve problems on a range of topics. If you are pressed for time, try more straightforward problems. If you are ambitious, push the envelope a bit. You are encouraged to talk to each other about the problems. (Write up your solutions individually.) You are also encouraged to talk to me about them. Ideally, you should find out who did problems that you didn't do. Make sure you read all the problems, because we will be be making use of many of these results.

Facts we'll use (short proofs)

Three of these count for one problem.

A1. Show that if (S) is the ideal generated by S, then $V(S) = V((S))$. Thus when looking at vanishing sets, it suffices to consider vanishing sets of ideals.

A2. (a) Show that $\emptyset$ and $\operatorname{Spec} R$ are both open.
(b) (The intersection of two open sets is open.) Check that $V(I_1 I_2) = V(I_1) \cup V(I_2)$.
(c) (The union of any collection of open sets is open.) If I_i is a collection of ideals (as i runs over some index set), check that $V(\sum_i I_i) = \cap_i V(I_i)$.

A3. If $I \subset R$ is an ideal, show that $V(\sqrt{I}) = V(I)$.

A4. Show that if R is an integral domain, then $\operatorname{Spec} R$ is an irreducible topological space. (Hint: look at the point $[(0)]$.)

A5. Show that the closed points of $\operatorname{Spec} R$ correspond to the maximal ideals.

A6. If $X = \operatorname{Spec} R$, show that $[\mathfrak{p}]$ is a specialization of $[\mathfrak{q}]$ if and only if $\mathfrak{q} \subset \mathfrak{p}$.

A7. If X is a finite union of quasicompact spaces, show that X is quasicompact.

A8. Suppose $f_i \in R$ for $i \in I$. Show that $\cup_{i \in I} D(f_i) = \operatorname{Spec} R$ if and only if $(f_i) = R$.

A9. Show that $D(f) \cap D(g) = D(fg)$. Hence the distinguished base is a *nice* base.

A10. Show that if $D(f) \subset D(g)$, then $f^n \in (g)$ for some n.

A11. Show that $f \in \mathfrak{N}$ if and only if $D(f) = \emptyset$.

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A12. Suppose $f \in R$. Show that under the identification of $D(f)$ in $\operatorname{Spec} R$ with $\operatorname{Spec} R_f$, there is a natural isomorphism of sheaves $(D(f), \mathcal{O}_{\operatorname{Spec} R}|_{D(f)}) \cong (\operatorname{Spec} R_f, \mathcal{O}_{\operatorname{Spec} R_f})$.

A13. Show that the disjoint union of a *finite* number of affine schemes is also an affine scheme. (Hint: say what the ring is.)

A14. An infinite disjoint union of (non-empty) affine schemes is not an affine scheme. (One-word hint: quasicompactness.)

A15. If X is a scheme, and U is any open subset, then prove that $(U, \mathcal{O}_X|_U)$ is also a scheme.

A16. Show that if X is a scheme, then the affine open sets form a base for the Zariski topology. (Warning: they don't form a nice base, as we'll see in a different exercise on this problem set.) However, in "most nice situations" this will be true, as we will later see, when we define the analogue of "Hausdorffness", called separatedness.)

Facts we'll use

B1. Show that $\operatorname{Spec} R$ is quasicompact.

B2. Suppose that $I, S \subset R$ are an ideal and multiplicative subset respectively. Show that the Zariski topology on $\operatorname{Spec} R/I$ (resp. $\operatorname{Spec} S^{-1}R$) is the subspace topology induced by inclusion in $\operatorname{Spec} R$. (Hint: compare closed subsets.)

B3. (a) Show that $V(I(S)) = \overline{S}$. Hence $V(I(S)) = S$ for a closed set S. (b) Show that if $I \subset R$ is an ideal, then $I(V(I)) = \sqrt{I}$.

B4. (Important!) Show that V and I give a bijection between *irreducible closed subsets* of $\operatorname{Spec} R$ and *prime* ideals of R. From this conclude that in $\operatorname{Spec} R$ there is a bijection between points of $\operatorname{Spec} R$ and irreducible closed subsets of $\operatorname{Spec} R$ (where a point determines an irreducible closed subset by taking the closure). Hence each irreducible closed subset has precisely one generic point.

B5. (Important!) Show that the distinguished opens form a base for the Zariski topology.

B6. (a) Recall that sections of the structure sheaf on the base were defined by $\mathcal{O}_{\operatorname{Spec} R}(D(f)) = R_f$. Verify that this is well-defined, i.e. if $D(f) = D(f')$ then $R_f \cong R_{f'}$.
(b) Recall that restriction maps on the base were defined as follows. If $D(f) \subset D(g)$, then we have shown that $f^n \in (g)$, i.e. we can write $f^n = ag$, so there is a natural map $R_g \to R_f$ given by $r/g^m \mapsto (ra^m)/(f^{mn})$, and we define

$$\operatorname{res}_{D(g),D(f)} : \mathcal{O}_{\operatorname{Spec} R(D(g))} \to \mathcal{O}_{\operatorname{Spec} R(D(f))}$$

to be this map. Show that $\operatorname{res}_{D(g),D(f)}$ is well-defined, i.e. that it is independent of the choice of a and n, and if $D(f) = D(f')$ and $D(g) = D(g')$, then

$$\begin{array}{ccc} R_g & \xrightarrow{\operatorname{res}_{D(g),D(f)}} & R_f \\ \downarrow \sim & & \downarrow \sim \\ R_{g'} & \xrightarrow{\operatorname{res}_{D(g),D(f)}} & R_{f'} \end{array}$$

commutes.

B7. Show that the structure sheaf satisfies "identity on the distinguished base". Show that it satisfies "gluability on the distinguished base". (We used this to show that the structure sheaf is actually a sheaf.)

B8. Suppose M is an R-module. Show that the following construction describes a sheaf $\tilde{M}$ on the distinguished base. To $D(f)$ we associate $M_f = M \otimes_R R_f$; the restriction map is the "obvious" one.

B9. Show that the stalk of $\mathcal{O}_{\operatorname{Spec} R}$ at the point $[\mathfrak{p}]$ is the ring $R_{\mathfrak{p}}$. (Hint: use distinguished open sets in the direct limit you use to define the stalk. In the course of doing this, you'll discover a useful principle. In the concrete definition of stalk, the elements were sections of the sheaf over *some* open set containing our point, and two sections over different open sets were considered the same if they agreed on some smaller open set. In fact, you can just consider elements of your base when doing this. I think this is called a cofinal system in the directed set, but I might be mistaken.) This is yet another reason to like the notion of a sheaf on a base.

B10. (Important!) Figure out how to define projective n-space $\mathbb{P}^n_k$. Glue together $n+1$ opens each isomorphic to $\mathbb{A}^n_k$. Show that the only global sections of the structure sheaf are the constants, and hence that $\mathbb{P}^n_k$ is not affine if $n > 0$. (Hint: you might fear that you will need some delicate interplay among all of your affine opens, but you will only need two of your opens to see this. There is even some geometric intuition behind this: the complement of the union of two opens has codimension 2. But "Hartogs' Theorem" says that any function defined on this union extends to be a function on all of projective space. Because we're expecting to see only constants as functions on all of projective space, we should already see this for this union of our two affine open sets.)

Practice with the concepts

C1. Verify that $[(y - x^2)] \in \mathbb{A}^2_k$ is a generic point for $V(y - x^2)$.

C2. Suppose $X \subset \mathbb{A}^3_k$ is the union of the three axes. Give generators for the ideal $I(X)$.

C3. Describe a natural isomorphism $(k[x,y]/(xy))_x \cong k[x]_x$.

C4. Suppose we have a polynomial $f(x) \in k[x]$. Instead, we work in $k[x,\epsilon]/\epsilon^2$. What then is $f(x+\epsilon)$? (Do a couple of examples, and you will see the pattern. For example, if $f(x) = 3x^3 + 2x$, we get $f(x+\epsilon) = (3x^3+2x) + \epsilon(9x^2+2)$. Prove the pattern!) Useful tip: the dual numbers are a good source of (counter)examples, being the "smallest ring with nilpotents". They will also end up being important in defining differential information.

C5. Show that the affine base of the Zariski topology isn't necessarily a nice base. (Hint: look at the affine plane with the doubled origin.)

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FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 4

RAVI VAKIL

This set is due Monday, November 7. It covers (roughly) classes 8 and 9. Read all of these problems, and hand in six solutions. Two A problems count for one solution. One B problem counts for one solution. Try to solve problems on a range of topics. If you are pressed for time, try more straightforward problems. If you are ambitious, push the envelope a bit. You are encouraged to talk to each other about the problems. (Write up your solutions individually.) You are also encouraged to talk to me about them. Ideally, you should find out who did problems that you didn't do. Make sure you read all the problems, because we will be be making use of many of these results.

A1. Show that $\mathbb{P}^n_k$ is irreducible.

A2. You showed earlier that for affine schemes, there is a bijection between irreducible closed subsets and points. Show that this is true of schemes as well.

A3. Prove the following. If R is Noetherian, then $\operatorname{Spec} R$ is a Noetherian topological space. If X is a scheme that has a finite cover $X = \cup_{i=1}^n \operatorname{Spec} R_i$ where R_i is Noetherian, then X is a Noetherian topological space. Thus $\mathbb{P}^n_k$ and $\mathbb{P}^n_{\mathbb{Z}}$ are Noetherian topological spaces: we built them by gluing together a finite number of Spec's of Noetherian rings.

A4. If R is any ring, show that the irreducible components of $\operatorname{Spec} R$ are in bijection with the minimal primes of R. (Here minimality is with respect to inclusion.)

A5. Show that an irreducible topological space is connected.

A6. Show that a finite union of affine schemes is quasicompact. (Hence $\mathbb{P}^n_k$ is quasicompact.) Show that every closed subset of an affine scheme is quasicompact. Show that every closed subset of a quasicompact scheme is quasicompact.

A7. Show that a scheme is reduced if and only if none of the stalks have nilpotents. Hence show that if f and g are two functions on a reduced scheme that agree at all points, then $f = g$.

A8. Show that an affine scheme $\operatorname{Spec} R$ is integral if and only if R is an integral domain.

A9. Show that a scheme X is integral if and only if it is irreducible and reduced.

A10. Suppose X is an integral scheme. Then X (being irreducible) has a generic point η. Suppose $\operatorname{Spec} R$ is any non-empty affine open subset of X. Show that the stalk at η,

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$\mathcal{O}_{X,\eta}$, is naturally $\operatorname{Frac} R$. This is called the *function field* of X. It can be computed on any non-empty open set of X (as any such open set contains the generic point).

A11. Suppose X is an integral scheme. Show that the restriction maps $\operatorname{res}_{U,V} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$ are inclusions so long as $V \neq \emptyset$. Suppose $\operatorname{Spec} R$ is any non-empty affine open subset of X (so R is an integral domain). Show that the natural map $\mathcal{O}_X(U) \to \mathcal{O}_{X,\eta} = \operatorname{Frac} R$ (where U is any non-empty open set) is an inclusion.

A12. Suppose $f(x,y)$ and $g(x,y)$ are two complex polynomials ($f, g \in \mathbb{C}[x,y]$). Suppose f and g have no common factors. Show that the system of equations $f(x,y) = g(x,y) = 0$ has a finite number of solutions.

A13. If R is a finitely generated domain over k, show that $\dim R[x] = \dim R + 1$. (In fact this is true if R is Noetherian. You're welcome to try to prove that. We'll prove it later in the class, and you may use this fact in later problem sets.)

A14. Show that the underlying topological space of a Noetherian scheme is Noetherian. Show that a Noetherian scheme has a finite number of irreducible components.

A15. Suppose X is an integral scheme, that can be covered by open subsets of the form $\operatorname{Spec} R$ where R is a finitely generated domain over k. Then $\dim X$ is the transcendence degree of the function field (the stalk at the generic point) $\mathcal{O}_{X,\eta}$ over k. Thus (as the generic point lies in all non-empty open sets) the dimension can be computed in any open set of X.

A16. What is the dimension of $\operatorname{Spec} k[w,x,y,z]/(wx - yz, x^{17} + y^{17})$? (Be careful to check the hypotheses before invoking Krull!)

A17. Suppose that R is a finitely generated domain over k, and $\mathfrak{p}$ is a prime ideal. Show that $\dim R_{\mathfrak{p}} = \dim R - \dim R/\mathfrak{p}$.

A18. Show that all open subsets of a Noetherian topological space (hence of a Noetherian scheme) are quasicompact.

A19. Check that our new definition of reduced (in terms of affine covers) agrees with our earlier definition. This definition is advantageous: our earlier definition required us to check that the ring of functions over *any* open set is nilpotent free. This lets us check in an affine cover. Hence for example $\mathbb{A}^n_k$ and $\mathbb{P}^n_k$ are reduced.

A20. If R is a unique factorization domain, show that R is integrally closed (in its fraction field $\operatorname{Frac}(R)$). Hence $\mathbb{A}^n_k$ and $\mathbb{P}^n_k$ are both normal.

A21. Suppose R is a ring, and $(f_1, \dots, f_n) = R$. Show that if R has no nonzero nilpotents (i.e. 0 is a radical ideal), then R_{f_i} also has no nonzero nilpotents. Show that if no R_{f_i} has a nonzero nilpotent, then neither does R.

A22. Suppose R is an integral domain. Show that if R is integrally closed, then so is R_f.

A23. Suppose X is a quasicompact scheme, and f is a function vanishing on all the points of X. Show that $f^n = 0$ for some n. Show that this can be false without the quasicompact hypothesis.

B1. Show that $\left(k[x,y]/(xy,x^2)\right)_y$ has no nilpotents. (Hint: show that it is isomorphic to another ring, by considering the geometric picture.)

B2. Give (with proof!) an example of a scheme that is connected but reducible.

B3. Show that $\dim \mathbb{A}^1_{\mathbb{Z}} = 2$.

B4. Suppose that R is a Unique Factorization Domain containing $1/2$, $f \in R$ has no repeated prime factors, and $z^2 - f$ is irreducible in $R[z]$. Show that $\operatorname{Spec} R[z]/(z^2-f)$ is normal. (Hint: one of Gauss' Lemmas.) Show that the following schemes are normal: $\operatorname{Spec} \mathbb{Z}[x]/(x^2-n)$ where n is a square-free integer congruent to $3 \pmod 4$; $\operatorname{Spec} k[x_1,\ldots,x_n]/x_1^2 + x_2^2 + \cdots + x_m^2$ where $\operatorname{char} k \neq 2$, $m \geq 3$; $\operatorname{Spec} k[w,x,y,z]/(wx-yz)$ where $\operatorname{char} k \neq 2$ and k is algebraically closed. Show that if f has repeated prime factors, then $\operatorname{Spec} R[z]/(z^2-f)$ is *not* normal.

B5. Show that $\operatorname{Spec} k[w,x,y,z]/(wz-xy, wy-x^2, xz-y^2)$ is an irreducible surface. (It is no harder to show that it is an integral surface.) We will see next week that this is the affine cone over the twisted cubic.

B6. Suppose $X = \operatorname{Spec} R$ where R is a Noetherian domain, and Z is an irreducible component of $V(r_1,\ldots,r_n)$, where $r_1,\ldots,r_n \in R$. Show that the height of (the prime associated to) Z is at most n. Conversely, suppose $X = \operatorname{Spec} R$ where R is a Noetherian domain, and Z is an irreducible subset of height n. Show that there are $f_1, \ldots, f_n \in R$ such that Z is an irreducible component of $V(f_1,\ldots,f_n)$.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 5

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This set is due Monday, November 14. It covers (roughly) classes 10, 11, and 12.

As you might have noticed, last week there were a **lot** of interesting problems worth trying — too many to do! (This is just because we've gone far enough that we can really explore interesting questions.) So please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in six solutions. If you are ambitious (and have the time), go for more. **Problems marked with "-" count for half a solution.** Problems marked with "+" may be harder or more fundamental, but still count for one solution. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems.

Class 8:

1. (a) Use dimension theory to prove a microscopically stronger version of the weak Nullstellensatz: Suppose $R = k[x_1, \dots, x_n]/I$, where k is an algebraically closed field and I is some ideal. Then the maximal ideals are precisely those of the form $(x_1 - a_1, \dots, x_n - a_n)$, where $a_i \in k$.
(b) Suppose $R = k[x_1, \dots, x_n]/I$ where k is not necessarily algebraically closed. Show that every maximal ideal of R has a residue field that is a finite extension of k. [Hint for both: the maximal ideals correspond to dimension 0 points, which correspond to transcendence degree 0 extensions of k, i.e. finite extensions of k. If k is algebraically closed, the maximal ideals correspond to surjections $f : k[x_1, \dots, x_n] \to k$. Fix one such surjection. Let $a_i = f(x_i)$, and show that the corresponding maximal ideal is $(x_1 - a_1, \dots, x_n - a_n)$.]

Class 10:

2+. Suppose R is a ring, and $(f_1, \dots, f_n) = R$. Suppose A is a ring, and R is an A-algebra. Show that if each R_{f_i} is a finitely-generated A-algebra, then so is R.

3. Show that an irreducible homogeneous polynomial in $n+1$ variables (over a field k) describes an integral scheme of dimension $n-1$. We think of this as a "hypersurface in $\mathbb{P}^n_k$".

4. Show that $wx = yz, x^2 = wy, y^2 = xz$ describes an irreducible curve in $\mathbb{P}^3_k$ (the twisted cubic!).

5. Suppose S_* is a graded ring (with grading $\mathbb{Z}^{\geq 0}$). It is automatically a module over S_0. Now $S_+ := \oplus_{i>0} S_i$ is an ideal, which we will call the *irrelevant ideal*; suppose that it is a finitely generated ideal. Show that S_* is a finitely-generated S_0-algebra.

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6+. Recall the definition of the distinguished open subset $D(f)$ on $\operatorname{Proj} S_*$, where f is homogeneous of positive degree. Show that

$$(D(f), \mathcal{O}_{\operatorname{Proj} S_*}) \cong \operatorname{Spec}(S_f)_0$$

defines a sheaf on $\operatorname{Proj} S_*$. (We used this to define the structure sheaf $\mathcal{O}_{\operatorname{Proj} S_*}$ on $\operatorname{Proj} S_*$.)

7-. Show that $\operatorname{Proj} k[x_0, \ldots, x_n]$ is isomorphic to our earlier definition of $\mathbb{P}^n$.

8-. Show that $Y = \mathbb{P}^2 - (x^2 + y^2 + z^2 = 0)$ is affine, and find its corresponding ring (= find its ring of global sections).

Class 11:

9-. Show that $\mathbb{P}^0_A = \operatorname{Proj} A[T] \cong A$. Thus "$\operatorname{Spec} A$ is a projective A-scheme".

10. Show that all projective A-schemes are quasicompact. (Translation: show that any projective A-scheme is covered by a finite number of affine open sets.) Show that $\operatorname{Proj} S_*$ is finite type over $A = S_0$. If S_0 is a Noetherian ring, show that $\operatorname{Proj} S_*$ is a Noetherian scheme, and hence that $\operatorname{Proj} S_*$ has a finite number of irreducible components. Show that any quasiprojective scheme is locally of finite type over A. If A is Noetherian, show that any quasiprojective A-scheme is quasicompact, and hence of finite type over A.

11. Give an example of a quasiprojective A-scheme that is not quasicompact (necessarily for some non-Noetherian A).

12-. Show that $\mathbb{P}^n_k$ is normal. More generally, show that $\mathbb{P}^n_R$ is normal if R is a Unique Factorization Domain.

13+. Show that the projective cone of $\operatorname{Proj} S_*$ has an open subscheme $D(T)$ that is the affine cone, and that its complement $V(T)$ can be identified with $\operatorname{Proj} S_*$ (as a topological space). (More precisely, setting $T = 0$ "cuts out" a scheme isomorphic to $\operatorname{Proj} S_*$ — see if you can make that precise.)

14. If S_* is a finitely generated domain over k, and $\operatorname{Proj} S_*$ is non-empty show that $\dim \operatorname{Spec} S_* = \dim \operatorname{Proj} S_* + 1$.

15. Show that the irreducible subsets of dimension $n-1$ of $\mathbb{P}^n_k$ correspond to homogeneous irreducible polynomials modulo multiplication by non-zero scalars.

16+.

(a) Suppose I is any homogeneous ideal, and f is a homogeneous element. Suppose f vanishes on $V(I)$. Show that $f^n \in I$ for some n. (Hint: Mimic the proof in the affine case.)
(b) If $Z \subset \operatorname{Proj} S_*$, define $I(\cdot)$. Show that it is a homogeneous ideal. For any two subsets, show that $I(Z_1 \cup Z_2) = I(Z_1) \cap I(Z_2)$.
(c) For any homogeneous ideal I with $V(I) \neq \emptyset$, show that $I(V(I)) = \sqrt{I}$.
(d) For any subset $Z \subset \operatorname{Proj} S_*$, show that $V(I(Z)) = \overline{Z}$.

17. Show that the following are equivalent. (a) $V(I) = \emptyset$ (b) for any f_i (i in some index set) generating I, $\cup D(f_i) = \operatorname{Proj} S_*$ (c) $\sqrt{I} \supset S_+$.

18+. Show that $\operatorname{Proj} S_{n\cdot}$ is isomorphic to $\operatorname{Proj} S_*$.

For problems 19-21, suppose $S_* = k[x, y]$ (with the usual grading).

19. Show that $S_{2\cdot} \cong k[u, v, w]/(uw - v^2)$. (Thus the 2-uple Veronese embedding of $\mathbb{P}^1$ is as a conic in $\mathbb{P}^2$.)

20. Show that $\operatorname{Proj} S_{3\cdot}$ is the *twisted cubic* "in" $\mathbb{P}^3$. (The equations of the twisted cubic turn up in problems 4 and 39.)

21+. Show that $\operatorname{Proj} S_{d\cdot}$ is given by the equations that

$$\begin{pmatrix} y_0 & y_1 & \cdots & y_{d-1} \\ y_1 & y_2 & \cdots & y_d \end{pmatrix}$$

is rank 1 (i.e. that all the 2×2 minors vanish). This is called the *degree* d *rational normal curve* "in" $\mathbb{P}^d$.

22. Find the equations cutting out the *Veronese surface* $\operatorname{Proj} S_{2\cdot}$ where $S_* = k[x_0, x_1, x_2]$, which sits naturally in $\mathbb{P}^5$.

23. Show that $\mathbb{P}(m, n)$ is isomorphic to $\mathbb{P}^1$. Show that $\mathbb{P}(1, 1, 2) \cong \operatorname{Proj} k[u, v, w, z]/(uw - v^2)$. Hint: do this by looking at the even-graded parts of $k[x_0, x_1, x_2]$. (Picture: this is a projective cone over a conic curve.)

24+. (This is a handy exercise for later.) (a) *(Hypersurfaces meet everything of dimension at least* 1 *in projective space — unlike in affine space.)* Suppose X is a closed subset of $\mathbb{P}^n_k$ of dimension at least 1, and H a nonempty hypersurface in $\mathbb{P}^n_k$. Show that H meets X. (Hint: consider the affine cone, and note that the cone over H contains the origin. Use Krull's Principal Ideal Theorem.)
(b) (Definition: Subsets in $\mathbb{P}^n$ cut out by linear equations are called *linear subspaces*. Dimension 1, 2 linear subspaces are called *lines* and *planes* respectively.) Suppose $X \hookrightarrow \mathbb{P}^n_k$ is a closed subset of dimension r. Show that any codimension r linear space meets X. (Hint: Refine your argument in (a).)
(c) Show that there is a codimension $r + 1$ complete intersection (codimension $r + 1$ set that is the intersection of $r + 1$ hypersurfaces) missing X. (The key step: show that there is a hypersurface that doesn't contain every generic point of X.) If k is infinite, show that there is a codimension $r + 1$ linear subspace missing X. (The key step: show that there is a hyperplane not containing any generic point of a component of X.)

25. Describe all the lines on the quadric surface $wx - yz = 0$ in $\mathbb{P}^3_k$. (Hint: they come in two "families", called the *rulings* of the quadric surface.)

26. (This is intended for people who already know what derivations are.) In differential geometry, the tangent space at a point is sometimes defined as the vector space of derivations at that point. A derivation is a function that takes in functions near the point that vanish at the point, and gives elements of the field k, and satisfies the Leibniz rule

$(fg)' = f'g + g'f$. Show that this agrees with our definition of tangent space. (One direction was shown in class 11.)

27+. *(Nakayama's lemma version 3)* Suppose R is a ring, and I is an ideal of R contained in all maximal ideals. Suppose M is a *finitely generated* R-module, and $N \subset M$ is a submodule. If $N/IN \hookrightarrow M/IM$ an isomorphism, then $M = N$.

28+. *(Nakayama's lemma version 4)* Suppose $(R, \mathfrak{m})$ is a local ring. Suppose M is a finitely-generated R-module, and $f_1, \dots, f_n \in M$, with (the images of) $f_1, \dots, f_n$ generating $M/\mathfrak{m}M$. Then $f_1, \dots, f_n$ generate M. (In particular, taking $M = \mathfrak{m}$, if we have generators of $\mathfrak{m}/\mathfrak{m}^2$, they also generate $\mathfrak{m}$.)

Class 12:

29-. Show that if A is a Noetherian local ring, then A has finite dimension. (*Warning:* Noetherian rings in general could have infinite dimension.)

30+. *(the Jacobian criterion for checking nonsingularity)* Suppose k is an algebraically closed field, and X is a finite type k-scheme. Then locally it is of the form $\operatorname{Spec} k[x_1, \dots, x_n]/(f_1, \dots, f_r)$. Show that the Zariski tangent space at the closed point p (with residue field k, by the Nullstellensatz) is given by the cokernel of the Jacobian map $k^r \to k^n$ given by the Jacobian matrix

(1)
$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(p) & \cdots & \frac{\partial f_r}{\partial x_1}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial x_n}(p) & \cdots & \frac{\partial f_r}{\partial x_n}(p) \end{pmatrix}.$$

(This is just making precise our example of a curve in $\mathbb{A}^3$ cut out by a couple of equations, where we picked off the linear terms .) Possible hint: "translate p to the origin," and consider linear terms.

31. Show that the singular *closed* points of the hypersurface $f(x_1, \dots, x_n) = 0$ in $\mathbb{A}^n_k$ are given by the equations $f = \frac{\partial f}{\partial x_1} = \cdots = \frac{\partial f}{\partial x_n} = 0$.

32. Show that $\mathbb{A}^1$ and $\mathbb{A}^2$ are nonsingular. (Make sure to check nonsingularity at the non-closed points! Fortunately you know what all the points of $\mathbb{A}^2$ are; this is trickier for $\mathbb{A}^3$.) You are not allowed to use the fact that regular local rings remain regular under localization.

33. Show that $\operatorname{Spec} \mathbb{Z}$ is a nonsingular curve.

34. Note that $\mathbb{Z}[i]$ is dimension 1, as $\mathbb{Z}[x]$ has dimension 2 (problem set exercise), and is a domain, and $x^2 + 1$ is not 0, so $\mathbb{Z}[x]/(x^2+1)$ has dimension 1 by Krull. Show that $\operatorname{Spec} \mathbb{Z}[i]$ is a nonsingular curve. (This exercise is intended for people who know about the primes in the Gaussian integers $\mathbb{Z}[i]$.)

35. Show that there is one singular point of $\operatorname{Spec} \mathbb{Z}[2i]$, and describe it.

36. *(the Euler test for projective hypersurfaces)* There is an analogous Jacobian criterion for hypersurfaces $f = 0$ in $\mathbb{P}^n_k$. Show that the singular *closed* points correspond to the locus $f = \frac{\partial f}{\partial x_1} = \cdots = \frac{\partial f}{\partial x_n} = 0$. If the degree of the hypersurface is not divisible by the characteristic of any of the residue fields (e.g. if we are working over a field of characteristic 0), show that it suffices to check $\frac{\partial f}{\partial x_1} = \cdots = \frac{\partial f}{\partial x_n} = 0$. (Hint: show that f lies in the ideal $(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})$). (Fact: this will give the singular points in general. I don't want to prove this, and I won't use it.)

37-. Suppose k is algebraically closed. Show that $y^2 z = x^3 - xz^2$ in $\mathbb{P}^2_k$ is an irreducible nonsingular curve. (This is for practice.) Warning: I didn't say $\operatorname{char} k = 0$.

38-. Find all the singular closed points of the following plane curves. Here we work over a field of characteristic 0 for convenience.

(a) $y^2 = x^2 + x^3$. This is called a *node*.
(b) $y^2 = x^3$. This is called a *cusp*.
(c) $y^2 = x^4$. This is called a *tacnode*.

39. Show that the twisted cubic $\operatorname{Proj} k[w, x, y, z]/(wz - xy, wy - x^2, xz - y^2)$ is nonsingular. (You can do this by using the fact that it is isomorphic to $\mathbb{P}^1$. I'd prefer you to do this with the explicit equations, for the sake of practice.)

40-. Show that the only dimension 0 Noetherian regular local rings are fields. (Hint: Nakayama.)

41-. Consider the following two examples:
(i) *(the 5-adic valuation)* $K = \mathbb{Q}$, $\nu(r)$ is the "power of 5 appearing in r", e.g. $\nu(35/2) = 1$, $\nu(27/125) = -3$.
(ii) $K = k(x)$, $\nu(f)$ is the "power of x appearing in f.
Describe the valuation rings in those two examples.

42. Show that $0 \cup \{x \in K^* : \nu(x) \geq 1\}$ is the unique maximal ideal of the valuation ring. (Hint: show that everything in the complement is invertible.) Thus the valuation ring is a local ring.

43+. Show that every discrete valuation ring is a Noetherian regular local ring of dimension 1. (This was part of our long theorem showing that many things were equivalent.)

44-. Suppose R is a Noetherian local domain of dimension 1. Show that R is a principal ideal domain if and only if it is a discrete valuation ring.

45-. Show that there is only one discrete valuation on a discrete valuation ring.

46. Suppose X is a regular integral Noetherian scheme, and $f \in \operatorname{Frac}(\Gamma(X, \mathcal{O}_X))^*$ is a nonzero element of its function field. Show that f has a finite number of zeros and poles.

47+. Suppose A is a subring of a ring B, and $x \in B$. Suppose there is a faithful $A[x]$-module M that is finitely generated as an A-module. Show that x is integral over A. (Hint: look carefully at the proof of Nakayama's Lemma version 1 in the Class 11 notes, and change a few words.)

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FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 6

RAVI VAKIL

This set is due Wednesday, November 30. It covers (roughly) classes 13 and 14.

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in six solutions. If you are ambitious (and have the time), go for more. Problems marked with "-" count for half a solution. Problems marked with "+" may be harder or more fundamental, but still count for one solution. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. I'm happy to give hints, and some of these problems require hints!

Class 13:

1. Show that $(x, z) \subset k[w, x, y, z]/(wz - xy)$ is a height 1 ideal that is not principal. (Make sure you have a picture of this in your head!)

2. Suppose X is an integral Noetherian scheme, and $f \in \mathrm{Frac}(\Gamma(X, \mathcal{O}_X))^*$ is a non-zero element of its function field. Show that f has a finite number of zeros and poles. (Hint: reduce to $X = \operatorname{Spec} R$. If $f = f_1/f_2$, where $f_i \in R$, prove the result for f_i.)

3. Let R be the subring $k[x^3, x^2, xy, y] \subset k[x, y]$. (The idea behind this example: I'm allowing all monomials in $k[x, y]$ except for x.) Show that it is not integrally closed (easy — consider the "missing x"). Show that it is regular in codimension 1 (hint: show it is dimension 2, and when you throw out the origin you get something nonsingular, by inverting x^2 and y respectively, and considering R_{x^2} and R_y).

4. You have checked that if $k = \mathbb{C}$, then $k[w, x, y, z]/(wx - yz)$ is integrally closed (PS4, problem B5). Show that it is not a unique factorization domain. (The most obvious possibility is to do this "directly", but this might be hard. Another possibility, faster but less intuitive, is to prove the intermediate result that *in a unique factorization domain, any height* 1 *prime is principal*, and considering Exercise 1.)

5. Show that on a Noetherian scheme, you can check nonsingularity by checking at closed points. (Caution: a scheme in general needn't have any closed points!) You are allowed to use the unproved fact from the notes, that any localization of a regular local ring is regular.

6. Show that a nonsingular locally Noetherian scheme is irreducible if and only if it is connected. (I'm not sure if this fact requires Noetherianness.)

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7-. Show that there is a nonsingular hypersurface of degree d. Show that there is a Zariski-open subset of the space of hypersurfaces of degree d. The two previous sentences combine to show that the nonsingular hypersurfaces form a Zariski-open set. Translation: almost all hypersurfaces are smooth.

8-. Suppose $(R, \mathfrak{m}, k)$ is a regular Noetherian local ring of dimension n. Show that $\dim_k(\mathfrak{m}^i/\mathfrak{m}^{i+1}) = \binom{n+i-1}{i}$.

9. Show that fact 2 in the "good facts to know about regular local rings" implies that $(R, \mathfrak{m})$ is a domain. (Hint: show that if $f, g \neq 0$, then $fg \neq 0$, by considering the leading terms.)

Note that we have proved this fact (referred to in the previous problem) if $(R, \mathfrak{m})$ is a Noetherian local ring containing its residue field k. The next three exercises fill out the proof in the notes. Do them only if you are fairly happy with other things.

10. If S is a Noetherian ring, show that $S[[t]]$ is Noetherian. (Hint: Suppose $I \subset S[[t]]$ is an ideal. Let $I_n \subset S$ be the coefficients of t^n that appear in the elements of I form an ideal. Show that $I_n \subset I_{n+1}$, and that I is determined by $(I_0, I_1, I_2, \dots)$.)

11. Show that $\dim k[[t_1, \dots, t_n]]$ is dimension n. (Hint: find a chain of $n+1$ prime ideals to show that the dimension is at least n. For the other inequality, use Krull.)

12. If R is a Noetherian local ring, show that $\hat{R} := \lim_{\leftarrow} R/\mathfrak{m}^n$ is a Noetherian local ring. (Hint: Suppose $\mathfrak{m}/\mathfrak{m}^2$ is finite-dimensional over k, say generated by $x_1, \dots, x_n$. Describe a surjective map $k[[t_1, \dots, t_n]] \to \hat{R}$.)

13. Show that a section of a sheaf on the distinguished affine base is determined by the section's germs.

14+. Recall Theorem 4.2(a) in the class 13 notes, which states that a sheaf on the distinguished affine base $\mathcal{F}^b$ determines a unique sheaf $\mathcal{F}$, which when restricted to the affine base is $\mathcal{F}^b$. We defined

$$\mathcal{F}(U) := \{(f_x \in \mathcal{F}^b_x)_{x \in U} : \forall x \in U, \exists U_X \text{ with } x \subset U_x \subset U, F^x \in \mathcal{F}^b(U_x) : F^x_y = f_y \forall y \in U_x\}$$

where each U_x is in our base. In class I claimed that if U is in our base, that $\mathcal{F}(U) = \mathcal{F}^b(U)$. We clearly have a map $\mathcal{F}^b(U) \to \mathcal{F}(U)$. Prove that it is an isomorphism.

15+. Show that a sheaf of $\mathcal{O}_X$-modules on "the distinguished affine base" yields an $\mathcal{O}_X$-module.

Class 14:

16+. (a first example of the total complex of a double complex) Suppose $0 \to A \to B \to C$ is exact. Define the total complex

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle -\mathrm{id}} & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \end{array}$$

as

$$0 \to A \to A \oplus B \to B \oplus C$$

in the "obvious" way. Show that the total complex is also exact.

17. (a) Suppose $X = \operatorname{Spec} k[t]$. Let $\mathcal{F}$ be the skyscraper sheaf supported at the origin $[(t)]$, with group $k(t)$. Give this the structure of an $\mathcal{O}_X$-module. Show that this is not a quasicoherent sheaf. (More generally, if X is an integral scheme, and $p \in X$ that is not the generic point, we could take the skyscraper sheaf at p with group the function field of X. Except in a silly circumstances, this sheaf won't be quasicoherent.)
(b) Suppose $X = \operatorname{Spec} k[t]$. Let $\mathcal{F}$ be the skyscraper sheaf supported at the generic point $[(0)]$, with group $k(t)$. Give this the structure of an $\mathcal{O}_X$-module. Show that this *is* a quasicoherent sheaf. Describe the restriction maps in the distinguished topology of X.

18+. (Important Exercise for later) Suppose X is a Noetherian scheme. Suppose $\mathcal{F}$ is a quasicoherent sheaf on X, and let $f \in \Gamma(X, \mathcal{O}_X)$ be a function on X. Let $R = \Gamma(X, \mathcal{O}_X)$ for convenience. Show that the restriction map $\operatorname{res}_{X_f \subset X} : \Gamma(X, \mathcal{F}_X) \to \Gamma(X_f, \mathcal{F}_X)$ (here X_f is the open subset of X where f doesn't vanish) is precisely localization. In other words show that there is an isomorphism $\Gamma(X, \mathcal{F})_f \to \Gamma(X_f, \mathcal{F})$ making the following diagram commute.

$$\begin{array}{ccc} \Gamma(X, \mathcal{F}) & \xrightarrow{\operatorname{res}_{X_f \subset X}} & \Gamma(X_f, \mathcal{F}) \\ & {\scriptstyle \otimes_R R_f} \searrow \quad \nearrow {\scriptstyle \sim} & \\ & \Gamma(X, \mathcal{F})_f & \end{array}$$

All that you should need in your argument is that X admits a cover by a finite number of open sets, and that their pairwise intersections are each quasicompact. We will later rephrase this as saying that X is quasicompact and quasiseparated. (Hint: cover by affine open sets. Use the sheaf property. A nice way to formalize this is the following. Apply the exact functor $\otimes_R R_f$ to the exact sequence

$$0 \to \Gamma(X, \mathcal{F}) \to \oplus_i \Gamma(U_i, \mathcal{F}) \to \oplus \Gamma(U_{ijk}, \mathcal{F})$$

where the U_i form a finite cover of X and U_{ijk} form an affine cover of $U_i \cap U_j$.)

19-. Give a counterexample to show that the above statement need not hold if X is not quasicompact. (Possible hint: take an infinite disjoint union of affine schemes.)

20. (This is for arithmetically-minded people only — I won't define my terms.) Prove that a fractional ideal on a ring of integers in a number field yields an invertible sheaf. Show that any two that differ by a principal ideal yield the same invertible sheaf. (Thus we have described a map from the class group of the number field to the Picard group of its ring of integers. We will later see that this is an isomorphism.)

21+. Show that you can check exactness of a sequence of quasicoherent sheaves on an affine cover. (In particular, taking sections over an affine open $\operatorname{Spec} R$ is an exact functor from the category of quasicoherent sheaves on X to the category of R-modules. Recall that taking sections is only left-exact in general. Similarly, you can check surjectivity on an affine cover unlike sheaves in general.)

22+. If $\mathcal{F}$ and $\mathcal{G}$ are quasicoherent sheaves, show that $\mathcal{F} \otimes \mathcal{G}$ is given by the following information: If $\operatorname{Spec} R$ is an affine open, and $\Gamma(\operatorname{Spec} R, \mathcal{F}) = M$ and $\Gamma(\operatorname{Spec} R, \mathcal{G}) = N$, then $\Gamma(\operatorname{Spec} R, \mathcal{F} \otimes \mathcal{G}) = M \otimes N$, and the restriction map $\Gamma(\operatorname{Spec} R, \mathcal{F} \otimes \mathcal{G}) \to \Gamma(\operatorname{Spec} R_f, \mathcal{F} \otimes \mathcal{G})$ is precisely the localization map $M \otimes_R N \to (M \otimes_R N)_f \cong M_f \otimes_{R_f} N_f$. (We are using the algebraic fact that that $(M \otimes_R N)_f \cong M_f \otimes_{R_f} N_f$. You can prove this by universal property if you want, or by using the explicit construction.)

23. If $\mathcal{F}$ and $\mathcal{G}$ are locally free sheaves, show that $\mathcal{F} \otimes \mathcal{G}$ is locally free. (Possible hint for this, and later exercises: check on sufficiently small affine open sets.)

24. Prove the following.
(a) Tensoring by a quasicoherent sheaf is right-exact. More precisely, if $\mathcal{F}$ is a quasicoherent sheaf, and $\mathcal{G}' \to \mathcal{G} \to \mathcal{G}'' \to 0$ is an exact sequence of quasicoherent sheaves, then so is $\mathcal{G}' \otimes \mathcal{F} \to \mathcal{G} \otimes \mathcal{F} \to \mathcal{G}'' \otimes \mathcal{F} \to 0$ is exact.
(b) Tensoring by a locally free sheaf is exact. More precisely, if $\mathcal{F}$ is a quasicoherent sheaf, and $\mathcal{G}' \to \mathcal{G} \to \mathcal{G}''$ is an exact sequence of quasicoherent sheaves, then then so is $\mathcal{G}' \otimes \mathcal{F} \to \mathcal{G} \otimes \mathcal{F} \to \mathcal{G}'' \otimes \mathcal{F}$.
(c) The stalk of the tensor product of quasicoherent sheaves at a point is the tensor product of the stalks.
(d) Invertible sheaves on a scheme X (up to isomorphism) form a group. This is called the Picard group of X, and is denoted $\operatorname{Pic} X$. For arithmetic people: this group, for the Spec of the ring of integers R in a number field, is the class group of R.

25. Show that sheaf Hom, $\underline{\operatorname{Hom}}$, is quasicoherent, and is what you think it might be. (Describe it on affine opens, and show that it behaves well with respect to localization with respect to f. To show that $\operatorname{Hom}_A(M, N)_f \cong \operatorname{Hom}_{A_f}(M_f, N_f)$, take a "partial resolution" $A^q \to A^p \to M \to 0$, and apply $\operatorname{Hom}(\cdot, N)$ and localize.) ($\underline{\operatorname{Hom}}$ was defined earlier, and was the subject of a homework problem.) Show that $\underline{\operatorname{Hom}}$ is a left-exact functor in both variables.

26+. Show that if $\mathcal{F}$ is locally free then $\mathcal{F}^\vee$ is locally free, and that there is a canonical isomorphism $(\mathcal{F}^\vee)^\vee \cong \mathcal{F}$. (Caution: your argument showing that if there is a canonical isomorphism $(\mathcal{F}^\vee)^\vee \cong \mathcal{F}$ better not also show that there is a canonical isomorphism $\mathcal{F}^\vee \cong \mathcal{F}$! We'll see an example soon of a locally free $\mathcal{F}$ that is not isomorphic to its dual. The example will be the line bundle $\mathcal{O}(1)$ on $\mathbb{P}^1$.)

27. The direct sum of quasicoherent sheaves is what you think it is.

For the next exercises, recall the following. If M is an A-module, then the *tensor algebra* $T^*(M)$ is a non-commutative algebra, graded by $\mathbb{Z}^{\geq 0}$, defined as follows. $T^0(M) = A$, $T^n(M) = M \otimes_A \cdots \otimes_A M$ (where n terms appear in the product), and multiplication is what you expect. The *symmetric algebra* $\operatorname{Sym}^* M$ is a symmetric algebra, graded by $\mathbb{Z}^{\geq 0}$, defined as the quotient of $T^*(M)$ by the (two-sided) ideal generated by all elements of the form $x \otimes y - y \otimes x$ for all $x, y \in M$. Thus $\mathrm{Sym}^n M$ is the quotient of $M \otimes \cdots \otimes M$ by the relations of the form $m_1 \otimes \cdots \otimes m_n - m_1' \otimes \cdots \otimes m_n'$ where $(m_1', \ldots, m_n')$ is a rearrangement of $(m_1, \ldots, m_n)$. The *exterior algebra* $\wedge^* M$ is defined to be the quotient of T^*M by the (two-sided) ideal generated by all elements of the form $x \otimes y + y \otimes x$ for all $x, y \in M$. Thus $\wedge^n M$ is the quotient of $M \otimes \cdots \otimes M$ by the relations of the form $m_1 \otimes \cdots \otimes m_n - (-1)^{\mathrm{sgn}} m_1' \otimes \cdots \otimes m_n'$ where $(m_1', \ldots, m_n')$ is a rearrangement of $(m_1, \ldots, m_n)$, and the sgn is even if the rearrangement is an even permutation, and odd if the rearrangement is an odd permutation. (It is a "skew-commutative" A-algebra.) It is most correct to write $T^*_A(M)$, $\mathrm{Sym}^*_A(M)$, and $\wedge^*_A(M)$, but the "base ring" is usually omitted for convenience.

28. If $\mathcal{F}$ is a quasicoherent sheaf, then define the quasicoherent sheaves $T^n\mathcal{F}$, $\mathrm{Sym}^n \mathcal{F}$, and $\wedge^n\mathcal{F}$. If $\mathcal{F}$ is locally free of rank m, show that $T^n\mathcal{F}$, $\mathrm{Sym}^n \mathcal{F}$, and $\wedge^n\mathcal{F}$ are locally free, and find their ranks.

29+. If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of locally free sheaves, then for any r, there is a filtration of $\mathrm{Sym}^r \mathcal{F}$:

$$\mathrm{Sym}^r \mathcal{F} = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^r \supset F^{r+1} = 0$$

with quotients

$$F^p / F^{p+1} \cong (\mathrm{Sym}^p \mathcal{F}') \otimes (\mathrm{Sym}^{r-p} \mathcal{F}'')$$

for each p.

30. Suppose $\mathcal{F}$ is locally free of rank n. Then $\wedge^n\mathcal{F}$ is called the *determinant (line) bundle*. Show that $\wedge^r\mathcal{F} \times \wedge^{n-r}\mathcal{F} \to \wedge^n\mathcal{F}$ is a perfect pairing for all r.

31+. If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of locally free sheaves, then for any r, there is a filtration of $\wedge^r\mathcal{F}$:

$$\wedge^r\mathcal{F} = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^r \supset F^{r+1} = 0$$

with quotients

$$F^p / F^{p+1} \cong (\wedge^p \mathcal{F}') \otimes (\wedge^{r-p} \mathcal{F}'')$$

for each p. In particular, $\det \mathcal{F} = (\det \mathcal{F}') \otimes (\det \mathcal{F}'')$.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 7

RAVI VAKIL

This set is due Wednesday, December 7. It covers (roughly) classes 15 and 16.

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in six solutions, *including # 23*. If you are ambitious (and have the time), go for more. Problems marked with "-" count for half a solution. Problems marked with "+" may be harder or more fundamental, but still count for one solution. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. I'm happy to give hints, and some of these problems require hints!

Class 15:

You are not allowed to try the next four problems if you already know how to do them!

1. M Noetherian implies that any submodule of M is a finitely generated R-module. Hence for example if R is a Noetherian ring then finitely generated = Noetherian.

2. If $0 \to M' \to M \to M'' \to 0$ is exact, then M' and M'' are Noetherian if and only if M is Noetherian. (Hint: Given an ascending chain in M, we get two simultaneous ascending chains in M' and M''.)

3. A Noetherian as an A-module implies A^n is a Noetherian A-module.

4. If A is a Noetherian ring and M is a finitely generated A-module, then any submodule of M is finitely generated. (Hint: suppose $M' \hookrightarrow M$ and $A^n \twoheadrightarrow M$. Construct N with

$$\begin{array}{ccc} N & \hookrightarrow & A^n \\ \downarrow & & \downarrow \\ M' & \hookrightarrow & M \end{array}$$

.)

5-. Show A is coherent (as an A-module) if and only if the notion of finitely presented agrees with the notion of coherent.

6. If $f \in A$, show that if M is a finitely generated (resp. finitely presented, coherent) A-module, then M_f is a finitely generated (resp. finitely presented, coherent) A_f-module. (Hint: localization is exact.)

7. If $(f_1, \ldots, f_n) = A$, and M_{f_i} is finitely generated (resp. coherent) A_{f_i}-module for all i, then M is a finitely generated (resp. coherent) A-module.

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8. *(Exercise on support of a sheaf)* Show that the support of a finite type quasicoherent sheaf on a scheme is a closed subset. (Hint: Reduce to an affine open set. Choose a finite set of generators of the corresponding module.) Show that the support of a quasicoherent sheaf need not be closed. (Hint: If $A = \mathbb{C}[t]$, then $\mathbb{C}[t]/(t-a)$ is an A-module supported at a. Consider $\oplus_{a \in \mathbb{C}} \mathbb{C}[t]/(t-a)$.)

9. *(Exercise on rank)*

(a) If $m_1, \ldots, m_n$ are generators at P, they are generators in an open neighborhood of P. (Hint: Consider $\operatorname{coker} A^n \xrightarrow{(f_1,\ldots,f_n)} M$ and Exercise 8.)

(b) Show that at any point, $\operatorname{rank}(\mathcal{F} \oplus \mathcal{G}) = \operatorname{rank}(\mathcal{F}) + \operatorname{rank}(\mathcal{G})$ and $\operatorname{rank}(\mathcal{F} \otimes \mathcal{G}) = \operatorname{rank} \mathcal{F} \operatorname{rank} \mathcal{G}$ at any point. (Hint: Show that direct sums and fibered products commute with ring quotients and localizations, i.e. $(M \oplus N) \otimes_R (R/I) \cong M/IM \oplus N/IN$, $(M \otimes_R N) \otimes_R (R/I) \cong (M \otimes_R R/I) \otimes_{R/I} (N \otimes_R R/I) \cong M/IM \otimes_{R/I} N/IM$, etc.) Thanks to Jack Hall for improving this problem.

(c) Show that rank is an upper semicontinuous function on X. (Hint: Generators at P are generators nearby.)

10. If X is reduced, $\mathcal{F}$ is coherent, and the rank is constant, show that $\mathcal{F}$ is locally free. (Hint: choose a point $p \in X$, and choose generators of the stalk $\mathcal{F}_p$. Let U be an open set where the generators are sections, so we have a map $\phi : \mathcal{O}_U^{\oplus n} \to \mathcal{F}|_U$. The cokernel and kernel of ϕ are supported on closed sets by Exercise 8. Show that these closed subsets don't include p. Make sure you use the reduced hypothesis!) Thus coherent sheaves are locally free on a dense open set. Show that this can be false if X is not reduced. (Hint: $\operatorname{Spec} k[x]/x^2$, $M = k$.)

11. *(Geometric Nakayama)* Suppose X is a scheme, and $\mathcal{F}$ is a finite type quasicoherent sheaf. Show that if $\mathcal{F}_x \otimes k(x) = 0$, then there exists V such that $\mathcal{F}|_V = 0$. Better: if I have a set that generates the fiber, it defines the stalk.

12. *(Reason for the name "invertible" sheaf)* Suppose $\mathcal{F}$ and $\mathcal{G}$ are finite type sheaves such that $\mathcal{F} \otimes \mathcal{G} \cong \mathcal{O}_X$. Then $\mathcal{F}$ and $\mathcal{G}$ are both invertible (Hint: Nakayama.) This is the reason for the adjective "invertible" these sheaves are the invertible elements of the monoid of finite type sheaves. This exercise is a little less important.

13. *(A non-quasicoherent sheaf of ideals)* Let $X = \operatorname{Spec} k[x]_{(x)}$, the germ of the affine line at the origin, which has two points, the closed point and the generic point η. Define $\mathcal{I}(X) = \{0\} \subset \mathcal{O}_X(X) = k[x]_{(x)}$, and $\mathcal{I}(\eta) = k(x) = \mathcal{O}_X(\eta)$. Show that $\mathcal{I}$ is not a quasicoherent sheaf of ideals.

14. *(Sections of locally free sheaves cut out closed subschemes)* Suppose $\mathcal{F}$ is a locally free sheaf on a scheme X, and s is a section of $\mathcal{F}$. Describe how $s = 0$ "cuts out" a closed subscheme.

15. *(Reduction of a scheme)*

(a) X^{red} has the same underlying topological space as X: there is a natural homeomorphism of the underlying topological spaces $X^{red} \cong X$. Picture: taking the reduction may be interpreted as shearing off the fuzz on the space.

(b) Give an example to show that it is *not* true that $\Gamma(X^{red}, \mathcal{O}_{X^{red}}) = \Gamma(X, \mathcal{O}_X)/\sqrt{\Gamma(X, \mathcal{O}_X)}$. (Hint: $\coprod_{n>0} \operatorname{Spec} k[t]/(t^n)$ with global section $(t, t, t, \dots)$.) Motivation for this exercise: this *is* true on each affine open set.

Class 16:

16. Describe the scheme-theoretic intersection of $(y - x^2)$ and y in $\mathbb{A}^2$. Draw a picture.

17. Suppose we have an effective Cartier divisor, a closed subscheme locally cut out by a single equation. As described in class, this gives an invertible sheaf with a canonical section. Show that this section vanishes along our actual effective Cartier divisor.

18. Describe the invertible sheaf corresponding to an effective Cartier divisor in terms of transition functions. More precisely, on any affine open set where the effective Cartier divisor is cut out by a single equation, the invertible sheaf is trivial. Determine the transition functions between two such overlapping affine open sets. Verify that there is indeed a canonical section of this invertible sheaf, by describing it.

19. Show that $\widetilde{M_*} \otimes \widetilde{N_*} \cong \widetilde{M_* \otimes_{S_*} N_*}$. (Hint: describe the isomorphism of sections over each $D(f)$, and show that this isomorphism behaves well with respect to smaller distinguished opens.)

20. *(Closed immersions in projective S_0-schemes)* Show that if I_* is a graded ideal of S_*, show that we get a closed immersion $\operatorname{Proj} S_*/I_* \hookrightarrow \operatorname{Proj} S_*$.

21. Suppose S_* is generated over S_0 by $f_1, \dots, f_n$. Suppose $d = \operatorname{lcm}(\deg f_1, \dots, \deg f_n)$. Show that S_{d*} is generated in "new" degree 1 (= "old" degree d). (Hint: I like to show this by induction on the size of the set $\{\deg f_1, \dots, \deg f_n\}$.) This is handy, because we can stick every Proj in some projective space via the construction of previous exercise.

22. If S_* is generated in degree 1, show that $\mathcal{O}_{\operatorname{Proj} S_*}(n)$ is an invertible sheaf.

23. (Mandatory exercise — I am happy to walk you through it, see the notes.) Calculate $\dim_k \Gamma(\mathbb{P}^m_k, \mathcal{O}_{\mathbb{P}^m_k}(n))$.

24. Show that $\mathcal{F}(n) \cong \mathcal{F} \otimes \mathcal{O}(n)$.

25. Show that $\mathcal{O}(m+n) \cong \mathcal{O}(m) \otimes \mathcal{O}(n)$.

26. Show that if $m \neq n$, then $\mathcal{O}_{\mathbb{P}^l_k}(m)$ is not isomorphic to $\mathcal{O}_{\mathbb{P}^l_k}(n)$ if $l > 0$. (Hence we have described a countable number of invertible sheaves (line bundles) that are non-isomorphic. We will see later that these are *all* the line bundles on projective space $\mathbb{P}^n_k$.)

27. If quasicoherent sheaves $\mathcal{L}$ and $\mathcal{M}$ are generated by global sections at a point p, then so is $\mathcal{L} \otimes \mathcal{M}$. (This exercise is less important, but is good practice for the concept.)

28. An invertible sheaf $\mathcal{L}$ on X is generated by global sections if and only if for any point $x \in X$, there is a section of $\mathcal{L}$ not vanishing at x. (Hint: Nakayama.)

29+. (Important! A theorem of Serre. See the notes for extensive hints.) Suppose S_0 is a Noetherian ring, and S_* is generated in degree 1. Let $\mathcal{F}$ be any finite type sheaf on $\operatorname{Proj} S_*$. Then for some integer n_0, for all $n \geq n_0$, $\mathcal{F}(n)$ can be generated by a finite number of global sections.

30. Show that Γ_* gives a functor from the category of quasicoherent sheaves on $\operatorname{Proj} S_*$ to the category of graded S_*-modules. (In other words, show that if $\mathcal{F} \to \mathcal{G}$ is a morphism of quasicoherent sheaves on $\operatorname{Proj} S_*$, describe the natural map $\Gamma_*(\mathcal{F}) \to \Gamma_*(\mathcal{G})$, and show that such natural maps respect the identity and composition.)

31. Show that the canonical map $M_* \to \Gamma_* \widetilde{M_*}$ need not be injective, nor need it be surjective. (Hint: $S_* = k[x]$, $M_* = k[x]/x^2$ or $M_* = \{$ polynomials with no constant terms $\}$.)

32. Describe the natural map $\widetilde{\Gamma_* \mathcal{F}} \to \mathcal{F}$ as follows. First describe it over $D(f)$. Note that sections of the left side are of the form m/f^n where $m \in \Gamma_{n \deg f} \mathcal{F}$, and $m/f^n = m'/f^{n'}$ if there is some N with $f^N(f^{n'} m - f^n m') = 0$. Show that your map behaves well on overlaps $D(f) \cap D(g) = D(fg)$.

33+. Show that the natural map $\widetilde{\Gamma_* \mathcal{F}} \to \mathcal{F}$ is an isomorphism, by showing that it is an isomorphism over $D(f)$ for any f. Do this by first showing that it is surjective. This will require following some of the steps of the proof of Serre's theorem (a previous exercise on this set). Then show that it is injective. (This is longer, but worth it.)

34. *(Γ_* and $\sim$ are adjoint functors)* Prove part of the statement that Γ_* and $\sim$ are adjoint functors, by describing a natural bijection $\operatorname{Hom}(M_*, \Gamma_*(\mathcal{F})) \cong \operatorname{Hom}(\widetilde{M_*}, \mathcal{F})$. For the map from left to right, start with a morphism $M_* \to \Gamma_*(\mathcal{F})$. Apply $\sim$, and postcompose with the isomorphism $\widetilde{\Gamma_* \mathcal{F}} \to \mathcal{F}$, to obtain

$$\widetilde{M_*} \to \widetilde{\Gamma_* \mathcal{F}} \to \mathcal{F}.$$

Do something similar to get from right to left. Show that "both compositions are the identity in the appropriate category". (Is there a clever way to do that?)

Coherence: These twenty problems are only for people who are curious about the notion of coherence for general rings. Others should just skip these. (This is the one exception of my injunction to read all problems.)

A. Show that coherent implies finitely presented implies finitely generated.

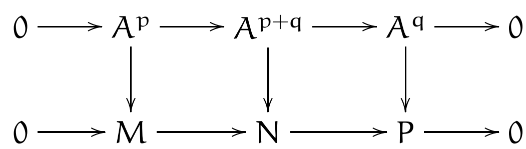
B. Show that 0 is coherent.

Suppose for problems C–I that

$$0 \to M \to N \to P \to 0 \tag{1}$$

is an exact sequence of A-modules.

Hint ⋆. Here is a *hint* which applies to several of the problems: try to write



and possibly use the snake lemma.

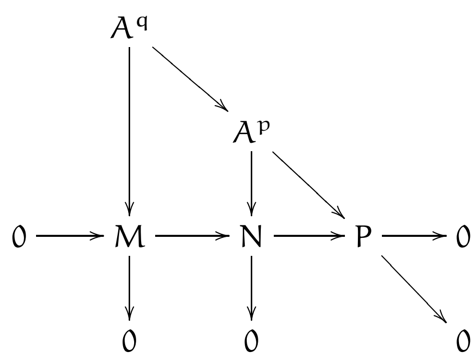
C. Show that N finitely generated implies P finitely generated. (You will only need right-exactness of (1).)

D. Show that M, P finitely generated implies N finitely generated. (Possible hint: ⋆.) (You will only need right-exactness of (1).)

E. Show that N, P finitely generated need not imply M finitely generated. (Hint: if I is an ideal, we have $0 \to I \to A \to A/I \to 0$.)

F. Show that N coherent, M finitely generated implies M coherent. (You will only need left-exactness of (1).)

G. Show that N, P coherent implies M coherent. Hint for (i) in the definition of coherence:



(You will only need left-exactness of (1).)

H. Show that M finitely generated and N coherent implies P coherent. (Hint for (ii) in the definition of coherence: ⋆. You will only right-exactness of (1).)

I. Show that M, P coherent implies N coherent. (Hint: ⋆.)

At this point, we have shown that if two of (1) are coherent, the third is as well.

J. Show that a finite direct sum of coherent modules is coherent.

K. Suppose M is finitely generated, N coherent. Then if $\phi : M \to N$ is any map, then show that $\operatorname{Im} \phi$ is coherent.

L. Show that the kernel and cokernel of maps of coherent modules are coherent.

At this point, we have verified that coherent A-modules form an abelian subcategory of the category of A-modules. (Things you have to check: 0 should be in this set; it should be closed under finite sums; and it should be closed under taking kernels and cokernels.)

M. Suppose M and N are coherent submodules of the coherent module P. Show that $M + N$ and $M \cap N$ are coherent. (Hint: consider the right map $M \oplus N \to P$.)

N. Show that if A is coherent (as an A-module) then finitely presented modules are coherent. (Of course, if finitely presented modules are coherent, then A is coherent, as A is finitely presented!) (This is also # 5.)

O. If M is finitely presented and N is coherent, show that $\mathrm{Hom}(M, N)$ is coherent. (Hint: Hom is left-exact in its first entry.)

P. If M is finitely presented, and N is coherent, show that $M \otimes N$ is coherent.

Q. If $f \in A$, show that if M is a finitely generated (resp. finitely presented, coherent) A-module, then M_f is a finitely generated (resp. finitely presented, coherent) A_f-module. Hint: localization is exact. (This is also # 6.)

R. Suppose $(f_1, \dots, f_n) = A$. Show that if M_{f_i} is finitely generated for all i, then M is too. (Hint: Say M_{f_i} is generated by $m_{ij} \in M$ as an A_{f_i}-module. Show that the m_{ij} generate M. To check surjectivity $\oplus_{i,j} A \to M$, it suffices to check "on $D(f_i)$" for all i.) (This is half of # 7.)

S. Suppose $(f_1, \dots, f_n) = A$. Show that if M_{f_i} is coherent for all i, then M is too. (Hint from Rob Easton: if $\phi : A^2 \to M$, then $(\ker \phi)_{f_i} = \ker(\phi_{f_i})$, which is finitely generated for all i. Then apply the previous exercise.) (This is the other half of # 7.)

T. Show that the ring $A := k[x_1, x_2, \dots]$ is not coherent over itself. (Hint: consider $A \to A$ with $x_1, x_2, \dots \mapsto 0$.) Thus we have an example of a finitely presented module that is not coherent; a surjection of finitely presented modules whose kernel is not even finitely generated; hence an example showing that finitely presented modules don't form an abelian category.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 8

RAVI VAKIL

This set is due Wednesday, December 14, in my mailbox. (I will accept it, and other older sets, until Friday, December 16. That will likely be a hard deadline, because I may not be around to pick up any later sets.) It covers (roughly) classes 17 and 18.

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in four solutions. If you are ambitious (and have the time), go for more. Problems marked with "-" count for half a solution. Problems marked with "+" may be harder or more fundamental, but still count for one solution. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. I'm happy to give hints, and some of these problems require hints!

Class 17:

1. Show that if $\mathfrak{q}$ is primary, then $\sqrt{\mathfrak{q}}$ is prime.

2-. Show that if $\mathfrak{q}$ and $\mathfrak{q}'$ are $\mathfrak{p}$-primary, then so is $\mathfrak{q} \cap \mathfrak{q}'$.

3-. (reality check) Find all the primary ideals in $\mathbb{Z}$.

4+. Suppose A is a Noetherian ring. Show that every proper ideal $I \neq A$ has a primary decomposition. (Hint: Noetherian induction.)

5. Find a minimal primary decomposition of (x^2, xy).

6+. (a) If $\mathfrak{p}$, $\mathfrak{p}_1$, ..., $\mathfrak{p}_n$ are prime ideals, and $\mathfrak{p} = \cap \mathfrak{p}_i$, show that $\mathfrak{p} = \mathfrak{p}_i$ for some i. (Hint: assume otherwise, choose $f_i \in \mathfrak{p}_i - \mathfrak{p}$, and consider $\prod f_i$.)
(b) If $\mathfrak{p} \supset \cap \mathfrak{p}_i$, then $\mathfrak{p} \supset \mathfrak{p}_i$ for some i.
(c) Suppose $I \subseteq \cup^n \mathfrak{p}_i$. Show that $I \subset \mathfrak{p}_i$ for some i. (Hint: by induction on n.)

7. Show that these associated primes behave well with respect to localization. In other words if A is a Noetherian ring, and S is a multiplicative subset (so, as we've seen, there is an inclusion-preserving correspondence between the primes of $S^{-1}A$ and those primes of A not meeting S), then the associated primes of $S^{-1}A$ are just the associated primes of A not meeting S.

8. Show that the minimal primes of 0 are associated primes. (We have now proved important fact **(1)**.) (Hint: suppose $\mathfrak{p} \supset \cap_{i=1}^n \mathfrak{q}_i$. Then $\mathfrak{p} = \sqrt{\mathfrak{p}} \supset \sqrt{\cap_{i=1}^n \mathfrak{q}_i} = \cap_{i=1}^n \sqrt{\mathfrak{q}_i} = \cap_{i=1}^n \mathfrak{p}_i$, so by Exercise 6(b), $\mathfrak{p} \supset \mathfrak{p}_i$ for some i. If $\mathfrak{p}$ is minimal, then as $\mathfrak{p} \supset \mathfrak{p}_i \supset (0)$, we must have

Date: Monday, December 9, 2005. One-character update December 19.

$\mathfrak{p} = \mathfrak{p}_i$.) Show that there can be other associated primes that are not minimal. (Hint: see an earlier exercise.)

9. Show that if A is reduced, then the only associated primes are the minimal primes.

10. Verify the inclusions and equalities

$$D = \cup_{x \neq 0}(0 : x) \subseteq \cup_{x \neq 0}\sqrt{(0 : x)} \subseteq D.$$

11. Suppose f and g are two global sections of a Noetherian normal scheme with the same poles and zeros. Show that each is a unit times the other.

Class 18:

12. If $W \subset X$ and $Y \subset Z$ are both open immersions of ringed spaces, show that any morphism of ringed spaces $X \to Y$ induces a morphism of ringed spaces $W \to Z$.

13. Show that morphisms of ringed spaces glue. In other words, suppose X and Y are ringed spaces, $X = \cup_i U_i$ is an open cover of X, and we have morphisms of ringed spaces $f_i : U_i \to Y$ that "agree on the overlaps", i.e. $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$. Show that there is a unique morphism of ringed spaces $f : X \to Y$ such that $f|_{U_i} = f_i$. (Long ago we had an exercise proving this for topological spaces.)

14. (Easy but important) Given a morphism of ringed spaces $f : X \to Y$ with $f(p) = q$, show that there is a map of stalks $(\mathcal{O}_Y)_q \to (\mathcal{O}_X)_p$.

15. If $f^\# : S \to R$ is a morphism of rings, verify that $R_{f^\# s} \cong R \otimes_S S_s$.

16. Show that morphisms of locally ringed spaces glue (Hint: Basically, the proof of the corresponding exercise for ringed spaces works.)

17+ (easy but important) (a) Show that $\operatorname{Spec} R$ is a locally ringed space. (b) The morphism of ringed spaces $f : \operatorname{Spec} R \to \operatorname{Spec} S$ defined by a ring morphism $f^\# S \to R$ is a morphism of locally ringed spaces.

18++ (Important practice!) Make sense of the following sentence: "$\mathbb{A}^{n+1} - \vec{0} \to \mathbb{P}^n$ given by $(x_0, x_1, \dots, x_{n+1}) \mapsto [x_0; x_1; \dots; x_n]$ is a morphism of schemes." Caution: you can't just say where points go; you have to say where functions go. So you'll have to divide these up into affines, and describe the maps, and check that they glue.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 9

RAVI VAKIL

This set is due Tuesday, January 24, in Jarod Alper's mailbox. It covers (roughly) classes 19 through 22. (This is a long one, because I'm giving you the option of doing some problems from the end of last quarter.)

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in four solutions. If you are ambitious (and have the time), go for more. Problems marked with "-" count for half a solution. Problems marked with "+" may be harder or more fundamental, but still count for one solution. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. I'm happy to give hints, and some of these problems require hints!

Class 19:

1+. Show that morphisms $X \to \operatorname{Spec} A$ are in natural bijection with ring morphisms $A \to \Gamma(X, \mathcal{O}_X)$. (Hint: Show that this is true when X is affine. Use the fact that morphisms glue.)

2. Show that $\operatorname{Spec} \mathbb{Z}$ is the final object in the category of schemes. In other words, if X is any scheme, there exists a unique morphism to $\operatorname{Spec} \mathbb{Z}$. (Hence the category of schemes is isomorphic to the category of $\mathbb{Z}$-schemes.)

3. Show that morphisms $X \to \operatorname{Spec} \mathbb{Z}[t]$ correspond to global sections of the structure sheaf.

4. Show that global sections of $\mathcal{O}_X^*$ correspond naturally to maps to $\operatorname{Spec} \mathbb{Z}[t, t^{-1}]$. ($\operatorname{Spec} \mathbb{Z}[t, t^{-1}]$ is a *group scheme*.)

5+. Suppose X is a finite type k-scheme. Describe a natural bijection $\operatorname{Hom}(\operatorname{Spec} k[\epsilon]/\epsilon^2, X)$ to the data of a k-valued point (a point whose residue field is k, necessarily closed) and a tangent vector at that point.

6. Suppose $i : U \to Z$ is an open immersion, and $f : Y \to Z$ is any morphism. Show that $U \times_Z Y$ exists. (Hint: I'll even tell you what it is: $(f^{-1}(U), \mathcal{O}_Y|_{f^{-1}(U)})$.)

7-. Show that open immersions are monomorphisms.

8+. Suppose $Y \to Z$ is a closed immersion, and $X \to Z$ is any morphism. Show that the fibered product $X \times_Y Z$ exists, by explicitly describing it. Show that the projection $X \times_Y Z \to Y$ is a closed immersion. We say that "closed immersions are preserved by

Date: Monday, January 16, 2005. Updated January 23. Minor update January 25.

base change" or "closed immersions are preserved by fibered product". (Base change is another word for fibered products.)

9. Show that closed immersions are monomorphisms.

10. *(quasicompactness is affine-local on the target)* Show that a morphism $f : X \to Y$ is quasicompact if there is cover of Y by open affine sets U_i such that $f^{-1}(U_i)$ is quasicompact. (Hint: affine communication lemma!)

11. Show that the composition of two quasicompact morphisms is quasicompact.

12. *(the notions "locally of finite type" and "finite type" is affine-local on the target)* Show that a morphism $f : X \to Y$ is locally of finite type if there is a cover of Y by open affine sets $\operatorname{Spec} R_i$ such that $f^{-1}(\operatorname{Spec} R_i)$ is locally of finite type over R_i.

13-. Show that a closed immersion is a morphism of finite type.

14-. Show that an open immersion is locally of finite type. Show that an open immersion into a Noetherian scheme is of finite type. More generally, show that a quasicompact open immersion is of finite type.

15-. Show that a composition of two morphisms of finite type is of finite type.

16. Suppose we have a composition of morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$, where f is quasicompact, and $g \circ f$ is finite type. Show that f is finite type.

17-. Suppose $f : X \to Y$ is finite type, and Y is Noetherian. Show that X is also Noetherian.

18. Suppose X is an affine scheme, and Y is a closed subscheme locally cut out by one equation (e.g. if X is an effective Cartier divisor). Show that $X - Y$ is affine. (This is clear if Y is globally cut out by one equation f; then if $X = \operatorname{Spec} R$ then $Y = \operatorname{Spec} R_f$. However, this is not always true.) Hint: affine locality of the notion of "affine morphism".

19. Here is an explicit consequence of the previous exercise. We showed earlier that on the cone over the smooth quadric surface $\operatorname{Spec} k[w, x, y, z]/(wz - xy)$, the cone over a ruling $w = x = 0$ is not cut out scheme-theoretically by a single equation, by considering Zariski-tangent spaces. We now show that it isn't even cut out set-theoretically by a single equation. For if it were, its complement would be affine. But then the closed subscheme of the complement cut out by $y = z = 0$ would be affine. But this is the scheme $y = z = 0$ (also known as the wx-plane) minus the point $w = x = 0$, which we've seen is non-affine. (For comparison, on the cone $\operatorname{Spec} k[x, y, z]/(xy - z^2)$, the ruling $x = z = 0$ is not cut out scheme-theoretically by a single equation, but it *is* cut out set-theoretically by $x = 0$.) Verify all of this!

20. *(the property of finiteness is affine-local on the target)* Show that a morphism $f : X \to Y$ is finite if there is a cover of Y by open affine sets $\operatorname{Spec} R$ such that $f^{-1}(\operatorname{Spec} R)$ is the spectrum of a finite R-algebra. (Hint: Use that $f_*\mathcal{O}_X$ is finite type.)

21-. Show that closed immersions are finite morphisms.

22. (a) Show that if a morphism is finite then it is quasifinite. (b) Show that the converse is not true. (Hint: $\mathbb{A}^1 - \{0\} \to \mathbb{A}^1$.)

23. Suppose X is a Noetherian scheme. Show that a subset of X is constructable if and only if it is the finite disjoint union of locally closed subsets.

24-. Show that the image of an irreducible scheme is irreducible.

Class 20:

25. Let $f : \operatorname{Spec} A \to \operatorname{Spec} B$ be a morphism of affine schemes, and suppose M is an A-module, so $\tilde{M}$ is a quasicoherent sheaf on $\operatorname{Spec} A$. Show that $f_* \tilde{M} \cong \widetilde{M_B}$. (Hint: There is only one reasonable way to proceed: look at distinguished opens!)

26. Give an example of a morphism of schemes $\pi : X \to Y$ and a quasicoherent sheaf $\mathcal{F}$ on X such that $\pi_* \mathcal{F}$ is not quasicoherent. (Answer: $Y = \mathbb{A}^1$, $X =$ countably many copies of $\mathbb{A}^1$. Then let $f = t$. X_t has a global section $(1/t, 1/t^2, 1/t^3, \ldots)$. The key point here is that infinite direct sums do not commute with localization.)

27. Suppose $f : X \to Y$ is a finite morphism of Noetherian schemes. If $\mathcal{F}$ is a coherent sheaf on X, show that $f_* \mathcal{F}$ is a coherent sheaf. (Hint: Show first that $f_* \mathcal{O}_X$ is finite type = locally finitely generated.)

28. Verify that the following is an example showing that pullback is not left-exact: consider the exact sequence of sheaves on $\mathbb{A}^1$, where p is the origin:

$$0 \to \mathcal{O}_{\mathbb{A}^1}(-p) \to \mathcal{O}_{\mathbb{A}^1} \to \mathcal{O}_p \to 0.$$

(This is a closed subscheme exact sequence; also an effective Cartier exact sequence. Algebraically, we have $k[t]$-modules $0 \to tk[t] \to k[t] \to k \to 0$.) Restrict to p.

Class 21:

29. The notion of integral morphism is well behaved with localization and quotient of B, and quotient of A (but not localization of A, witness $k[t] \to k[t]$, but $k[t] \to k[t]_{(t)}$). The notion of integral extension is well behaved with respect to localization and quotient of B, but not quotient of A (same example, $k[t] \to k[t]/(t)$).

30+. (a) Show that if B is an integral extension of A, and C is an integral extension of B, then C is an integral extension of A.
(b) Show that if B is a finite extension of A, and C is a finite extension of B, then C is an finite extension of A.

31-. Show that the special case of the going-up theorem where A is a field translates to: if $B \subset A$ is a subring with A integral over B, then B is a field. Prove this. (Hint: all you need to do is show that all nonzero elements in B have inverses in B. Here is the start: If $b \in B$, then $1/b \in A$, and this satisfies some integral equation over B.)

32+. *(sometimes also called the going-up theorem)* Show that if $\mathfrak{q}_1 \subset \mathfrak{q}_2 \subset \cdots \subset \mathfrak{q}_n$ is a chain of prime ideals of B, and $\mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_m$ is a chain of prime ideals of A such that $\mathfrak{p}_i$ "lies over" $\mathfrak{q}_i$ (and $m < n$), then the second chain can be extended to $\mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n$ so that this remains true.

33+. Show that if $f : \operatorname{Spec} A \to \operatorname{Spec} B$ corresponds to an integral *extension of rings*, then $\dim \operatorname{Spec} A = \dim \operatorname{Spec} B$.

34. Show that finite morphisms are *closed*, i.e. the image of any closed subset is closed.

35. Show that integral ring extensions induce a surjective map of spectra.

36. Suppose X is a Noetherian scheme. Show that a subset of X is constructable if and only if it is the finite disjoint union of locally closed subsets. (This is admittedly the same as 23.)

37. Show that a dominant morphism of integral schemes $X \to Y$ induces an inclusion of function fields in the other direction.

38. If $\phi : A \to B$ is a ring morphism, show that the corresponding morphism of affine schemes $\operatorname{Spec} B \to \operatorname{Spec} A$ is dominant iff ϕ has nilpotent kernel.

39+. Reduce the proof of Chevalley's theorem to the following case: suppose $f : X = \operatorname{Spec} A \to Y = \operatorname{Spec} B$ is a dominant morphism, where A and B are domains, and f corresponds to $\phi : B \to B[x_1, \ldots, x_n]/I \cong A$. Show that the image of f contains a dense open subset of $\operatorname{Spec} B$. (See the class notes.)

Class 22:

40. Let $\phi : X \to \mathbb{P}^n_A$ be a morphism of A-schemes, corresponding to an invertible sheaf $\mathcal{L}$ on X and sections $s_0, \ldots, s_n \in \Gamma(X, \mathcal{L})$ as above. Then ϕ is a closed immersion iff (1) each open set $X_i = X_{s_i}$ is affine, and (2) for each i, the map of rings $A[y_0, \ldots, y_n] \to \Gamma(X_i, \mathcal{O}_{X_i})$ given by $y_j \mapsto s_j/s_i$ is surjective.

41. *(Automorphisms of projective space)* Show that all the automorphisms of projective space $\mathbb{P}^n_k$ correspond to $(n+1)\times(n+1)$ invertible matrices over k, modulo scalars (also known as $PGL_{n+1}(k)$). (Hint: Suppose $f : \mathbb{P}^n_k \to \mathbb{P}^n_k$ is an automorphism. Show that $f^*\mathcal{O}(1) \cong \mathcal{O}(1)$. Show that $f^* : \Gamma(\mathbb{P}^n, \mathcal{O}(1)) \to \Gamma(\mathbb{P}^n, \mathcal{O}(1))$ is an isomorphism.)

42. Show that any map from projective space to a smaller projective space is constant. (Fun!)

43. Prove that $\mathbb{A}^n_R \cong \mathbb{A}^n_{\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} R$. Prove that $\mathbb{P}^n_R \cong \mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} R$.

44. Show that for finite-type schemes over $\mathbb{C}$, the complex-valued points of the fibered product correspond to the fibered product of the complex-valued points. (You will just use the fact that $\mathbb{C}$ is algebraically closed.)

45-. Describe $\operatorname{Spec} \mathbb{C} \times_{\operatorname{Spec} \mathbb{R}} \operatorname{Spec} \mathbb{C}$.

46. Consider the morphism of schemes $X = \operatorname{Spec} k[t] \to Y = \operatorname{Spec} k[u]$ corresponding to $k[u] \to k[t]$, $u = t^2$ (where the characteristic of k is not 2). Show that $X \times_Y X$ has 2 irreducible components. Compare what is happening above the generic point of Y to the previous exercise.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 10

RAVI VAKIL

This set is due Thursday, February 2, in Jarod Alper's mailbox. It covers (roughly) classes 23 and 24.

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in six solutions. If you are ambitious (and have the time), go for more. Problems marked with "-" count for half a solution. Problems marked with "+" may be harder or more fundamental, but still count for one solution. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. I'm happy to give hints, and some of these problems require hints!

0. Here is something I would like to see worked out. Show that the points of $\operatorname{Spec} \overline{\mathbb{Q}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ are in natural bijection with $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, and the Zariski topology on the former agrees with the profinite topology on the latter.

Class 23:

1-. Show that for the morphism $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{R}$, all geometric fibers consist of two reduced points.

2+. Show that the notion of "morphism locally of finite type" is preserved by base change. Show that the notion of "affine morphism" is preserved by base change. Show that the notion of "finite morphism" is preserved by base change.

3+. Show that the notion of "morphism of finite type" is preserved by base change.

4. Show that the notion of "quasicompact morphism" is preserved by base change.

5. Show that the notion of "quasifinite morphism" (= finite type + finite fibers) is preserved by base change. (Note: the notion of "finite fibers" is not preserved by base change. $\operatorname{Spec} \overline{\mathbb{Q}} \to \operatorname{Spec} \mathbb{Q}$ has finite fibers, but $\operatorname{Spec} \overline{\mathbb{Q}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} \to \operatorname{Spec} \overline{\mathbb{Q}}$ has one point for each element of $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.)

6. Show that surjectivity is preserved by base change (or fibered product). In other words, if $X \to Y$ is a surjective morphism, then for any $Z \to Y$, $X \times_Y Z \to Z$ is surjective. (You may end up using the fact that for any fields k_1 and k_2 containing k_3, $k_1 \otimes_{k_3} k_2$ is non-zero, and also the axiom of choice.)

Date: Tuesday, January 24, 2006. Minor update October 26, 2006.

7-. Show that the notion of "irreducible" is not necessarily preserved by base change. Show that the notion of "connected" is not necessarily preserved by base change. (Hint: $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$, $\mathbb{Q}[i] \otimes_{\mathbb{Q}} \mathbb{Q}[i]$.)

8. Show that $\operatorname{Spec} \mathbb{C}$ is not a geometrically irreducible $\mathbb{R}$-scheme. If $\operatorname{char} k = p$, show that $\operatorname{Spec} k(u)$ is not a geometrically reduced $\operatorname{Spec} k(u^p)$-scheme.

9. Show that the notion of geometrically irreducible (resp. connected, reduced, integral) fibers behaves well with respect to base change.

10. Suppose that l/k is a finite field extension. Show that a k-scheme X is normal if and only if $X \times_{\operatorname{Spec} k} \operatorname{Spec} l$ is normal. Hence deduce that if k is any field, then $\operatorname{Spec} k[w, x, y, z]/(wz - xy)$ is normal. Hint: we showed earlier (Problem B4 on set 4) that $\operatorname{Spec} k[a, b, c, d]/(a^2 + b^2 + c^2 + d^2)$ is normal. (This is less important, but helps us understand this example.)

11. Show that the Segre scheme (the image of the Segre morphism) is cut out by the equations corresponding to

$$\operatorname{rank} \begin{pmatrix} a_{00} & \cdots & a_{0n} \\ \vdots & \ddots & \vdots \\ a_{m0} & \cdots & a_{mn} \end{pmatrix} = 1,$$

i.e. that all 2×2 minors vanish. (Hint: suppose you have a polynomial in the a_{ij} that becomes zero upon the substitution $a_{ij} = x_i y_j$. Give a recipe for subtracting polynomials of the form monomial times 2×2 minor so that the end result is 0.)

12. Show that $X^{red} \to X$ satisfies the following universal property: any morphism from a reduced scheme Y to X factors uniquely through X^{red}.

$$\begin{array}{ccc} Y & \overset{\exists!}{\dashrightarrow} & X^{red} \\ & \searrow & \swarrow \\ & X & \end{array} .$$

(Do this exercise if you want to see how this sort of argument works in general.)

13. Show that $\nu : \operatorname{Spec} \tilde{R} \to \operatorname{Spec} R$ satisfies the universal property of normalization. We used this to show that normalization exists.

14. Show that normalizations exist for any quasiaffine X (i.e. any X that can be expressed as an open subset of an affine scheme). Show that normalizations exist in general.

Class 24:

15. Show that the normalization morphism is surjective. (Hint: Going-up!)

16. Show that $\dim \tilde{X} = \dim X$ (hint: see our going-up discussion).

17. Show that if X is an integral finite-type k-scheme, then its normalization $\nu : \tilde{X} \to X$ is a finite morphism.

18. Explain how to generalize the notion of normalization to the case where X is a reduced Noetherian scheme (with possibly more than one component). This basically requires defining a universal property. I'm not sure what the "perfect" definition, but all reasonable universal properties should lead to the same space.

19. Show that if X is an integral finite type k-scheme, then its non-normal points form a closed subset. (This is a bit trickier. Hint: consider where $\nu_* \mathcal{O}_{\tilde{X}}$ has rank 1.) I haven't thought through all the details recently, so I hope I've stated this correctly.

20. (Good practice with the concept.) Suppose $X = \operatorname{Spec} \mathbb{Z}[15i]$. Describe the normalization $\tilde{X} \to X$. (Hint: it isn't hard to find an integral extension of $\mathbb{Z}[15i]$ that is integrally closed. By the above discussion, you've then found the normalization!) Over what points of X is the normalization not an isomorphism?

21. (This is an important generalization!) Suppose X is an integral scheme. Define the *normalization of* X, $\nu : \tilde{X} \to X$, *in a given finite field extension of the function field of* X. Show that $\tilde{X}$ is normal. (This will be hard-wired into your definition.) Show that if either X is itself normal, or X is finite type over a field k, then the normalization in a finite field extension is a finite morphism.

22. Suppose $X = \operatorname{Spec} \mathbb{Z}$ (with function field $\mathbb{Q}$). Find its integral closure in the field extension $\mathbb{Q}(i)$.

23. (a) Suppose $X = \operatorname{Spec} k[x]$ (with function field $k(x)$). Find its integral closure in the field extension $k(y)$, where $y^2 = x^2 + x$. (We get a Dedekind domain.)
(b) Suppose $X = \mathbb{P}^1$, with distinguished open $\operatorname{Spec} k[x]$. Find its integral closure in the field extension $k(y)$, where $y^2 = x^2 + x$. (Part (a) involves computing the normalization over one affine open set; now figure out what happens over the "other".)

24. Show that if $f : Z \to X$ is an affine morphism, then we have a natural isomorphism $Z \cong \underline{\operatorname{Spec}}\, f_* \mathcal{O}_Z$ of X-schemes.

25. ($\underline{\operatorname{Spec}}$ behaves well with respect to base change) Suppose $f : Z \to X$ is any morphism, and $\mathcal{A}$ is a quasicoherent sheaf of algebras on X. Show that there is a natural isomorphism $Z \times_X \underline{\operatorname{Spec}}\, \mathcal{A} \cong \underline{\operatorname{Spec}}\, f^* \mathcal{A}$.

26. If $\mathcal{F}$ is a locally free sheaf, show that $\underline{\operatorname{Spec}} \operatorname{Sym} \mathcal{F}^*$ is a vector bundle, i.e. that given any point $p \in X$, there is a neighborhood $p \in U \subset X$ such that $\underline{\operatorname{Spec}} \operatorname{Sym} \mathcal{F}^*|_U \cong \mathbb{A}^1_U$. Show that $\mathcal{F}$ is a sheaf of sections of it.

27. Suppose $f : \underline{\operatorname{Spec}}\, \mathcal{A} \to X$ is a morphism. Show that the category of quasicoherent sheaves on $\underline{\operatorname{Spec}}\, A$ is "essentially the same" (=equivalent) as the category of quasicoherent sheaves on X with the structure of $\mathcal{A}$-modules (quasicoherent $\mathcal{A}$-modules on X).

28. Complete this argument that if $X = \operatorname{Spec} A$, then $(\operatorname{Proj} \mathcal{S}_*, \mathcal{O}(1))$ satisfies the universal property.

29. Show that $(\mathrm{Proj}\,\mathcal{S}_*, \mathcal{O}(1))$ exists in general, by following the analogous universal property argument: show that it exists for X quasiaffine, then in general.

30. ($\underline{\mathrm{Proj}}$ behaves well with respect to base change) Suppose $\mathcal{S}_*$ is a quasicoherent sheaf of graded algebras on X satisfying the required hypotheses above for $\underline{\mathrm{Proj}}\mathcal{S}_*$ to exist. Let $f : Y \to X$ be any morphism. Give a natural isomorphism

$$(\underline{\mathrm{Proj}} f^*\mathcal{S}_*, \mathcal{O}_{\underline{\mathrm{Proj}} f^*\mathcal{S}_*}(1)) \cong (Y \times_X \underline{\mathrm{Proj}}\mathcal{S}_*, g^*\mathcal{O}_{\underline{\mathrm{Proj}}\mathcal{S}_*}(1)) \cong$$

where g is the natural morphism in the base change diagram

$$\begin{array}{ccc} Y \times_X \underline{\mathrm{Proj}}\mathcal{S}_* & \xrightarrow{g} & \underline{\mathrm{Proj}}\mathcal{S}_* \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X. \end{array}$$

31. $\underline{\mathrm{Proj}}(\mathcal{S}_*[t]) \cong \underline{\mathrm{Spec}}\,\mathcal{S}_* \coprod \underline{\mathrm{Proj}}\mathcal{S}_*$, where $\underline{\mathrm{Spec}}\,\mathcal{S}_*$ is an open subscheme, and $\underline{\mathrm{Proj}}\mathcal{S}_*$ is a closed subscheme. Show that $\underline{\mathrm{Proj}}\mathcal{S}*$ is an effective Cartier divisor, corresponding to the invertible sheaf $\mathcal{O}_{\underline{\mathrm{Proj}}N}(1)$. (This is the generalization of the projective and affine cone. At some point I should give an explicit reference to our earlier exercise on this.)

32. Suppose $\mathcal{L}$ is an invertible sheaf on X, and $\mathcal{S}_*$ is a quasicoherent sheaf of graded algebras on X satisfying the required hypotheses above for $\underline{\mathrm{Proj}}\mathcal{S}_*$ to exist. Define $\mathcal{S}'_* = \oplus_{n=0}\mathcal{S}_n \otimes \mathcal{L}_n$. Give a natural isomorphism of X-schemes

$$(\underline{\mathrm{Proj}}\mathcal{S}'_*, \mathcal{O}_{\underline{\mathrm{Proj}}\mathcal{S}'_*}(1)) \cong (\underline{\mathrm{Proj}}\mathcal{S}_*, \mathcal{O}_{\underline{\mathrm{Proj}}\mathcal{S}_*}(1) \otimes \pi^*\mathcal{L}),$$

where $\pi : \underline{\mathrm{Proj}}\mathcal{S}_* \to X$ is the structure morphism. In other words, informally speaking, the $\underline{\mathrm{Proj}}$ is the same, but the $\mathcal{O}(1)$ is twisted by $\mathcal{L}$.

33. Show that closed immersions are projective morphisms. (Hint: Suppose the closed immersion $X \to Y$ corresponds to $\mathcal{O}_Y \to \mathcal{O}_X$. Consider $\mathcal{S}_0 = \mathcal{O}_X$, $\mathcal{S}_i = \mathcal{O}_Y$ for $i > 1$.)

34. (suggested by Kirsten) Suppose $f : X \hookrightarrow \mathbb{P}^n_S$ where S is some scheme. Show that the structure morphism $\pi : X \to S$ is a projective morphism as follows: let $\mathcal{L} = f^*\mathcal{O}_{\mathbb{P}^n_S}(1)$, and show that $X = \underline{\mathrm{Proj}}\pi_*\mathcal{L}^{\otimes n}$.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 11

RAVI VAKIL

This set is due Thursday, February 9, in Jarod Alper's mailbox. It covers (roughly) classes 25 and 26.

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in six solutions. If you are ambitious (and have the time), go for more. Problems marked with "-" count for half a solution. Problems marked with "+" may be harder or more fundamental, but still count for one solution. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. I'm happy to give hints, and some of these problems require hints!

Class 25:

1. Verify that the following definition of "induced reduced subscheme structure" is well-defined. Suppose X is a scheme, and S is a *closed subset* of X. Then there is a unique reduced closed subscheme Z of X "supported on S". More precisely, it can be defined by the following universal property: for any morphism from a *reduced* scheme Y to X, whose image lies in S (as a set), this morphism factors through Z uniquely. Over an affine $X = \operatorname{Spec} R$, we get $\operatorname{Spec} R/I(S)$. (For example, if S is the entire underlying set of X, we get X^{red}.)

2+. Show that open immersions and closed immersions are separated. (Answer: Show that monomorphisms are separated. Open and closed immersions are monomorphisms, by earlier exercises. Alternatively, show this by hand.)

3+. Show that every morphism of affine schemes is separated. (Hint: this was essentially done in the notes if you know where to look.)

4. Complete the proof that $\mathbb{P}^n_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$ is separated, by verifying the last sentence in the proof.

5. Show that the line with doubled origin X is not separated, by verifying that the image of the diagonal morphism is not closed. (Another argument is given below, in Exercise 12.)

6. Show that any morphism from a Noetherian scheme is quasicompact. Hence show that any morphism from a Noetherian scheme is quasiseparated.

Date: Tuesday, January 31, 2006. Update February 27 (changes to 18). Update February 12 (change to 15).

7+. Show that $f : X \to Y$ is quasiseparated if and only if for any affine open $\operatorname{Spec} R$ of Y, and two affine open subsets U and V of X mapping to $\operatorname{Spec} R$, $U \cap V$ is a *finite* union of affine open sets.

8. Here is an example of a nonquasiseparated scheme. Let $X = \operatorname{Spec} k[x_1, x_2, \dots]$, and let U be $X - \mathfrak{m}$ where $\mathfrak{m}$ is the maximal ideal $(x_1, x_2, \dots)$. Take two copies of X, glued along U. Show that the result is not quasiseparated.

9. Prove that the condition of being quasiseparated is local on the target. (Hint: the condition of being quasicompact is local on the target by an earlier exercise.)

10-. Show that a k-scheme is separated (over k) iff it is separated over $\mathbb{Z}$.

11+ (the locus where two morphisms agree) We can now make sense of the following statement. Suppose

$$f, g : \quad \begin{array}{ccc} X & \longrightarrow & Y \\ & \searrow \quad \swarrow & \\ & Z & \end{array}$$

are two morphisms over Z. Then the locus on X where f and g agree is a locally closed subscheme of X. If $Y \to Z$ is separated, then the locus is a closed subscheme of X. More precisely, define V to be the following fibered product:

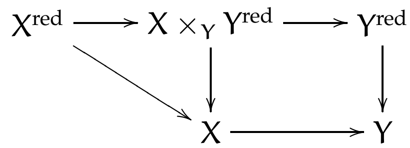
$$\begin{array}{ccc} V & \longrightarrow & Y \\ \downarrow & & \downarrow \delta \\ X & \xrightarrow{(f,g)} & Y \times_Z Y. \end{array}$$

As δ is a locally closed immersion, $V \to X$ is too. Then if $h : W \to X$ is any scheme such that $g \circ h = f \circ h$, then h factors through V. (Put differently: we are describing $V \hookrightarrow X$ by way of a universal property. Taking this as the definition, it is not a priori clear that V is a locally closed subscheme of X, or even that it exists.) Now we come to the exercise: prove this (the sentence before the parentheses). (Hint: we get a map $g \circ h = f \circ h : W \to Y$. Use the definition of fibered product to get $W \to V$.)

12. Show that the line with doubled origin X is not separated, by finding two morphisms $f_1, f_2 : W \to X$ whose domain of agreement is not a closed subscheme. (Another argument was given above, in Exercise 5.)

13. Suppose $\pi : Y \to X$ is a morphism, and $s : X \to Y$ is a *section* of a morphism, i.e. $\pi \circ s$ is the identity on X. Show that s is a locally closed immersion. Show that if π is separated, then s is a closed immersion. (This generalizes Proposition 1.12 in the Class 25 notes.)

14-. Suppose P is a class of morphisms such that closed immersions are in P, and P is closed under fibered product and composition. Show that if $X \to Y$ is in P then $X^{\mathrm{red}} \to Y^{\mathrm{red}}$ is in P. (Two examples are the classes of separated morphisms and quasiseparated morphisms.) (Hint:



)

15. Suppose $\pi : X \to Y$ is a morphism or a ring R, Y is a separated R-scheme, U is an affine open subset of X, and V is an affine open subset of Y. Show that $U \cap \pi^{-1}V$ is an affine open subset of X. (Hint: this generalizes Proposition 1.9 of the Class 25 notes. Use Proposition 1.12 or 1.13.) This will be used in the proof of the Leray spectral sequence.

16. Make this precise: show that the line with the doubled origin fails the valuative criterion for separatedness.

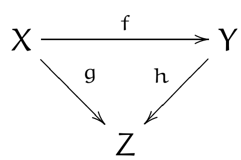
Class 26:

17-. Show that $\mathbb{A}^1_{\mathbb{C}} \to \mathbb{C}$ is not proper.

18. Show that finite morphisms are projective. (There was something that I didn't check in the notes.) More explicitly, if $X \to Y$ is finite, then I described a sheaf of graded algebras $\mathcal{S}_*$ on Y, and claimed that $X = \underline{\text{Proj}}\mathcal{S}_*$. Verify that this is indeed the case. What is $\mathcal{O}_{\underline{\text{Proj}}\mathcal{S}_*}(1)$?

19-. Suppose (1) is a commutative diagram, and f is surjective, g is proper, and h is separated and finite type. Show that h is proper.

(1)



(I'm not sure that this is useful, but I know that if I forget to mention it, it will come back to haunt me!)

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FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 12

RAVI VAKIL

This set is due Thursday, February 16, in Jarod Alper's mailbox. It covers (roughly) classes 27 and 28.

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in five solutions. If you are ambitious (and have the time), go for more. Problems marked with "-" count for half a solution. Problems marked with "+" may be harder or more fundamental, but still count for one solution. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. I'm happy to give hints, and some of these problems require hints!

Class 27:

1+. (Scheme-theoretic closure and scheme-theoretic image) If $f : W \to Y$ is any morphism, define the scheme-theoretic image as the smallest closed subscheme $Z \to Y$ so that f factors through $Z \hookrightarrow Y$. Show that this is well-defined. (One possible hint: use a universal property argument.) If Y is affine, the ideal sheaf corresponds to the functions on Y that are zero when pulled back to Z. Show that the closed set underlying the image subscheme may be strictly larger than the closure of the set-theoretic image: consider $\operatorname{Spec} k(t) \to \operatorname{Spec} k[t]$. (We define the scheme-theoretic closure of a locally closed subscheme $W \hookrightarrow Y$ as the scheme-theoretic image of the morphism.)

2-. Show that rational functions on an integral scheme correspond to rational maps to $\mathbb{A}^1_{\mathbb{Z}}$.

3-. Show that you can compose two rational maps $f : X \dashrightarrow Y$, $g : Y \dashrightarrow Z$ if f is dominant.

4. We define the *graph* of a rational map $f : X \dashrightarrow Y$ as follows: let (U, f') be any representative of this rational map (so $f' : U \to Y$ is a morphism). Let Γ_f be the scheme-theoretic closure of $\Gamma_{f'} \hookrightarrow U \times Y \hookrightarrow X \times Y$, where the first map is a closed immersion, and the second is an open immersion. Show that this is independent of the choice of U.

5. Let K be a finitely generated field extension of transcendence degree m over k. Show there exists an irreducible k-variety W with function field K. (Hint: let $x_1, \ldots, x_n$ be generators for K over k. Consider the map $\operatorname{Spec} K \to \operatorname{Spec} k[t_1, \ldots, t_n]$ given by the ring map $t_i \mapsto x_i$. Take the scheme-theoretic closure of the image.)

Date: Tuesday, February 7, 2006. Updated March 8.

6+. Prove the following. Suppose X and Y are integral and separated (our standard hypotheses from last day). Then X and Y are birational if and only if there is a dense=non-empty open subscheme U of X and a dense=non-empty open subscheme V of Y such that $U \cong Y$. (Feel free to consult Iitaka, or Hartshorne Chapter I Corollary 4.5.)

7. Use the class discussion to find a "formula" for all Pythagorean triples.

8. Show that the conic $x^2+y^2 = z^2$ in $\mathbb{P}^2_k$ is isomorphic to $\mathbb{P}^1_k$ for any field k of characteristic not 2. (Presumably this is true for any ring in which 2 is invertible too...)

9. Find all rational solutions to the $y^2 = x^3 + x^2$, by finding a birational map to $\mathbb{A}^1$, mimicking what worked with the conic.

10. Find a birational map from the quadric $Q = \{x^2 + y^2 = w^2 + z^2\}$ to $\mathbb{P}^2$. Use this to find all rational points on Q. (This illustrates a good way of solving Diophantine equations. You will find a dense open subset of Q that is isomorphic to a dense open subset of $\mathbb{P}^2$, where you can easily find all the rational points. There will be a closed subset of Q where the rational map is not defined, or not an isomorphism, but you can deal with this subset in an ad hoc fashion.)

11. *(a first view of a blow-up)* Let k be an algebraically closed field. (We make this hypothesis in order to not need any fancy facts on nonsingularity.) Consider the rational map $\mathbb{A}^2_k \dashrightarrow \mathbb{P}^1_k$ given by $(x, y) \mapsto [x; y]$. I think you have shown earlier that this rational map cannot be extended over the origin. Consider the graph of the birational map, which we denote $\mathrm{Bl}_{(0,0)} \mathbb{A}^2_k$. It is a subscheme of $\mathbb{A}^2_k \times \mathbb{P}^1_k$. Show that if the coordinates on $\mathbb{A}^2$ are x, y, and the coordinates on $\mathbb{P}^1$ are u, v, this subscheme is cut out in $\mathbb{A}^2 \times \mathbb{P}^1$ by the single equation $xv = yu$. Show that $\mathrm{Bl}_{(0,0)} \mathbb{A}^2_k$ is nonsingular. Describe the fiber of the morphism $\mathrm{Bl}_{(0,0)} \mathbb{A}^2_k \to \mathbb{P}^1_k$ over each closed point of $\mathbb{P}^1_k$. Describe the fiber of the morphism $\mathrm{Bl}_{(0,0)} \mathbb{A}^2_k \to \mathbb{A}^2_k$. Show that the fiber over $(0, 0)$ is an effective Cartier divisor. It is called the *exceptional divisor*.

12. *(the Cremona transformation, a useful classical construction)* Consider the rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, given by $[x; y; z] \to [1/x; 1/y; 1/z]$. What is the the domain of definition? (It is bigger than the locus where $xyz \neq 0$!) You will observe that you can extend it over codimension 1 sets. This will again foreshadow a result we will soon prove.

Class 28:

13. *(Useful practice!)* Suppose X is a Noetherian k-scheme, and Z is an irreducible codimension 1 subvariety whose generic point is a nonsingular point of X (so the local ring $\mathcal{O}_{X,Z}$ is a discrete valuation ring). Suppose $X \dashrightarrow Y$ is a rational map to a projective k-scheme. Show that the domain of definition of the rational map includes a dense open subset of Z. In other words, rational maps from Noetherian k-schemes to projective k-schemes can be extended over nonsingular codimension 1 sets. See problem 12 to see this principle in action. (By the easy direction of the valuative criterion of separatedness, or the theorem of uniqueness of extensions of maps from reduced schemes to separated schemes — Theorem 3.3 of Class 27 — this map is unique.)

14. Show that all nonsingular proper curves are projective. (We may eventually see that all reduced proper curves over k are projective, but I'm not sure; this will use the Riemann-Roch theorem, and I may just prove it for projective curves.)

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FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 13

RAVI VAKIL

This set is due Thursday, February 23, in Jarod Alper's mailbox. It covers (roughly) classes 29 and 30.

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in five solutions. If you are ambitious (and have the time), go for more. Problems marked with "-" count for half a solution. Problems marked with "+" may be harder or more fundamental, but still count for one solution. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. Some of these problems require hints, and I'm happy to give them!

Class 29:

1+. (This was discussed in class 29, but I've put it in the class 27 notes, because it belongs more naturally there.) Suppose $W \hookrightarrow Y$ is a locally closed immersion. The scheme-theoretic closure is the smallest closed subscheme of Y containing W. Show that this notion is well-defined. More generally, if $f : W \to Y$ is any morphism, define the scheme-theoretic image as the smallest closed subscheme $Z \to Y$ so that f factors through $Z \hookrightarrow Y$. Show that this is well-defined. (One possible hint: use a universal property argument.) If Y is affine, the ideal sheaf corresponds to the functions on Y that are zero when pulled back to Z. Show that the closed set underlying the image subscheme may be strictly larger than the closure of the set-theoretic image: consider $\coprod_{n \geq 0} \operatorname{Spec} k[t]/t^n \to \operatorname{Spec} k[t]$. (I suspect that such a pathology cannot occur for finite type morphisms of Noetherian schemes, but I haven't investigated.)

2. Suppose $f : C \to C'$ is a degree d morphism of integral projective nonsingular curves, and $\mathcal{L}$ is an invertible sheaf on C'. Show that $\deg_C f^*\mathcal{L} = d \deg_{C'} \mathcal{L}$.

3. (for those who like working with non-Noetherian schemes) Suppose R is a ring that is coherent over itself. (In other words, R is a coherent R-module.) Show that for any coherent sheaf $\mathcal{F}$ on a projective R-scheme where R is Noetherian, $h^i(X, \mathcal{F})$ is a finitely generated R-module. (Hint: induct downwards as before. The order is as follows: $H^n(\mathbb{P}^n_R, \mathcal{F})$ finitely generated, $H^n(\mathbb{P}^n_R, \mathcal{G})$ finitely generated, $H^n(\mathbb{P}^n_R, \mathcal{F})$ coherent, $H^n(\mathbb{P}^n_R, \mathcal{G})$ coherent, $H^{n-1}(\mathbb{P}^n_R, \mathcal{F})$ finitely generated, $H^{n-1}(\mathbb{P}^n_R, \mathcal{G})$ finitely generated, etc.)

4+ (This is important!) Suppose $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is a short exact sequence of sheaves on a topological space, and $\mathcal{U}$ is an open cover such that on any intersection the sections of $\mathcal{F}_2$ surject onto $\mathcal{F}_3$. Show that we get a long exact sequence of cohomology. (Note that this applies in our case!)

Date: Tuesday, February 14, 2006. Updated March 8, 2006.

5. If D is an effective Cartier divisor on a projective nonsingular curve, say $D = \sum n_i p_i$, prove that $\deg D = \sum n_i \deg p_i$, where $\deg p_i$ is the degree of the field extension of the residue field at p_i over k.

Class 30:

6. Suppose $V \subset U$ are open subsets of X. Show that we have restriction morphisms $H^i(U, \mathcal{F}) \to H^i(V, \mathcal{F})$ (if U and V are quasicompact, and U hence V is separated). Show that restrictions commute. Hence if X is a Noetherian space, $H^i(\cdot, \mathcal{F})$ this is a contravariant functor from the category $\mathrm{Top}(X)$ to abelian groups. (The same argument will show more generally that for any map $f : X \to Y$, there exist natural maps $H^i(X, \mathcal{F}) \to H^i(X, f^* \mathcal{F})$; I should have asked this instead.)

7. Show that if $\mathcal{F} \to \mathcal{G}$ is a morphism of quasicoherent sheaves on separated and quasicompact X then we have natural maps $H^i(X, \mathcal{F}) \to H^i(X, \mathcal{G})$. Hence $H^i(X, \cdot)$ is a covariant functor from quasicoherent sheaves on X to abelian groups (or even R-modules).

8. Verify that $H^{n-1}(\mathbb{P}^{n-1}_R, \mathcal{F}') \to H^n(\mathbb{P}^n_R, \mathcal{F})$ is injective. (Hint: one possibility is by verifying that it is the map on Laurent monomials we claimed when proving that cohomology of $\mathcal{O}(m)$ is what we wanted it to be. In particular, this fact was used in that proof, so you can't use that theorem!)

9. Suppose X is a projective k-scheme. Show that Euler characteristic is additive in exact sequences. In other words, if $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is an exact sequence of coherent sheaves on X, then $\chi(X, \mathcal{G}) = \chi(X, \mathcal{F}) + \chi(X, \mathcal{H})$. (Hint: consider the long exact sequence in cohomology.) More generally, if

$$0 \to \mathcal{F}_1 \to \cdots \to \mathcal{F}_n \to 0$$

is an exact sequence of sheaves, show that

$$\sum_{i=1}^{n} (-1)^i \chi(X, \mathcal{F}_i) = 0.$$

10. *The Riemann-Roch theorem for line bundles on nonsingular projective curves over* k. Suppose $\mathcal{F}$ is an invertible sheaf on C. Show that $\chi(\mathcal{L}) = \deg \mathcal{L} + \chi(C, \mathcal{O}_C)$. (Possible hint: Write $\mathcal{L}$ as the difference of two effective Cartier divisors, $\mathcal{L} \cong \mathcal{O}(Z - P)$. Describe two exact sequences $0 \to \mathcal{L}(-Z) \to \mathcal{L} \to \mathcal{O}_Z \otimes \mathcal{L} \to 0$ and $0 \to \mathcal{O}_C(-P) \to \mathcal{O}_C \to \mathcal{O}_P \to 0$, where $\mathcal{L}(-Z) \cong \mathcal{O}_C(P)$.)

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FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 14

RAVI VAKIL

This set is due Thursday, March 2, in Jarod Alper's mailbox. It covers (roughly) classes 31 and 32.

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in six solutions. If you are ambitious (and have the time), go for more. Problems marked with "-" count for half a solution. Problems marked with "+" may be harder or more fundamental, but still count for one solution. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. Some of these problems require hints, and I'm happy to give them!

Class 31:

1-. Prove the base case of Theorem 1.1 of Class 31. If you choose to do the case $k = -1$, explain precisely why what you are proving is the base case!

2-. Consider the short exact sequence of A-modules $0 \longrightarrow M \xrightarrow{\times f} M \longrightarrow K' \longrightarrow 0$. Show that $\operatorname{Supp} K' = \operatorname{Supp}(M) \cap \operatorname{Supp}(f)$.

3-. Show that the twisted cubic (in $\mathbb{P}^3$) has Hilbert polynomial $3m + 1$.

4. (a) Find the Hilbert polynomial for the dth Veronese embedding of $\mathbb{P}^n$ (i.e. the closed immersion of $\mathbb{P}^n$ in a bigger projective space by way of the line bundle $\mathcal{O}(d)$)
(b) Find the degree of the dth Veronese embedding of $\mathbb{P}^n$.

5-. Show that the degree of a degree d hypersurface is d (preventing a notational crisis).

6. Suppose a curve C is embedded in projective space via an invertible sheaf of degree d. (In other words, this line bundle determines a closed immersion.) Show that the degree of C under this embedding is d (preventing another notational crisis). (Hint: Riemann-Roch.)

7+. *(Bezout's theorem)* Suppose X is a projective scheme of dimension at least 1, and H is a degree d hypersurface not containing any associated points of X. (For example, if X is a projective variety, then we are just requiring H not to contain any irreducible components of X.) Show that $\deg H \cap X = d \deg X$.

8-. Determine the degree of the d-fold Veronese embedding of $\mathbb{P}^n$ in a different way as follows. Let $v_d : \mathbb{P}^n \to \mathbb{P}^N$ be the Veronese embedding. To find the degree of the image,

Date: Tuesday, February 21, 2006. Minor update April 9.

we intersect it with n hyperplanes in $\mathbb{P}^N$ (scheme-theoretically), and find the number of intersection points (counted with multiplicity). But the pullback of a hyperplane in $\mathbb{P}^N$ to $\mathbb{P}^n$ is a degree d hypersurface. Perform this intersection in $\mathbb{P}^n$, and use Bezout's theorem. (If already you know the answer by the earlier exercise on the degree of the Veronese embedding, this will be easier.)

9+. Show that if X is a complete intersection of dimension r in $\mathbb{P}^n$, then $H^i(X, \mathcal{O}_X(m)) = 0$ for all $0 < i < r$ and all m. Show that if $r > 0$, then $H^0(\mathbb{P}^n, \mathcal{O}(m)) \to H^0(X, \mathcal{O}(m))$ is surjective.

10-. Show that complete intersections of positive dimension are connected. (Hint: show $h^0(X, \mathcal{O}_X) = 1$.)

11-. Find the genus of the intersection of 2 quadrics in $\mathbb{P}^3$. (We get curves of more genera by generalizing this!)

12-. Show that the rational normal curve of degree d in $\mathbb{P}^d$ is *not* a complete intersection if $d > 2$.

13-. Show that the union of 2 distinct planes in $\mathbb{P}^4$ is not a complete intersection. (This is the first appearance of another universal counterexample!) Hint: it is connected, but you can slice with another plane and get something not connected.

14. Show that if π is affine, then for $i > 0$, $R^i\pi_*\mathcal{F} = 0$. Moreover, if Y is quasicompact and separated, show that the natural morphism $H^i(X, \mathcal{F}) \to H^i(Y, f_*\mathcal{F})$ is an isomorphism. (A special case of the first sentence is a special case we showed earlier, when π is a closed immersion. Hint: use any affine cover on Y, which will induce an affine cover of X.)

Class 32:

15+. *(Important algebra exercise)* Suppose $M_1 \xrightarrow{\alpha} M_2 \xrightarrow{\beta} M_3$ is a complex of A-modules (i.e. $\beta \circ \alpha = 0$), and N is an A-module. (a) Describe a natural homomorphism of the cohomology of the complex, tensored with N, with the cohomology of the complex you get when you tensor with N $H(M_*) \otimes_A B \to H(M_* \otimes_A N)$, i.e.

$$\left(\frac{\ker \beta}{\operatorname{im} \alpha}\right) \otimes_A N \to \frac{\ker(\beta \otimes N)}{\operatorname{im}(\alpha \otimes N)}.$$

I always forget which way this map is supposed to go.
(b) If N is *flat*, i.e. $\otimes N$ is an exact functor, show that the morphism defined above is an isomorphism. (Hint: This is actually a categorical question: if M_* is an exact sequence in an abelian category, and F is a right-exact functor, then (a) there is a natural morphism $FH(M_*) \to H(FM_*)$, and (b) if F is an exact functor, this morphism is an isomorphism.)

16+. *(Higher pushforwards and base change)* (a) Suppose $f : Z \to Y$ is any morphism, and $\pi : X \to Y$ as usual is quasicompact and separated. Suppose $\mathcal{F}$ is a quasicoherent sheaf on X. Let

$$\begin{array}{ccc} W & \xrightarrow{f'} & X \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ Z & \xrightarrow{f} & Y \end{array}$$

is a fiber diagram. Describe a natural morphism $f^*(R^i\pi_*\mathcal{F}) \to R^i\pi'_*(f')^*\mathcal{F}$.
(b) If $f : Z \to Y$ is an affine morphism, and for a cover $\operatorname{Spec} A_i$ of Y, where $f^{-1}(\operatorname{Spec} A_i) = \operatorname{Spec} B_i$, B_i is a *flat* A-algebra, show that the natural morphism of (a) is an isomorphism. (You can likely generalize this immediately, but this will lead us into the concept of flat morphisms, and we'll hold off discussing this notion for a while.)

17+. *(The projection formula)* Suppose $\pi : X \to Y$ is quasicompact and separated, and $\mathcal{E}$, $\mathcal{F}$ are quasicoherent sheaves on X and Y respectively. (a) Describe a natural morphism

$$(R^i\pi_*\mathcal{E}) \otimes \mathcal{F} \to R^i\pi_*(\mathcal{E} \otimes \pi^*\mathcal{F}).$$

(b) If $\mathcal{F}$ is locally free, show that this natural morphism is an isomorphism.

18. Consider the open immersion $\pi : \mathbb{A}^n - 0 \to \mathbb{A}^n$. By direct calculation, show that $R^{n-1}f_*\mathcal{O}_{\mathbb{A}^n - 0} \neq 0$.

19+. *(Semicontinuity of fiber dimension of projective morphisms)* Suppose $\pi : X \to Y$ is a projective morphism where $\mathcal{O}_Y$ is coherent. Show that $\{y \in Y : \dim f^{-1}(y) > k\}$ is a Zariski-closed subset. In other words, the dimension of the fiber "jumps over Zariski-closed subsets". (You can interpret the case $k = -1$ as the fact that projective morphisms are closed.) This exercise is rather important for having a sense of how projective morphisms behave! (Hint: see the notes.)

20. Suppose $f : X \to Y$ is a projective morphism, with $\mathcal{O}(1)$ on X. Suppose Y is quasicompact and $\mathcal{O}_Y$ is coherent. Let $\mathcal{F}$ be coherent on X. Show that

(a) $f^*f_*\mathcal{F}(n) \to \mathcal{F}(n)$ is surjective for $n \gg 0$. (First show that there is a natural map for any n! Hint: by adjointness of f_* with f_*.) Translation: for $n \gg 0$, $\mathcal{F}(n)$ is relatively generated by global sections.
(b) For $i > 0$ and $n \gg 0$, $R^if_*\mathcal{F}(n) = 0$.

21-. Show that $H^0(A^*) = E_\infty^{0,0} = E_2^{0,0}$ and

$$0 \to E_2^{1,0} \to H^1(A^*) \to E_2^{0,1} \to E_2^{2,0} \to H^2(A^*).$$

(Here take the spectral sequence starting with the vertical arrows.)

22. Suppose we are working in the category of vector spaces over a field k, and $\oplus_{p,q}E_2^{p,q}$ is a finite-dimensional vector space. Show that $\chi(H^*(A^*))$ is well-defined, and equals $\sum_{p,q}(-1)^{p+q}E_2^{p,q}$. (It will sometimes happen that $\oplus E_0^{p,q}$ will be an infinite-dimensional vector space, but that $E_2^{p,q}$ will be finite-dimensional!)

23. By looking at our spectral sequence proof of the five lemma, prove a subtler version of the five lemma, where one of the isomorphisms can instead just be required to be an injection, and another can instead just be required to be a surjection. (I'm deliberately not telling you which ones, so you can see how the spectral sequence is telling you how to improve the result.) I've heard this called the "subtle five lemma", but I like calling it the $4\frac{1}{2}$-lemma.

24. If β and δ (in (1)) are injective, and α is surjective, show that γ is injective. State the dual statement. (The proof of the dual statement will be essentially the same.)

$$\begin{array}{ccccccccc} F & \longrightarrow & G & \longrightarrow & H & \longrightarrow & I & \longrightarrow & J \\ {\scriptstyle\alpha}\uparrow & & {\scriptstyle\beta}\uparrow & & {\scriptstyle\gamma}\uparrow & & {\scriptstyle\delta}\uparrow & & {\scriptstyle\epsilon}\uparrow \\ A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \end{array} \tag{1}$$

25. Use spectral sequences to show that a short exact sequence of complexes gives a long exact sequence in cohomology.

26. Suppose $\mu : A^* \to B^*$ is a morphism of complexes. Suppose C^* is the single complex associated to the double complex $A^* \to B^*$. (C^* is called the *mapping cone* of μ.) Show that there is a long exact sequence of complexes:

$$\cdots \to H^{i-1}(C^*) \to H^i(A^*) \to H^i(B^*) \to H^i(C^*) \to H^{i+1}(A^*) \to \cdots .$$

(There is a slight notational ambiguity here; depending on how you index your double complex, your long exact sequence might look slightly different.) In particular, people often use the fact μ induces an isomorphism on cohomology if and only if the mapping cone is exact.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 15

RAVI VAKIL

This set is due Thursday, March 9, in Jarod Alper's mailbox. It covers (roughly) classes 33 and 34.

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in five solutions. If you are ambitious (and have the time), go for more. Problems marked with "-" count for half a solution. Problems marked with "+" may be harder or more fundamental, but still count for one solution. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. Some of these problems require hints, and I'm happy to give them!

Class 33:

1. *(for people who like non-algebraically closed fields)* Suppose that X is a quasicompact separated k-scheme, where k is a field. Suppose $\mathcal{F}$ is a quasicoherent sheaf on X. Let $X_{\overline{k}} = X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$, and $f : X_{\overline{k}} \to X$ the projection. Describe a natural isomorphism $H^i(X, \mathcal{F}) \otimes_k \overline{k} \to H^i(X_{\overline{k}}, f^*\mathcal{F})$. Recall that a k-scheme X is *geometrically integral* if $X_{\overline{k}}$ is integral. Show that if X is geometrically integral, then $H^0(X, \mathcal{O}_X) \cong k$. (This is a clue that $\mathbb{P}^1_{\mathbb{C}}$ is not a geometrically integral $\mathbb{R}$-scheme.)

2. Suppose Y is any scheme, and $\pi : \mathbb{P}^n_Y \to Y$ is the trivial projective bundle over Y. Show that $\pi_* \mathcal{O}_{\mathbb{P}^n_Y} \cong \mathcal{O}_Y$. More generally, show that $R^j \pi_* \mathcal{O}(m)$ is a finite rank free sheaf on Y, and is 0 if $j \neq 0, n$. Find the rank otherwise.

3. Let A be any ring. Suppose a is a negative integer and b is a positive integer. Show that $H^i(\mathbb{P}^m_A \times_A \mathbb{P}^n_A, \mathcal{O}(a, b))$ is 0 unless $i = m$, in which case it is a free A-module. Find the rank of this free A-module. (Hint: Use the previous exercise, and the projection formula, which was Exercise 1.3 of class 32, and exercise 17 of problem set 14.)

4. (a) Find the genus of a curve in class $(2, n)$ on $\mathbb{P}^1_k \times_k \mathbb{P}^1_k$. (A curve in class $(2, n)$ is any effective Cartier divisor corresponding to invertible sheaf $\mathcal{O}(2, n)$. Equivalently, it is a curve whose ideal sheaf is isomorphic to $\mathcal{O}(-2, -n)$. Equivalently, it is a curve cut out by a non-zero form of bidegree $(2, n)$.)
(b) Suppose for convenience that k is algebraically closed of characteristic not 2. Show that there exists an integral nonsingular curve in class $(2, n)$ on $\mathbb{P}^1_k \times \mathbb{P}^1_k$ for each $n > 0$.

5. Suppose X and Y are projective k-schemes, and $\mathcal{F}$ and $\mathcal{G}$ are coherent sheaves on X and Y respectively. Recall that if $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are the two projections, then $\mathcal{F} \boxtimes \mathcal{G} := \pi_1^* \mathcal{F} \otimes \pi_2^* \mathcal{G}$. Prove the following, adding additional hypotheses if you find

Date: Tuesday, February 28, 2006. Updated March 6.

them necessary.
(a) Show that $H^0(X \times Y, \mathcal{F} \boxtimes \mathcal{G}) = H^0(X, \mathcal{F}) \otimes H^0(Y, \mathcal{G})$.
(b) Show that $H^{\dim X + \dim Y}(X \times Y, \mathcal{F} \boxtimes \mathcal{G}) = H^{\dim X}(X, \mathcal{F}) \otimes_k H^{\dim Y}(Y, \mathcal{G})$.
(c) Show that $\chi(X \times Y, \mathcal{F} \boxtimes \mathcal{G}) = \chi(X, \mathcal{F})\chi(Y, \mathcal{G})$.

Class 34:

6-. Show that the following two morphisms are projective morphisms that are injective on points, but that are not injective on tangent vectors.
(a) the normalization of the cusp $y^2 = x^3$ in the plane
(b) the Frobenius morphism from $\mathbb{A}^1$ to $\mathbb{A}^1$, given by $k[t] \to k[u]$, $u \to t^p$, where k has characteristic p.

7. Suppose $\mathcal{L}$ is a degree $2g-2$ invertible sheaf. Show that it has $g-1$ or g sections, and it has g sections if and only if $\mathcal{L} \cong \mathcal{K}$.

8. Suppose C is a genus 0 curve (projective, geometrically integral and nonsingular). Show that C has a point of degree at most 2.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 16

RAVI VAKIL

This set is due Thursday, March 16, in Jarod Alper's mailbox. It covers (roughly) classes 35 and 36.

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in six solutions. If you are ambitious (and have the time), go for more. Problems marked with "-" count for half a solution. Problems marked with "+" may be harder or more fundamental, but still count for one solution. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. Some of these problems require hints, and I'm happy to give them!

Class 35:

1-. Show that a curve C of genus at least 1 admits a degree 2 cover of $\mathbb{P}^1$ if and only if it has a degree 2 invertible sheaf with precisely 2 sections.

2. Show that the nonhyperelliptic curves of genus 3 form a family of dimension 6. (Hint: Count the dimension of the family of nonsingular quartics, and quotient by $\operatorname{Aut} \mathbb{P}^2 = PGL(3)$.) This (and all other moduli dimension-counting arguments) should be interpreted as: "make a plausibility argument", as we haven't yet defined these moduli spaces.

3. Suppose C is a genus g curve. Show that if C is not hyperelliptic, then the canonical bundle gives a closed immersion $C \hookrightarrow \mathbb{P}^{g-1}$. (In the hyperelliptic case, we have already seen that the canonical bundle gives us a double cover of a rational normal curve.) Hint: follow the genus 3 case. Such a curve is called a *canonical curve*.

4-. Suppose C is a curve of genus $g > 1$, over a field k that is not algebraically closed. Show that C has a closed point of degree at most $2g - 2$ over the base field. (For comparison: if $g = 1$, there is no such bound!)

5. Suppose $X \subset Y \subset \mathbb{P}^n$ are a sequence of closed subschemes, where X and Y have the same Hilbert polynomial. Show that $X = Y$. (Hint: consider the exact sequence

$$0 \to \mathcal{I}_{X/Y} \to \mathcal{O}_Y \to \mathcal{O}_X \to 0.$$

Show that if the Hilbert polynomial of $\mathcal{I}_{X/Y}$ is 0, then $\mathcal{I}_{X/Y}$ must be the 0 sheaf.)

6. Suppose that C is a complete intersection of a quadric surface with a cubic surface. Show that $\mathcal{O}_C(1)$ has 4 sections. (Hint: long exact sequences!)

Date: Tuesday, February 28, 2006. Superficial update June 26.

7. Show that nonhyperelliptic curves of genus 4 "form a family of dimension $9 = 3g - 3$". (Again, this isn't a mathematically well-formed question. So just give a plausibility argument.)

8. Suppose C is a nonhyperelliptic genus 5 curve. The canonical curve is degree 8 in $\mathbb{P}^4$. Show that it lies on a three-dimensional vector space of quadrics (i.e. it lies on 3 independent quadrics). Show that a nonsingular complete intersection of 3 quadrics is a canonical genus 5 curve.

9. Show that the complete intersections of 3 quadrics in $\mathbb{P}^4$ form a family of dimension $12 = 3 \times 5 - 3$.

10-. Show that if $C \subset \mathbb{P}^{g-1}$ is a canonical curve of genus $g \geq 6$, then C is *not* a complete intersection. (Hint: Bezout.)

Class 36:

11. (a) Suppose C is a projective curve. Show that $C - p$ is affine. (Hint: show that $n \gg 0$, $\mathcal{O}(np)$ gives an embedding of C into some projective space $\mathbb{P}^m$, and that there is some hyperplane H meeting C precisely at p. Then $C - p$ is a closed subscheme of $\mathbb{P}^n - H$.)
(b) If C is a geometrically integral nonsingular curve over a field k (i.e. all of our standing assumptions, minus projectivity), show that it is projective or affine.

12. Suppose (E, p) is an elliptic curve. Show that $\mathcal{O}(4p)$ embeds E in $\mathbb{P}^3$ as the complete intersection of two quadrics.

13+. Verify that the axiomatic definition and the functorial definition of a group object in a category are the same.

14+. Suppose (E, p) is an elliptic curve. Show that (E, p) is a group scheme. You may assume that we've defined the multiplication morphism, as sketched in class and in the notes. (Caution! we've stated that only the closed points form a group — the group Pic^0. So there is something to show here. The main idea is that with varieties, lots of things can be checked on closed points. First assume that $k = \overline{k}$, so the closed points are dimension 1 points. Then the associativity diagram is commutative on closed points; argue that it is hence commutative. Ditto for the other categorical requirements. Finally, deal with the case where k is not algebraically closed, by working over the algebraic closure.)

15-. Show that $\mathbb{A}^1_k$ is a group scheme under addition, and $\mathbb{G}_m$ is a group scheme under multiplication. You'll see that the functorial description trumps the axiomatic description here! (Recall that $\mathrm{Hom}(X, \mathbb{A}^1_k)$ is canonically $\Gamma(X, \mathcal{O}_X)$, and $\mathrm{Hom}(X, \mathbb{G}_m)$ is canonically $\Gamma(X, \mathcal{O}_X)^*$.)

16. Define the group scheme $GL(n)$ over the integers.

17-. Define μ_n to be the kernel of the map of group schemes $\mathbb{G}_m \to \mathbb{G}_m$ that is "taking nth powers". In the case where n is a prime p, which is also $\mathrm{char}\, k$, describe μ_p. (I.e. how many points? How "big" = degree over k?)

18-. Define a *ring scheme*. Show that $\mathbb{A}^1_k$ is a ring scheme.

19. Because $\mathbb{A}^1_k$ is a group scheme, $k[t]$ is a Hopf algebra. Describe the comultiplication map $k[t] \to k[t] \otimes_k k[t]$.

20. Suppose X is a scheme, and L is the total space of a line bundle corresponding to invertible sheaf $\mathcal{L}$, so $L = \operatorname{Spec} \oplus_{n \geq 0} (\mathcal{L}^\vee)^{\otimes n}$. Show that $H^0(L, \mathcal{O}_L) = \oplus H^0(X, (\mathcal{L}^\vee)^{\otimes n})$.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 17

RAVI VAKIL

This set is due Thursday, April 20. You can hand it in to Rob Easton, in class or via his mailbox. It covers (roughly) classes 37 and 38.

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in five solutions. If you are ambitious (and have the time), go for more. Problems marked with "-" count for half a solution. Problems marked with "+" may be harder or more fundamental, but still count for one solution. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. Some of these problems require hints, and I'm happy to give them!

Class 37:

1+. In class I stated the following. Note that if A is generated over B (as an algebra) by $x_i \in A$ (where i lies in some index set, possibly infinite), subject to some relations r_j (where j lies in some index set, and each is a polynomial in some finite number of the x_i), then the A-module $\Omega_{A/B}$ is generated by the dx_i, subject to the relations (i)—(iii) and $dr_j = 0$. In short, we needn't take every single element of A; we can take a generating set. And we needn't take every single relation among these generating elements; we can take generators of the relations. Verify this.

2. *(localization of differentials)* If S is a multiplicative set of A, show that there is a natural isomorphism $\Omega_{S^{-1}A/B} \cong S^{-1}\Omega_{A/B}$. (Again, this should be believable from the intuitive picture of "vertical cotangent vectors".) If T is a multiplicative set of B, show that there is a natural isomorphism $\Omega_{S^{-1}A/T^{-1}B} \cong S^{-1}\Omega_{A/B}$ where S is the multiplicative set of A that is the image of the multiplicative set $T \subset B$.

3+. (a) *(pullback of differentials)* If

$$\begin{array}{ccc} A' & \longleftarrow & A \\ \uparrow & & \uparrow \\ B' & \longleftarrow & B \end{array}$$

is a commutative diagram, show that there is a natural homomorphism of A'-modules $\Omega_{A/B} \otimes_A A' \to \Omega_{A'/B'}$. An important special case is $B = B'$.
(b) *(differentials behave well with respect to base extension, affine case)* If furthermore the above diagram is a tensor diagram (i.e. $A' \cong B' \otimes_B A$) then show that $\Omega_{A/B} \otimes_A A' \to \Omega_{A'/B'}$ is an isomorphism.

4. Suppose k is a field, and K is a separable algebraic extension of k. Show that $\Omega_{K/k} = 0$.

Date: Tuesday, April 11, 2006. Minor update May 15.

5. *(Jacobian description of $\Omega_{A/B}$)* Suppose $A = B[x_1, \dots, x_n]/(f_1, \dots, f_r)$. Then $\Omega_{A/B} = \{\oplus_i B dx_i\}/\{df_j = 0\}$ maybe interpreted as the cokernel of the Jacobian matrix $J : A^{\oplus r} \to A^{\oplus n}$.

Class 38:

6. *(normal bundles to effective Cartier divisors)* Suppose $D \subset X$ is an effective Cartier divisor. Show that the conormal sheaf $\mathcal{N}^{\vee}_{D/X}$ is $\mathcal{O}(-D)|_D$ (and in particular is an invertible sheaf), and hence that the normal sheaf is $\mathcal{O}(D)|_D$. It may be surprising that the normal sheaf should be locally free if $X \cong \mathbb{A}^2$ and D is the union of the two axes (and more generally if X is nonsingular but D is singular), because you may be used to thinking that the normal bundle is isomorphic to a "tubular neighborhood".

7-. Suppose $f : X \to Y$ is locally of finite type, and X is locally Noetherian. Show that $\Omega_{X/Y}$ is a coherent sheaf on X.

8+. *(differentials on hyperelliptic curves)* Consider the double cover $f : C \to \mathbb{P}^1_k$ branched over $2g + 2$ distinct points. (We saw earlier that this curve has genus g.) Then $\Omega_{C/k}$ is again an invertible sheaf. What is its degree? (Hint: let x be a coordinate on one of the coordinate patches of $\mathbb{P}^1_k$. Consider $f^* dx$ on C, and count poles and zeros.) In class I gave a sketch showing that you should expect the answer to be $2g - 2$.

9. *(differentials on non-singular plane curves)* Suppose C is a nonsingular plane curve of degree d in $\mathbb{P}^2_k$, where k is algebraically closed. By considering coordinate patches, find the degree of $\Omega_{C/k}$. Make any reasonable simplifying assumption (so that you believe that your result still holds for "most" curves).

10. Suppose that C is a nonsingular projective curve over k such that $\Omega_{C/k}$ is an invertible sheaf. (We'll see that for nonsingular curves, the sheaf of differentials is always locally free. But we don't yet know that.) Let $C_{\overline{k}} = C \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$. Show that $\Omega_{C_{\overline{k}}/\overline{k}}$ is locally free, and that

$$\deg \Omega_{C_{\overline{k}}/\overline{k}} = \deg \Omega_{C/k}.$$

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FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 18

RAVI VAKIL

This set is due Thursday, May 4. You can hand it in to Rob Easton, in class or via his mailbox. It covers (roughly) classes 39, 40, 41, and 42.

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in seven solutions. If you are ambitious (and have the time), go for more. Problems marked with "-" count for half a solution. Problems marked with "+" may be harder or more fundamental, but still count for one solution. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. Some of these problems require hints, and I'm happy to give them!

Classes 39–40:

1. Show that $H^1(\mathbb{P}^n_A, T_{\mathbb{P}^n_A}) = 0$. (This later turns out to be an important calculation for the following reason. If X is a nonsingular variety, $H^1(X, T_X)$ parametrizes deformations of the variety. Thus projective space can't deform, and is "rigid".)

2. I discussed the Grassmannian, which "parametrizes" the space of vector spaces of dimension m in an $(n+1)$-dimensional vector space V (over our base field k). The case $m = 1$ is $\mathbb{P}^n$. Over $G(m, n+1)$ we have a short exact sequence of locally free sheaves

$$0 \to S \to V \otimes \mathcal{O}_{G(m,n+1)} \to Q \to 0$$

where $V \otimes \mathcal{O}_{G(m,n+1)}$ is a trivial bundle, and S is the "universal subbundle" (such that over a point $[V' \subset V]$ of the Grassmannian $G(m, n+1)$, $S|_{[V' \subset V]}$ is V). Then

$$\Omega_{G(m,n+1)/k} \cong \underline{\mathrm{Hom}}(Q, S). \tag{1}$$

In the case of projective space, $m = 1$, $S = \mathcal{O}(-1)$. Verify (1) in this case.

3+. Show that if k is separably closed, then $X_{\overline{k}}$ is nonsingular if and only if X is nonsingular.

4-. Show that Bertini's theorem still holds even if the variety X is singular in dimension 0.

5. Suppose $C \subset \mathbb{P}^2$ is a nonsingular conic over a field of characteristic not 2. Show that the dual variety is also a conic. (More precisely, suppose C is cut out by $f(x_0, x_1, x_2) = 0$. Show that $\{(a_0, a_1, a_2) : a_0x_0 + a_1x_1 + a_2x_2 = 0\}$ is cut out by a quadratic equation.) Thus for example, through a general point in the plane, there are two tangents to C. (The points on a line in the dual plane corresponds to those lines through a point of the original plane.)

Date: Tuesday, April 25, 2006. Updated June 26.

6. *(interpreting the ramification divisor in terms of number of preimages)* Suppose all the ramification above $y \in Y$ is tame. Show that the degree of the branch divisor at y is $\deg(f : X \to Y) - \#f^{-1}(y)$. Thus the multiplicity of the branch divisor counts the extent to which the number of preimages is less than the degree.

7. *(degree of dual curves)* Describe the degree of the dual to a nonsingular degree d plane curve C as follows. Pick a general point $p \in \mathbb{P}^2$. Find the number of tangents to C through p, by noting that projection from p gives a degree d map to $\mathbb{P}^1$ (why?) by a curve of known genus (you've calculated this before), and that ramification of this cover of $\mathbb{P}^1$ corresponds to a tangents through p. (Feel free to make assumptions, e.g. that for a general p this branched cover has the simplest possible branching — this should be a back-of-an-envelope calculation.)

8. *(Artin-Schreier covers)* In characteristic 0, the only connected unbranched cover of $\mathbb{A}^1$ is the isomorphism $\mathbb{A}^1 \xrightarrow{\sim} \mathbb{A}^1$; that was an earlier example/exercise, when we discussed Riemann-Hurwitz the first time. In positive characteristic, this needn't be true, because of wild ramification over ∞. Show that the morphism corresponding to $k[x] \to k[x,y]/(y^p - x^p - y)$ is such a map. (Once the theory of the algebraic fundamental group is developed, this translates to: "$\mathbb{A}^1$ is not simply connected in characteristic p.")

Classes 41–42:

9-. If $N' \to N \to N''$ is exact and M is a flat A-module, show that $M \otimes_A N' \to M \otimes_A N \to M \otimes_A N''$ is exact. Hence *any* exact sequence of A-modules remains exact upon tensoring with M. (We've seen things like this before, so this should be fairly straightforward.)

10-. *(localizations are flat)*. Suppose that S is a multiplicative subset of B. Show that $B \to S^{-1}B$ is a flat ring morphism.

11-. Suppose that A is a ring, $\mathfrak{p}$ is a prime ideal, M is an $A_{\mathfrak{p}}$-module, and N is an A-module. Show that $M \otimes_A N$ is canonically isomorphic to $M \otimes_{A_{\mathfrak{p}}} N_{\mathfrak{p}}$.

12. (a) Prove that flatness is preserved by chase of base ring: If M flat A-module, $A \to B$ is a homomorphism, then $M \otimes_A B$ is a flat B-module.

(b) Prove transitivity of flatness: If B is a flat A-algebra, and M is B-flat, then it is also A-flat. (Hint: consider the natural isomorphism $(M \otimes_A B) \otimes_B \cdot \cong M \otimes_B (B \otimes_A \cdot)$.)

13. If X is a scheme, and η is the generic point for an irreducible component, show that the natural morphism $\operatorname{Spec} \mathcal{O}_{X,\eta} \to X$ is flat. (Hint: localization is flat.)

14. Show that $B \to A$ is faithfully flat if and only if $\operatorname{Spec} A \to \operatorname{Spec} B$ is faithfully flat. (Use the definitions in the notes!)

15. Show that two homotopic maps of complexes induce the same map on homology. (Do this only if you haven't seen this before!)

16. Show that any two lifts of resolutions of modules are homotopic (see the notes for a more precise statement).

17. The notion of an *injective object* in an abelian category is dual to the notion of a projective object. Define derived functors for (i) covariant left-exact functors (these are called right-derived functors), (ii) contravariant left-exact functors (also right-derived functors), and (iii) contravariant right-exact functors (these are called left-derived functors), making explicit the necessary assumptions of the category having enough injectives or projectives.

18+. If B is A-flat, then we get isomorphism $B \otimes \operatorname{Tor}_i^A(M, N) \cong \operatorname{Tor}_i^B(B \otimes M, B \otimes N)$. (Here is a fancier fact that experts may want to try: if B is not A-flat, we don't get an isomorphism; instead we get a spectral sequence.)

19. *(not too important, but good practice if you haven't played with* Tor *before)* If x is not a 0-divisor, show that $\operatorname{Tor}_i^A(A/x, M)$ is 0 for $i > 1$, and for $i = 0$, get M/xM, and for $i = 1$, get $(M : x)$ (those things sent to 0 upon multiplication by x).

20+. *(flatness over the dual numbers)* This fact is important in deformation theory and elsewhere. Show that M is flat over $k[t]/t^2$ if and only if the natural map $M/tM \to tM$ is an isomorphism.

21-. If $0 \to M_0 \to M_1 \to \cdots \to M_n \to 0$ is an exact sequence, and M_i is flat for $i > 0$, show that M_0 is flat too. (Hint: as always, break into short exact sequences.)

22+. *(flat limits are unique)* Suppose A is a discrete valuation ring, and let η be the generic point of $\operatorname{Spec} A$. Suppose X is proper over A, and Y is a closed subscheme of X_η. Show that there is only one closed subscheme Y' of X, proper over A, such that $Y'|_\eta = Y$, and Y' is flat over A.

23. *(an interesting explicit example of a flat limit)* Let $X = \mathbb{A}^3 \times \mathbb{A}^1 \to Y = \mathbb{A}^1$ over a field k, where the coordinates on $\mathbb{A}^3$ are x, y, and z, and the coordinates on $\mathbb{A}^1$ are t. Define X away from $t = 0$ as the union of the two lines $y = z = 0$ (the x-axis) and $x = z - t = 0$ (the y-axis translated by t). Find the flat limit at $t = 0$. (Hint: it is *not* the union of the two axes, although it includes it. The flat limit is non-reduced.)

24. Prove that flat and locally finite type morphisms of locally Noetherian schemes are open. (Hint: reduce to the affine case. Use Chevalley's theorem to show that the image is constructible. Reduce to target that is the spectrum of a local ring. Show that the generic point is hit.)

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FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 19

RAVI VAKIL

This set is due Thursday, May 18. You can hand it in to Rob Easton, in class or via his mailbox. It covers (roughly) classes 43, 44, 45, and 46.

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in eight solutions. If you are ambitious (and have the time), go for more. *Unlike previous sets, problems marked with "+" count for two solutions.* Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. Some of these problems require hints, and I'm happy to give them!

In lieu of completing this problems, you can prove the Cohomology and base change theorem.

Classes 43–44:

1. Prove the Riemann-Roch theorem for two $\mathbb{P}^1$'s glued together at a (reduced) point. (We needed this for our proof that a certain proper surface was nonprojective.)

2. *Gluing two schemes together along isomorphic closed subschemes.* Suppose X' and X'' are two schemes, with closed subschemes $W' \hookrightarrow X'$ and $W'' \hookrightarrow X''$, and an isomorphism $W' \to W''$. Show that we can glue together X' and X'' along $W' \cong W''$. More precisely, show that the following *coproduct* exists:

$$\begin{array}{ccc} W' \cong W'' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ X'' & \longrightarrow & ?. \end{array}$$

Hint: work by analogy with our product construction. If the coproduct exists, it is unique up to unique isomorphism. Start with judiciously chosen affine open subsets, and glue.

3. I alleged that a certain surface is proper over k (see the notes). Prove this. (Possible hint: show that the union of two proper schemes is proper.)

4. The Picard scheme $\operatorname{Pic} X/Y \to Y$ is a scheme over Y which represents the following functor: Given any $T \to Y$, we have the set of invertible sheaves on $X \times_Y T$, modulo those invertible sheaves pulled back from T. In other words, there is a natural bijection between

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diagrams of the form

$$\begin{array}{ccc} \mathcal{L} & & \\ \downarrow & & \\ X \times_T Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ T & \longrightarrow & Y \end{array}$$

and diagrams of the form

$$\begin{array}{ccc} & \nearrow & \mathrm{Pic}_{X/Y} \\ & & \downarrow \\ T & \longrightarrow & Y. \end{array}$$

It is a hard theorem (due to Grothendieck) that (at least if Y is reasonable, e.g. locally Noetherian — I haven't consulted the appropriate references) $\mathrm{Pic}\, X/Y \to Y$ exists, i.e. that this functor is representable. In fact $\mathrm{Pic}\, X/Y$ is of finite type. Problem: Given its existence, check that $\mathrm{Pic}_{X/Y}$ is a group scheme over Y, using our functorial definition of group schemes.

5. Show that the Picard scheme for $X \to Y$, where the morphism is flat and projective, and the fibers are geometrically integral, is separated over Y by showing that it satisfies the valuative criterion of separatedness.

Classes 45–46:

6. Suppose $\mathcal{F}$ is a coherent sheaf on X, $\pi : X \to Y$ projective, Y (hence X) Noetherian, and $\mathcal{F}$ flat over Y. Let $\phi^p : R^p\pi_*\mathcal{F} \otimes k(y) \to H^p(X_y, \mathcal{F}_y)$ be the natural morphism. Suppose $H^p(X_y, \mathcal{F}_y) = 0$ for all $y \in Y$. Show that ϕ^{p-1} is an isomorphism for all $y \in Y$. (Hint: cohomology and base change (b).)

7. With the same hypotheses as the previous problem, suppose $R^p\pi_*\mathcal{F} = 0$ for $p \geq p_0$. Show that $H^p(X_y, \mathcal{F}_y) = 0$ for all $y \in Y$, $k \geq k_0$. (Same hint. You can also do this directly from the key theorem presented in class.)

8+. *(Important!)* Suppose π is a projective flat family, each of whose fibers are (nonempty) integral schemes, or more generally whose fibers satisfy $h^0(X_y) = 1$. Then (*) holds. (Hint: consider

$$\mathcal{O}_Y \otimes k(y) \longrightarrow (\pi_*\mathcal{O}_X) \otimes k(y) \xrightarrow{\phi^0} H^0(X_y, \mathcal{O}_{X_y}) \cong k(y)\ .$$

The composition is surjective, hence ϕ^0 is surjective, hence it is an isomorphism (by the Cohomology and base change theorem (a)). Then thanks to the Cohomology and base change theorem (b), $\pi_*\mathcal{O}_X$ is locally free, thus of rank 1. If I have a map of invertible sheaves $\mathcal{O}_Y \to \pi_*\mathcal{O}_X$ that is an isomorphism on closed points, it is an isomorphism (everywhere) by Nakayama.)

9. *(the Hodge bundle; important in Gromov-Witten theory)* Suppose $\pi : X \to Y$ is a projective flat family, all of whose geometric fibers are connected reduced curves of arithmetic genus

g. Show that $R^1\pi_*\mathcal{O}_X$ is a locally free sheaf of rank g. This is called the *Hodge bundle*. [Hint: use cohomology and base change (b) twice, once with $p = 2$, and once with $p = 1$.]

10. Suppose $\pi : X \to Y$ satisfies (*). Show that if $\mathcal{M}$ is any invertible sheaf on Y, then the natural morphism $\mathcal{M} \to \pi_*\pi^*\mathcal{M}$ is an isomorphism. In particular, we can recover $\mathcal{M}$ from $\pi^*\mathcal{M}$ by pushing forward. (Hint: projection formula.)

11. Suppose X is an integral Noetherian scheme. Show that $\operatorname{Pic}(X \times \mathbb{P}^1) \cong \operatorname{Pic} X \times \mathbb{Z}$. (Side remark: If X is non-reduced, this is still true, see Hartshorne Exercise III.12.6(b). It need only be connected of finite type over k. Presumably locally Noetherian suffices.) Extend this to $X \times \mathbb{P}^n$. Extend this to any $\mathbb{P}^n$-bundle over X.

12. Suppose $X \to Y$ is the projectivization of a vector bundle $\mathcal{F}$ over a reduced locally Noetherian scheme (i.e. $X = \underline{\operatorname{Proj}} \operatorname{Sym}^* \mathcal{F}$). Then I think we've already shown in an exercise that it is also the projectivization of $\mathcal{F} \otimes \mathcal{L}$. If Y is reduced and locally Noetherian, show that these are the only ways in which it is the projectivization of a vector bundle. (Hint: note that you can recover $\mathcal{F}$ by pushing forward $\mathcal{O}(1)$.)

13. Suppose $\pi : X \to Y$ is a projective flat morphism over a Noetherian integral scheme, all of whose geometric fibers are isomorphic to $\mathbb{P}^n$ (over the appropriate field). Show that this is a projective bundle if and only if there is an invertible sheaf on X that restricts to $\mathcal{O}(1)$ on all the fibers. (One direction is clear: if it is a projective bundle, then it has a projective $\mathcal{O}(1)$. In the other direction, the candidate vector bundle is $\pi_*\mathcal{O}(1)$. Show that it is indeed a locally free sheaf of the desired rank. Show that its projectivization is indeed $\pi : X \to Y$.)

14. *An example of a Picard scheme* Show that the Picard scheme of $\mathbb{P}^1_k$ over k is isomorphic to $\mathbb{Z}$.

15+. *An example of a Picard scheme* Show that if E is an elliptic curve over k (a geometrically integral and nonsingular genus 1 curve with a marked k-point), then $\operatorname{Pic} E$ is isomorphic to $E \times \mathbb{Z}$. Hint: Choose a marked point p. (You'll note that this isn't canonical.) Describe the candidate universal invertible sheaf on $E \times \mathbb{Z}$. Given an invertible sheaf on $E \times X$, where X is an arbitrary Noetherian scheme, describe the morphism $X \to E \times \mathbb{Z}$.

16. By a similar argument as we showed that abelian varieties are commutative, show that any map $f : A \to A'$ from one abelian variety to another is a group homomorphism followed by a translation. (Hint: reduce quickly to the case where f sends the identity to the identity. Then show that "$f(x + y) - f(x) - f(y) = e$".)

17. Prove the following. Suppose $f : X \to Y$ is a flat finite-type morphism of locally Noetherian schemes, and Y is irreducible. Then the following are equivalent.

- Every irreducible component of X has dimension $\dim Y + n$.
- For any point $y \in Y$ (not necessarily closed!), every irreducible component of the fiber X_y has dimension n.

18+. Show that if $f : X \to Y$ is a flat morphism of finite type k-schemes (or localizations thereof), then any associated point of X must map to an associated point of Y. (I find this an important point when visualizing flatness!) Hint: use a variant of an argument in the notes. (See the statement of this problem in the notes for more details.)

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FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 20

RAVI VAKIL

This set is due Thursday, May 25. You can hand it in to Rob Easton, in class or via his mailbox. It covers (roughly) classes 47 and 48.

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in five solutions. If you are ambitious (and have the time), go for more. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. Some of these problems require hints, and I'm happy to give them!

1. *(for those who know what a Cohen-Macaulay scheme is)* Suppose $\pi : X \to Y$ is a map of locally Noetherian schemes, where both X and Y are equidimensional, and Y is nonsingular. Show that if any two of the following hold, then the third does as well:

- π is flat.
- X is Cohen-Macaulay.
- Every fiber X_y is Cohen-Macaulay of the expected dimension.

2. *(generated $\otimes$ generated = generated for finite type sheaves)* Suppose $\mathcal{F}$ and $\mathcal{G}$ are finite type sheaves on a scheme X that are generated by global sections. Show that $\mathcal{F} \otimes \mathcal{G}$ is also generated by global sections. In particular, if $\mathcal{L}$ and $\mathcal{M}$ are invertible sheaves on a scheme X, and both $\mathcal{L}$ and $\mathcal{M}$ are base-point-free, then so is $\mathcal{L} \otimes \mathcal{M}$. (This is often summarized as "base-point-free + base-point-free = base-point-free". The symbols $+$ is used rather than $\otimes$, because Pic is an abelian group.)

3. *(very ample + very ample = very ample)* If $\mathcal{L}$ and $\mathcal{M}$ are invertible sheaves on a scheme X, and both $\mathcal{L}$ and $\mathcal{M}$ are base-point-free, then so is $\mathcal{L} \otimes \mathcal{M}$. Hint: Segre. In particular, tensor powers of a very ample invertible sheaf are very ample.

4+. *(very ample + relatively generated = very ample)*. Suppose $\mathcal{L}$ is very ample, and $\mathcal{M}$ is relatively generated, both on $X \to Y$. Show that $\mathcal{L} \otimes \mathcal{M}$ is very ample. (Hint: Reduce to the case where the target is affine. $\mathcal{L}$ induces a map to $\mathbb{P}^n_A$, and this corresponds to $n+1$ sections $s_0, \dots, s_n$ of $\mathcal{L}$. We also have a finite number m of sections $t_1, \dots, t_m$ of $\mathcal{M}$ which generate the stalks. Consider the $(n+1)m$ sections of $\mathcal{L} \otimes \mathcal{M}$ given by $s_i t_j$. Show that these sections are base-point-free, and hence induce a morphism to $\mathbb{P}^{(n+1)m-1}$. Show that it is a closed immersion.)

5. Suppose $\pi : X \to Y$ is proper and Y is quasicompact. Show that if $\mathcal{L}$ is relatively ample on X, then some tensor power of $\mathcal{L}$ is very ample.

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6. State and prove Serre's criterion for relative ampleness (where the target is quasicompact) by adapting the statement of Serre's criterion for ampleness. **Whoops! Ziyu and Rob point out that I used Serre's criterion as the *definition* of ampleness (and similarly, relative ampleness). Thus this exercise is nonsense.**

7. Use Serre's criterion for ampleness to prove that the pullback of ample sheaf on a projective scheme by a finite morphism is ample. Hence if a base-point-free invertible sheaf on a proper scheme induces a morphism to projective space that is finite onto its image, then it is is ample.

8. In class, we proved the following: Suppose $\pi : X \to \operatorname{Spec} B$ is proper, $\mathcal{L}$ ample, and $\mathcal{M}$ invertible. Then $\mathcal{L}^{\otimes n} \otimes \mathcal{M}$ is very ample for $n \gg 0$. Give and prove the corresponding statement for a relatively ample invertible sheaf over a quasicompact base.

9. Suppose X a projective k-scheme. Show that every invertible sheaf is the difference of two *effective* Cartier divisors. Thus the groupification of the semigroup of effective Cartier divisors is the Picard group. Hence if you want to prove something about Cartier divisors on such a thing, you can study effective Cartier divisors. (This is false if projective is replaced by proper — ask Sam Payne for an example.)

10. Suppose C is a generically reduced projective k-curve. Then we can define degree of an invertible sheaf $\mathcal{M}$ as follows. Show that $\mathcal{M}$ has a meromorphic section that is regular at every singular point of C. Thus our old definition (number of zeros minus number of poles, using facts about discrete valuation rings) applies. Prove the Riemann-Roch theorem for generically reduced projective curves. (Hint: our original proof essentially will carry through without change.)

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FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 21

RAVI VAKIL

This set is due Thursday June 8. You can hand it in to Rob Easton, for example via his mailbox. It covers (roughly) classes 49 and 50.

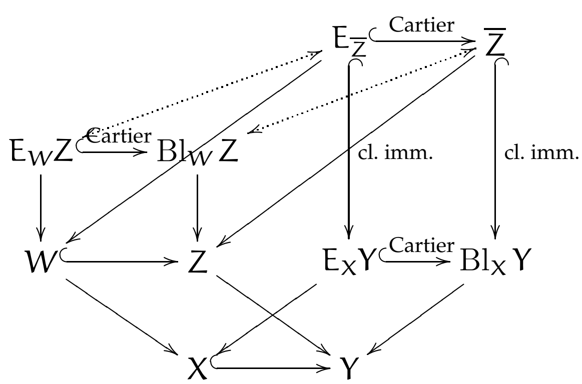
Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in five solutions. If you are ambitious (and have the time), go for more. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. Some of these problems require hints, and I'm happy to give them!

1. Suppose X is an open subscheme of Y, cut out by a finite type sheaf of ideals. If U is an open subset of Y, show that $\mathrm{Bl}_{U\cap X} U \cong \beta^{-1}(U)$, where $\beta : \mathrm{Bl}_X Y \to Y$ is the blow-up. (Hint: show $\beta^{-1}(U)$ satisfies the universal property!)

2. *(The blow up can be computed locally.)* Show that if Y_α is an open cover of Y (as α runs over some index set), and the blow-up of Y_α along $X \cap Y_\alpha$ exists, then the blow-up of Y along X exists.

3. *(The blow-up preserves irreducibility and reducedness.)* Show that if Y is irreducible, and X doesn't contain the generic point of Y, then $\mathrm{Bl}_X Y$ is irreducible. Show that if Y is reduced, then $\mathrm{Bl}_X Y$ is reduced.

4+. Prove the blow-up closure lemma (see the class notes). Hint: obviously, construct maps in both directions, using the universal property. The following diagram may or may not help.



5. If Y and Z are closed subschemes of a given scheme X, show that $\mathrm{Bl}_Y Y \cup Z \cong \mathrm{Bl}_{Y\cap Z} Z$. (In particular, if you blow up a scheme along an irreducible component, the irreducible component is blown out of existence.)

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6. Consider the curve $y^2 = x^3 + x^2$ inside the plane $\mathbb{A}^2_k$. Blow up the origin, and compute the total and proper transform of the curve. (By the blow-up closure lemma, the latter is the blow-up of the nodal curve at the origin.) Check that the proper transform is nonsingular. (All but the last sentence were done in class.)

7. Describe both the total and proper transform of the curve C given by $y = x^2 - x$ in $\mathrm{Bl}_{(0,0)} \mathbb{A}^2$. Verify that the proper transform of C is isomorphic to C. Interpret the intersection of the proper transform of C with the exceptional divisor E as the slope of C at the origin.

8. *(blowing up a cuspidal plane curve)* Describe the proper transform of the cuspidal curve C' given by $y^2 = x^3$ in the plane $\mathbb{A}^2_k$. Show that it is nonsingular. Show that the proper transform of C meets the exceptional divisor E at one point, and is tangent to E there.

9. (a) Desingularize the tacnode $y^2 = x^4$ by blowing up the plane at the origin (and taking the proper transform), and then blowing up the resulting surface once more.
(b) Desingularize $y^8 - x^5 = 0$ in the same way. How many blow-ups do you need?
(c) Do (a) instead in one step by blowing up (y, x^2).

10. Blowing up something nonreduced in nonsingular can give you something singular, as shown in this example. Describe the blow up of the ideal (x, y^2) in $\mathbb{A}^2_k$. What singularity do you get? (Hint: it appears in a nearby exercise.)

11. Blow up the cone point $z^2 = x^2 + y^2$ at the origin. Show that the resulting surface is nonsingular. Show that the exceptional divisor is isomorphic to $\mathbb{P}^1$.

12+. If $X \hookrightarrow \mathbb{P}^n$ is a projective scheme, show that the exceptional divisor of the blow up the affine cone over X at the origin is isomorphic to X, and that its normal bundle is $\mathcal{O}_X(-1)$. (In the case $X = \mathbb{P}^1$, we recover the blow-up of the plane at a point. In particular, we again recover the important fact that the normal bundle to the exceptional divisor is $\mathcal{O}(-1)$.)

13. Show that the multiplicity of the exceptional divisor in the total transform of a subscheme of $\mathbb{A}^n$ when you blow up the origin is the lowest degree that appears in a defining equation of the subscheme. (For example, in the case of the nodal and cuspidal curves above, Example **??** and Exercise respectively, the exceptional divisor appears with multiplicity 2.) This is called the *multiplicity* of the singularity.

14. Suppose Y is the cone $x^2 + y^2 = z^2$, and X is the ruling of the cone $x = 0, y = z$. Show that $\mathrm{Bl}_X Y$ is nonsingular. (In this case we are blowing up a codimension 1 locus that is not a Cartier divisor. Note that it *is* Cartier away from the cone point, so you should expect your answer to be an isomorphism away from the cone point.)

15+. *(blow-ups resolve base loci of rational maps to projective space)* Suppose we have a scheme Y, an invertible sheaf $\mathcal{L}$, and a number of sections $s_0, \ldots, s_n$ of $\mathcal{L}$. Then away from the closed subscheme X cut out by $s_0 = \cdots = s_n = 0$, these sections give a morphism to $\mathbb{P}^n$. Show that this morphism extends to a morphism $\mathrm{Bl}_X Y \to \mathbb{P}^n$, where this morphism corresponds to the invertible sheaf $(\pi^*\mathcal{L})(-E_X Y)$, where $\pi : \mathrm{Bl}_X Y \to Y$ is the blow-up morphism. In other words, "blowing up the base scheme resolves this rational map". (Hint: it suffices to consider an affine open subset of Y where $\mathcal{L}$ is trivial.)

16. Blow up (xy, z) in $\mathbb{A}^3$, and verify that the exceptional divisor is indeed the projectivized normal bundle.

17. Suppose X is an irreducible nonsingular subvariety of a nonsingular variety Y, of codimension at least 2. Describe a natural isomorphism $\operatorname{Pic}\operatorname{Bl}_X Y \cong \operatorname{Pic} Y \oplus \mathbb{Z}$. (Hint: compare divisors on $\operatorname{Bl}_X Y$ and Y. Show that the exceptional divisor $E_X Y$ gives a non-torsion element of $\operatorname{Pic}(\operatorname{Bl}_X Y)$ by describing a $\mathbb{P}^1$ on $\operatorname{Bl}_X Y$ which has intersection number -1 with $E_X Y$.)

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