FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 19

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This set is due Thursday, May 18. You can hand it in to Rob Easton, in class or via his mailbox. It covers (roughly) classes 43, 44, 45, and 46.

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in eight solutions. If you are ambitious (and have the time), go for more. *Unlike previous sets, problems marked with "+" count for two solutions.* Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. Some of these problems require hints, and I'm happy to give them!

In lieu of completing this problems, you can prove the Cohomology and base change theorem.

Classes 43–44:

1. Prove the Riemann-Roch theorem for two $\mathbb{P}^1$'s glued together at a (reduced) point. (We needed this for our proof that a certain proper surface was nonprojective.)

2. *Gluing two schemes together along isomorphic closed subschemes.* Suppose X' and X'' are two schemes, with closed subschemes $W' \hookrightarrow X'$ and $W'' \hookrightarrow X''$, and an isomorphism $W' \to W''$. Show that we can glue together X' and X'' along $W' \cong W''$. More precisely, show that the following *coproduct* exists:

$$\begin{array}{ccc} W' \cong W'' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ X'' & \longrightarrow & ?. \end{array}$$

Hint: work by analogy with our product construction. If the coproduct exists, it is unique up to unique isomorphism. Start with judiciously chosen affine open subsets, and glue.

3. I alleged that a certain surface is proper over k (see the notes). Prove this. (Possible hint: show that the union of two proper schemes is proper.)

4. The Picard scheme $\operatorname{Pic} X/Y \to Y$ is a scheme over Y which represents the following functor: Given any $T \to Y$, we have the set of invertible sheaves on $X \times_Y T$, modulo those invertible sheaves pulled back from T. In other words, there is a natural bijection between

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diagrams of the form

$$\begin{array}{ccc} \mathcal{L} & & \\ \downarrow & & \\ X \times_T Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ T & \longrightarrow & Y \end{array}$$

and diagrams of the form

$$\begin{array}{ccc} & \nearrow & \mathrm{Pic}_{X/Y} \\ & & \downarrow \\ T & \longrightarrow & Y. \end{array}$$

It is a hard theorem (due to Grothendieck) that (at least if Y is reasonable, e.g. locally Noetherian — I haven't consulted the appropriate references) $\mathrm{Pic}\, X/Y \to Y$ exists, i.e. that this functor is representable. In fact $\mathrm{Pic}\, X/Y$ is of finite type. Problem: Given its existence, check that $\mathrm{Pic}_{X/Y}$ is a group scheme over Y, using our functorial definition of group schemes.

5. Show that the Picard scheme for $X \to Y$, where the morphism is flat and projective, and the fibers are geometrically integral, is separated over Y by showing that it satisfies the valuative criterion of separatedness.

Classes 45–46:

6. Suppose $\mathcal{F}$ is a coherent sheaf on X, $\pi : X \to Y$ projective, Y (hence X) Noetherian, and $\mathcal{F}$ flat over Y. Let $\phi^p : R^p\pi_*\mathcal{F} \otimes k(y) \to H^p(X_y, \mathcal{F}_y)$ be the natural morphism. Suppose $H^p(X_y, \mathcal{F}_y) = 0$ for all $y \in Y$. Show that ϕ^{p-1} is an isomorphism for all $y \in Y$. (Hint: cohomology and base change (b).)

7. With the same hypotheses as the previous problem, suppose $R^p\pi_*\mathcal{F} = 0$ for $p \geq p_0$. Show that $H^p(X_y, \mathcal{F}_y) = 0$ for all $y \in Y$, $k \geq k_0$. (Same hint. You can also do this directly from the key theorem presented in class.)

8+. *(Important!)* Suppose π is a projective flat family, each of whose fibers are (nonempty) integral schemes, or more generally whose fibers satisfy $h^0(X_y) = 1$. Then (*) holds. (Hint: consider

$$\mathcal{O}_Y \otimes k(y) \longrightarrow (\pi_*\mathcal{O}_X) \otimes k(y) \xrightarrow{\phi^0} H^0(X_y, \mathcal{O}_{X_y}) \cong k(y) \ .$$

The composition is surjective, hence ϕ^0 is surjective, hence it is an isomorphism (by the Cohomology and base change theorem (a)). Then thanks to the Cohomology and base change theorem (b), $\pi_*\mathcal{O}_X$ is locally free, thus of rank 1. If I have a map of invertible sheaves $\mathcal{O}_Y \to \pi_*\mathcal{O}_X$ that is an isomorphism on closed points, it is an isomorphism (everywhere) by Nakayama.)

9. *(the Hodge bundle; important in Gromov-Witten theory)* Suppose $\pi : X \to Y$ is a projective flat family, all of whose geometric fibers are connected reduced curves of arithmetic genus

g. Show that $R^1\pi_*\mathcal{O}_X$ is a locally free sheaf of rank g. This is called the *Hodge bundle*. [Hint: use cohomology and base change (b) twice, once with $p = 2$, and once with $p = 1$.]

10. Suppose $\pi : X \to Y$ satisfies (*). Show that if $\mathcal{M}$ is any invertible sheaf on Y, then the natural morphism $\mathcal{M} \to \pi_*\pi^*\mathcal{M}$ is an isomorphism. In particular, we can recover $\mathcal{M}$ from $\pi^*\mathcal{M}$ by pushing forward. (Hint: projection formula.)

11. Suppose X is an integral Noetherian scheme. Show that $\operatorname{Pic}(X \times \mathbb{P}^1) \cong \operatorname{Pic} X \times \mathbb{Z}$. (Side remark: If X is non-reduced, this is still true, see Hartshorne Exercise III.12.6(b). It need only be connected of finite type over k. Presumably locally Noetherian suffices.) Extend this to $X \times \mathbb{P}^n$. Extend this to any $\mathbb{P}^n$-bundle over X.

12. Suppose $X \to Y$ is the projectivization of a vector bundle $\mathcal{F}$ over a reduced locally Noetherian scheme (i.e. $X = \underline{\operatorname{Proj}} \operatorname{Sym}^* \mathcal{F}$). Then I think we've already shown in an exercise that it is also the projectivization of $\mathcal{F} \otimes \mathcal{L}$. If Y is reduced and locally Noetherian, show that these are the only ways in which it is the projectivization of a vector bundle. (Hint: note that you can recover $\mathcal{F}$ by pushing forward $\mathcal{O}(1)$.)

13. Suppose $\pi : X \to Y$ is a projective flat morphism over a Noetherian integral scheme, all of whose geometric fibers are isomorphic to $\mathbb{P}^n$ (over the appropriate field). Show that this is a projective bundle if and only if there is an invertible sheaf on X that restricts to $\mathcal{O}(1)$ on all the fibers. (One direction is clear: if it is a projective bundle, then it has a projective $\mathcal{O}(1)$. In the other direction, the candidate vector bundle is $\pi_*\mathcal{O}(1)$. Show that it is indeed a locally free sheaf of the desired rank. Show that its projectivization is indeed $\pi : X \to Y$.)

14. *An example of a Picard scheme* Show that the Picard scheme of $\mathbb{P}^1_k$ over k is isomorphic to $\mathbb{Z}$.

15+. *An example of a Picard scheme* Show that if E is an elliptic curve over k (a geometrically integral and nonsingular genus 1 curve with a marked k-point), then $\operatorname{Pic} E$ is isomorphic to $E \times \mathbb{Z}$. Hint: Choose a marked point p. (You'll note that this isn't canonical.) Describe the candidate universal invertible sheaf on $E \times \mathbb{Z}$. Given an invertible sheaf on $E \times X$, where X is an arbitrary Noetherian scheme, describe the morphism $X \to E \times \mathbb{Z}$.

16. By a similar argument as we showed that abelian varieties are commutative, show that any map $f : A \to A'$ from one abelian variety to another is a group homomorphism followed by a translation. (Hint: reduce quickly to the case where f sends the identity to the identity. Then show that "$f(x + y) - f(x) - f(y) = e$".)

17. Prove the following. Suppose $f : X \to Y$ is a flat finite-type morphism of locally Noetherian schemes, and Y is irreducible. Then the following are equivalent.

- Every irreducible component of X has dimension $\dim Y + n$.
- For any point $y \in Y$ (not necessarily closed!), every irreducible component of the fiber X_y has dimension n.

18+. Show that if $f : X \to Y$ is a flat morphism of finite type k-schemes (or localizations thereof), then any associated point of X must map to an associated point of Y. (I find this an important point when visualizing flatness!) Hint: use a variant of an argument in the notes. (See the statement of this problem in the notes for more details.)

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