FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 18

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This set is due Thursday, May 4. You can hand it in to Rob Easton, in class or via his mailbox. It covers (roughly) classes 39, 40, 41, and 42.

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in seven solutions. If you are ambitious (and have the time), go for more. Problems marked with "-" count for half a solution. Problems marked with "+" may be harder or more fundamental, but still count for one solution. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. Some of these problems require hints, and I'm happy to give them!

Classes 39–40:

1. Show that $H^1(\mathbb{P}^n_A, T_{\mathbb{P}^n_A}) = 0$. (This later turns out to be an important calculation for the following reason. If X is a nonsingular variety, $H^1(X, T_X)$ parametrizes deformations of the variety. Thus projective space can't deform, and is "rigid".)

2. I discussed the Grassmannian, which "parametrizes" the space of vector spaces of dimension m in an $(n+1)$-dimensional vector space V (over our base field k). The case $m = 1$ is $\mathbb{P}^n$. Over $G(m, n+1)$ we have a short exact sequence of locally free sheaves

$$0 \to S \to V \otimes \mathcal{O}_{G(m,n+1)} \to Q \to 0$$

where $V \otimes \mathcal{O}_{G(m,n+1)}$ is a trivial bundle, and S is the "universal subbundle" (such that over a point $[V' \subset V]$ of the Grassmannian $G(m, n+1)$, $S|_{[V' \subset V]}$ is V). Then

$$\Omega_{G(m,n+1)/k} \cong \underline{\mathrm{Hom}}(Q, S). \tag{1}$$

In the case of projective space, $m = 1$, $S = \mathcal{O}(-1)$. Verify (1) in this case.

3+. Show that if k is separably closed, then $X_{\overline{k}}$ is nonsingular if and only if X is nonsingular.

4-. Show that Bertini's theorem still holds even if the variety X is singular in dimension 0.

5. Suppose $C \subset \mathbb{P}^2$ is a nonsingular conic over a field of characteristic not 2. Show that the dual variety is also a conic. (More precisely, suppose C is cut out by $f(x_0, x_1, x_2) = 0$. Show that $\{(a_0, a_1, a_2) : a_0 x_0 + a_1 x_1 + a_2 x_2 = 0\}$ is cut out by a quadratic equation.) Thus for example, through a general point in the plane, there are two tangents to C. (The points on a line in the dual plane corresponds to those lines through a point of the original plane.)

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6. *(interpreting the ramification divisor in terms of number of preimages)* Suppose all the ramification above $y \in Y$ is tame. Show that the degree of the branch divisor at y is $\deg(f : X \to Y) - \#f^{-1}(y)$. Thus the multiplicity of the branch divisor counts the extent to which the number of preimages is less than the degree.

7. *(degree of dual curves)* Describe the degree of the dual to a nonsingular degree d plane curve C as follows. Pick a general point $p \in \mathbb{P}^2$. Find the number of tangents to C through p, by noting that projection from p gives a degree d map to $\mathbb{P}^1$ (why?) by a curve of known genus (you've calculated this before), and that ramification of this cover of $\mathbb{P}^1$ corresponds to a tangents through p. (Feel free to make assumptions, e.g. that for a general p this branched cover has the simplest possible branching — this should be a back-of-an-envelope calculation.)

8. *(Artin-Schreier covers)* In characteristic 0, the only connected unbranched cover of $\mathbb{A}^1$ is the isomorphism $\mathbb{A}^1 \xrightarrow{\sim} \mathbb{A}^1$; that was an earlier example/exercise, when we discussed Riemann-Hurwitz the first time. In positive characteristic, this needn't be true, because of wild ramification over ∞. Show that the morphism corresponding to $k[x] \to k[x,y]/(y^p - x^p - y)$ is such a map. (Once the theory of the algebraic fundamental group is developed, this translates to: "$\mathbb{A}^1$ is not simply connected in characteristic p.")

Classes 41–42:

9-. If $N' \to N \to N''$ is exact and M is a flat A-module, show that $M \otimes_A N' \to M \otimes_A N \to M \otimes_A N''$ is exact. Hence *any* exact sequence of A-modules remains exact upon tensoring with M. (We've seen things like this before, so this should be fairly straightforward.)

10-. *(localizations are flat)*. Suppose that S is a multiplicative subset of B. Show that $B \to S^{-1}B$ is a flat ring morphism.

11-. Suppose that A is a ring, $\mathfrak{p}$ is a prime ideal, M is an $A_{\mathfrak{p}}$-module, and N is an A-module. Show that $M \otimes_A N$ is canonically isomorphic to $M \otimes_{A_{\mathfrak{p}}} N_{\mathfrak{p}}$.

12. (a) Prove that flatness is preserved by chase of base ring: If M flat A-module, $A \to B$ is a homomorphism, then $M \otimes_A B$ is a flat B-module.

(b) Prove transitivity of flatness: If B is a flat A-algebra, and M is B-flat, then it is also A-flat. (Hint: consider the natural isomorphism $(M \otimes_A B) \otimes_B \cdot \cong M \otimes_B (B \otimes_A \cdot)$.)

13. If X is a scheme, and η is the generic point for an irreducible component, show that the natural morphism $\operatorname{Spec} \mathcal{O}_{X,\eta} \to X$ is flat. (Hint: localization is flat.)

14. Show that $B \to A$ is faithfully flat if and only if $\operatorname{Spec} A \to \operatorname{Spec} B$ is faithfully flat. (Use the definitions in the notes!)

15. Show that two homotopic maps of complexes induce the same map on homology. (Do this only if you haven't seen this before!)

16. Show that any two lifts of resolutions of modules are homotopic (see the notes for a more precise statement).

17. The notion of an *injective object* in an abelian category is dual to the notion of a projective object. Define derived functors for (i) covariant left-exact functors (these are called right-derived functors), (ii) contravariant left-exact functors (also right-derived functors), and (iii) contravariant right-exact functors (these are called left-derived functors), making explicit the necessary assumptions of the category having enough injectives or projectives.

18+. If B is A-flat, then we get isomorphism $B \otimes \operatorname{Tor}_i^A(M, N) \cong \operatorname{Tor}_i^B(B \otimes M, B \otimes N)$. (Here is a fancier fact that experts may want to try: if B is not A-flat, we don't get an isomorphism; instead we get a spectral sequence.)

19. *(not too important, but good practice if you haven't played with* Tor *before)* If x is not a 0-divisor, show that $\operatorname{Tor}_i^A(A/x, M)$ is 0 for $i > 1$, and for $i = 0$, get M/xM, and for $i = 1$, get $(M : x)$ (those things sent to 0 upon multiplication by x).

20+. *(flatness over the dual numbers)* This fact is important in deformation theory and elsewhere. Show that M is flat over $k[t]/t^2$ if and only if the natural map $M/tM \to tM$ is an isomorphism.

21-. If $0 \to M_0 \to M_1 \to \cdots \to M_n \to 0$ is an exact sequence, and M_i is flat for $i > 0$, show that M_0 is flat too. (Hint: as always, break into short exact sequences.)

22+. *(flat limits are unique)* Suppose A is a discrete valuation ring, and let η be the generic point of $\operatorname{Spec} A$. Suppose X is proper over A, and Y is a closed subscheme of X_η. Show that there is only one closed subscheme Y' of X, proper over A, such that $Y'|_\eta = Y$, and Y' is flat over A.

23. *(an interesting explicit example of a flat limit)* Let $X = \mathbb{A}^3 \times \mathbb{A}^1 \to Y = \mathbb{A}^1$ over a field k, where the coordinates on $\mathbb{A}^3$ are x, y, and z, and the coordinates on $\mathbb{A}^1$ are t. Define X away from $t = 0$ as the union of the two lines $y = z = 0$ (the x-axis) and $x = z - t = 0$ (the y-axis translated by t). Find the flat limit at $t = 0$. (Hint: it is *not* the union of the two axes, although it includes it. The flat limit is non-reduced.)

24. Prove that flat and locally finite type morphisms of locally Noetherian schemes are open. (Hint: reduce to the affine case. Use Chevalley's theorem to show that the image is constructible. Reduce to target that is the spectrum of a local ring. Show that the generic point is hit.)

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