FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 16

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This set is due Thursday, March 16, in Jarod Alper's mailbox. It covers (roughly) classes 35 and 36.

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in six solutions. If you are ambitious (and have the time), go for more. Problems marked with "-" count for half a solution. Problems marked with "+" may be harder or more fundamental, but still count for one solution. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. Some of these problems require hints, and I'm happy to give them!

Class 35:

1-. Show that a curve C of genus at least 1 admits a degree 2 cover of $\mathbb{P}^1$ if and only if it has a degree 2 invertible sheaf with precisely 2 sections.

2. Show that the nonhyperelliptic curves of genus 3 form a family of dimension 6. (Hint: Count the dimension of the family of nonsingular quartics, and quotient by $\operatorname{Aut} \mathbb{P}^2 = PGL(3)$.) This (and all other moduli dimension-counting arguments) should be interpreted as: "make a plausibility argument", as we haven't yet defined these moduli spaces.

3. Suppose C is a genus g curve. Show that if C is not hyperelliptic, then the canonical bundle gives a closed immersion $C \hookrightarrow \mathbb{P}^{g-1}$. (In the hyperelliptic case, we have already seen that the canonical bundle gives us a double cover of a rational normal curve.) Hint: follow the genus 3 case. Such a curve is called a *canonical curve*.

4-. Suppose C is a curve of genus $g > 1$, over a field k that is not algebraically closed. Show that C has a closed point of degree at most $2g - 2$ over the base field. (For comparison: if $g = 1$, there is no such bound!)

5. Suppose $X \subset Y \subset \mathbb{P}^n$ are a sequence of closed subschemes, where X and Y have the same Hilbert polynomial. Show that $X = Y$. (Hint: consider the exact sequence

$$0 \to \mathcal{I}_{X/Y} \to \mathcal{O}_Y \to \mathcal{O}_X \to 0.$$

Show that if the Hilbert polynomial of $\mathcal{I}_{X/Y}$ is 0, then $\mathcal{I}_{X/Y}$ must be the 0 sheaf.)

6. Suppose that C is a complete intersection of a quadric surface with a cubic surface. Show that $\mathcal{O}_C(1)$ has 4 sections. (Hint: long exact sequences!)

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7. Show that nonhyperelliptic curves of genus 4 "form a family of dimension $9 = 3g - 3$". (Again, this isn't a mathematically well-formed question. So just give a plausibility argument.)

8. Suppose C is a nonhyperelliptic genus 5 curve. The canonical curve is degree 8 in $\mathbb{P}^4$. Show that it lies on a three-dimensional vector space of quadrics (i.e. it lies on 3 independent quadrics). Show that a nonsingular complete intersection of 3 quadrics is a canonical genus 5 curve.

9. Show that the complete intersections of 3 quadrics in $\mathbb{P}^4$ form a family of dimension $12 = 3 \times 5 - 3$.

10-. Show that if $C \subset \mathbb{P}^{g-1}$ is a canonical curve of genus $g \geq 6$, then C is *not* a complete intersection. (Hint: Bezout.)

Class 36:

11. (a) Suppose C is a projective curve. Show that $C - p$ is affine. (Hint: show that $n \gg 0$, $\mathcal{O}(np)$ gives an embedding of C into some projective space $\mathbb{P}^m$, and that there is some hyperplane H meeting C precisely at p. Then $C - p$ is a closed subscheme of $\mathbb{P}^n - H$.)
(b) If C is a geometrically integral nonsingular curve over a field k (i.e. all of our standing assumptions, minus projectivity), show that it is projective or affine.

12. Suppose (E, p) is an elliptic curve. Show that $\mathcal{O}(4p)$ embeds E in $\mathbb{P}^3$ as the complete intersection of two quadrics.

13+. Verify that the axiomatic definition and the functorial definition of a group object in a category are the same.

14+. Suppose (E, p) is an elliptic curve. Show that (E, p) is a group scheme. You may assume that we've defined the multiplication morphism, as sketched in class and in the notes. (Caution! we've stated that only the closed points form a group — the group Pic^0. So there is something to show here. The main idea is that with varieties, lots of things can be checked on closed points. First assume that $k = \overline{k}$, so the closed points are dimension 1 points. Then the associativity diagram is commutative on closed points; argue that it is hence commutative. Ditto for the other categorical requirements. Finally, deal with the case where k is not algebraically closed, by working over the algebraic closure.)

15-. Show that $\mathbb{A}^1_k$ is a group scheme under addition, and $\mathbb{G}_m$ is a group scheme under multiplication. You'll see that the functorial description trumps the axiomatic description here! (Recall that $\mathrm{Hom}(X, \mathbb{A}^1_k)$ is canonically $\Gamma(X, \mathcal{O}_X)$, and $\mathrm{Hom}(X, \mathbb{G}_m)$ is canonically $\Gamma(X, \mathcal{O}_X)^*$.)

16. Define the group scheme $GL(n)$ over the integers.

17-. Define μ_n to be the kernel of the map of group schemes $\mathbb{G}_m \to \mathbb{G}_m$ that is "taking nth powers". In the case where n is a prime p, which is also $\mathrm{char}\, k$, describe μ_p. (I.e. how many points? How "big" = degree over k?)

18-. Define a *ring scheme*. Show that $\mathbb{A}^1_k$ is a ring scheme.

19. Because $\mathbb{A}^1_k$ is a group scheme, $k[t]$ is a Hopf algebra. Describe the comultiplication map $k[t] \to k[t] \otimes_k k[t]$.

20. Suppose X is a scheme, and L is the total space of a line bundle corresponding to invertible sheaf $\mathcal{L}$, so $L = \operatorname{Spec} \oplus_{n \geq 0} (\mathcal{L}^\vee)^{\otimes n}$. Show that $H^0(L, \mathcal{O}_L) = \oplus H^0(X, (\mathcal{L}^\vee)^{\otimes n})$.

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