# FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 4

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**This set is due Monday, November 7. It covers (roughly) classes 8 and 9.** Read all of these problems, and hand in six solutions. Two A problems count for one solution. One B problem counts for one solution. Try to solve problems on a range of topics. If you are pressed for time, try more straightforward problems. If you are ambitious, push the envelope a bit. You are encouraged to talk to each other about the problems. (Write up your solutions individually.) You are also encouraged to talk to me about them. Ideally, you should find out who did problems that you didn't do. Make sure you read all the problems, because we will be be making use of many of these results.

**A1.** Show that $\mathbb{P}^n_k$ is irreducible.

**A2.** You showed earlier that for affine schemes, there is a bijection between irreducible closed subsets and points. Show that this is true of schemes as well.

**A3.** Prove the following. If $R$ is Noetherian, then $\operatorname{Spec} R$ is a Noetherian topological space. If $X$ is a scheme that has a finite cover $X = \cup_{i=1}^n \operatorname{Spec} R_i$ where $R_i$ is Noetherian, then $X$ is a Noetherian topological space. Thus $\mathbb{P}^n_k$ and $\mathbb{P}^n_{\mathbb{Z}}$ are Noetherian topological spaces: we built them by gluing together a finite number of $\operatorname{Spec}$'s of Noetherian rings.

**A4.** If $R$ is any ring, show that the irreducible components of $\operatorname{Spec} R$ are in bijection with the minimal primes of $R$. (Here minimality is with respect to inclusion.)

**A5.** Show that an irreducible topological space is connected.

**A6.** Show that a finite union of affine schemes is quasicompact. (Hence $\mathbb{P}^n_k$ is quasicompact.) Show that every closed subset of an affine scheme is quasicompact. Show that every closed subset of a quasicompact scheme is quasicompact.

**A7.** Show that a scheme is reduced if and only if none of the stalks have nilpotents. Hence show that if $f$ and $g$ are two functions on a reduced scheme that agree at all points, then $f = g$.

**A8.** Show that an affine scheme $\operatorname{Spec} R$ is integral if and only if $R$ is an integral domain.

**A9.** Show that a scheme $X$ is integral if and only if it is irreducible and reduced.

**A10.** Suppose $X$ is an integral scheme. Then $X$ (being irreducible) has a generic point $\eta$. Suppose $\operatorname{Spec} R$ is any non-empty affine open subset of $X$. Show that the stalk at $\eta$,

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$\mathcal{O}_{X,\eta}$, is naturally $\operatorname{Frac} R$. This is called the *function field* of $X$. It can be computed on any non-empty open set of $X$ (as any such open set contains the generic point).

**A11.** Suppose $X$ is an integral scheme. Show that the restriction maps $\operatorname{res}_{U,V} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$ are inclusions so long as $V \neq \emptyset$. Suppose $\operatorname{Spec} R$ is any non-empty affine open subset of $X$ (so $R$ is an integral domain). Show that the natural map $\mathcal{O}_X(U) \to \mathcal{O}_{X,\eta} = \operatorname{Frac} R$ (where $U$ is any non-empty open set) is an inclusion.

**A12.** Suppose $f(x, y)$ and $g(x, y)$ are two complex polynomials ($f, g \in \mathbb{C}[x, y]$). Suppose $f$ and $g$ have no common factors. Show that the system of equations $f(x, y) = g(x, y) = 0$ has a finite number of solutions.

**A13.** If $R$ is a finitely generated domain over $k$, show that $\dim R[x] = \dim R + 1$. (In fact this is true if $R$ is Noetherian. You're welcome to try to prove that. We'll prove it later in the class, and you may use this fact in later problem sets.)

**A14.** Show that the underlying topological space of a Noetherian scheme is Noetherian. Show that a Noetherian scheme has a finite number of irreducible components.

**A15.** Suppose $X$ is an integral scheme, that can be covered by open subsets of the form $\operatorname{Spec} R$ where $R$ is a finitely generated domain over $k$. Then $\dim X$ is the transcendence degree of the function field (the stalk at the generic point) $\mathcal{O}_{X,\eta}$ over $k$. Thus (as the generic point lies in all non-empty open sets) the dimension can be computed in any open set of $X$.

**A16.** What is the dimension of $\operatorname{Spec} k[w, x, y, z]/(wx - yz, x^{17} + y^{17})$? (Be careful to check the hypotheses before invoking Krull!)

**A17.** Suppose that $R$ is a finitely generated domain over $k$, and $\mathfrak{p}$ is a prime ideal. Show that $\dim R_{\mathfrak{p}} = \dim R - \dim R/\mathfrak{p}$.

**A18.** Show that all open subsets of a Noetherian topological space (hence of a Noetherian scheme) are quasicompact.

**A19.** Check that our new definition of reduced (in terms of affine covers) agrees with our earlier definition. This definition is advantageous: our earlier definition required us to check that the ring of functions over *any* open set is nilpotent free. This lets us check in an affine cover. Hence for example $\mathbb{A}_k^n$ and $\mathbb{P}_k^n$ are reduced.

**A20.** If $R$ is a unique factorization domain, show that $R$ is integrally closed (in its fraction field $\operatorname{Frac}(R)$). Hence $\mathbb{A}_k^n$ and $\mathbb{P}_k^n$ are both normal.

**A21.** Suppose $R$ is a ring, and $(f_1, \ldots, f_n) = R$. Show that if $R$ has no nonzero nilpotents (i.e. $0$ is a radical ideal), then $R_{f_i}$ also has no nonzero nilpotents. Show that if no $R_{f_i}$ has a nonzero nilpotent, then neither does $R$.

**A22.** Suppose $R$ is an integral domain. Show that if $R$ is integrally closed, then so is $R_f$.

**A23.** Suppose $X$ is a quasicompact scheme, and $f$ is a function vanishing on all the points of $X$. Show that $f^n = 0$ for some $n$. Show that this can be false without the quasicompact hypothesis.

**B1.** Show that $\left(k[x,y]/(xy,x^2)\right)_y$ has no nilpotents. (Hint: show that it is isomorphic to another ring, by considering the geometric picture.)

**B2.** Give (with proof!) an example of a scheme that is connected but reducible.

**B3.** Show that $\dim \mathbb{A}^1_{\mathbb{Z}} = 2$.

**B4.** Suppose that $R$ is a Unique Factorization Domain containing $1/2$, $f \in R$ has no repeated prime factors, and $z^2 - f$ is irreducible in $R[z]$. Show that $\operatorname{Spec} R[z]/(z^2 - f)$ is normal. (Hint: one of Gauss' Lemmas.) Show that the following schemes are normal: $\operatorname{Spec} \mathbb{Z}[x]/(x^2 - n)$ where $n$ is a square-free integer congruent to $3 \pmod 4$; $\operatorname{Spec} k[x_1, \dots, x_n]/x_1^2 + x_2^2 + \cdots + x_m^2$ where $\operatorname{char} k \neq 2$, $m \geq 3$; $\operatorname{Spec} k[w,x,y,z]/(wx - yz)$ where $\operatorname{char} k \neq 2$ and $k$ is algebraically closed. Show that if $f$ has repeated prime factors, then $\operatorname{Spec} R[z]/(z^2 - f)$ is *not* normal.

**B5.** Show that $\operatorname{Spec} k[w,x,y,z]/(wz - xy, wy - x^2, xz - y^2)$ is an irreducible surface. (It is no harder to show that it is an integral surface.) We will see next week that this is the affine cone over the twisted cubic.

**B6.** Suppose $X = \operatorname{Spec} R$ where $R$ is a Noetherian domain, and $Z$ is an irreducible component of $V(r_1, \dots, r_n)$, where $r_1, \dots, r_n \in R$. Show that the height of (the prime associated to) $Z$ is at most $n$. Conversely, suppose $X = \operatorname{Spec} R$ where $R$ is a Noetherian domain, and $Z$ is an irreducible subset of height $n$. Show that there are $f_1, \dots, f_n \in R$ such that $Z$ is an irreducible component of $V(f_1, \dots, f_n)$.

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