FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 2

This set is due Monday, October 24. It covers classes 3 and 4. Read all of these problems, and hand in six solutions. The problems are arranged roughly in "chronological order", not by difficulty. Try to solve problems on a range of topics. If you are pressed for time, try more straightforward problems. If you are ambitious, push the envelope a bit. You are encouraged to talk to each other about the problems. (Write up your solutions individually.) You are also encouraged to talk to me about them. Ideally, you should find out who did problems that you didn't do.

1. Suppose

$$0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} A^n \xrightarrow{d^n} 0$$

is a complex of vector spaces (often called $A^\bullet$ for short), i.e. $d^i \circ d^{i-1} = 0$. Show that $\sum(-1)^i \dim A^i = \sum(-1)^i h^i(A^\bullet)$. (Recall that $h^i(A^\bullet) = \dim \ker(d^i)/\operatorname{im}(d^{i-1})$.) In particular, if $A^\bullet$ is exact, then $\sum(-1)^i \dim A^i = 0$. (If you haven't dealt much with cohomology, this will give you some practice. If you have, you shouldn't do this problem.)

Problems on presheaves and sheaves

2. Suppose $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves of abelian groups or $\mathcal{O}_X$-modules. If $\mathcal{H}$ is defined by the collection of data $\mathcal{H}(U) = \mathcal{G}(U)/\phi(\mathcal{F}(U))$ for all open U, show that $\mathcal{H}$ is a presheaf, and show that it is a cokernel in the category of presheaves. (I stated this as a fact in class, but you aren't allowed to appeal to authority.)

3. Suppose that $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \cdots \to \mathcal{F}_n \to 0$ is an *exact sequence of presheaves* of groups or $\mathcal{O}_X$-modules. Show that $0 \to \mathcal{F}_1(U) \to \mathcal{F}_2(U) \to \cdots \to \mathcal{F}_n(U) \to 0$ is also an exact sequence for all U.

4. (This problem sounds more confusing than it is!) Show that the presheaf kernel of a morphism of sheaves (of abelian groups, or $\mathcal{O}_X$-modules) is also sheaf. Show that it is the sheaf kernel (a kernel in the category of sheaves) as well. (This is one reason that kernels are easier than cokernels.)

5. The presheaf cokernel was defined in problem 2. Show that the sheafification of the presheaf cokernel is in fact the sheaf cokernel, by verifying that it satisfies the universal property.

6. Suppose $f : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves of abelian groups or $\mathcal{O}_X$-modules. Let $\operatorname{im} f$ be the sheafification of the "presheaf image". Show that there are natural isomorphisms $\operatorname{im} f \cong \mathcal{F}/\ker f$ and $\operatorname{coker} f \cong \mathcal{G}/\operatorname{im} f$. (This problem shows that this construction deserves to be called the "image".)

Date: Monday, October 17, 2005. (Corrected version October 20, 2005.)

7. Suppose $\mathcal{O}_X$ is a sheaf of rings on X. Define (categorically) what we should mean by tensor product of two presheaves or sheaves of $\mathcal{O}_X$-modules. Give an explicit construction, and show that it satisfies your categorical definition. *Hint:* take the "presheaf tensor product" — which needs to be defined — and sheafify. (This is admittedly a vague problem. If it is confusing, just ask. But it is good practice to turn your rough intuition into precise statements.)

8. Suppose $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H}$ is an exact sequence of sheaves (of abelian groups) on X. If $f : X \to Y$ is a continuous map, show that

$$0 \to f_*\mathcal{F} \to f_*\mathcal{G} \to f_*\mathcal{H}$$

is exact. Translation: pushforward is a left-exact functor. (The case of left-exactness of the global section functor can be interpreted as a special case of this, in the case where Y is a point.) Show that it needn't be exact on the right, i.e. that $f_*\mathcal{G} \to f_*\mathcal{H}$ needn't be surjective (= an epimorphism). (Hint: see the previous parenthetical comment, and think of your favorite short exact sequence of sheaves.)

The next three problems present some new concepts: gluing sheaves, sheaf homomorphisms, and flasque sheaves. I will feel comfortable using these concepts in class.

9. Suppose $X = \cup U_i$ is an open cover of X, and we have sheaves $\mathcal{F}_i$ on U_i along with isomorphisms $\phi_{ij} : \mathcal{F}_i|_{U_i \cap U_j} \to \mathcal{F}_j|_{U_i \cap U_j}$ that agree on triple overlaps (i.e. $\phi_{ij} \circ \phi_{jk} = \phi_{ij}$ on $U_i \cap U_j \cap U_k$). Show that these sheaves can be glued together into a unique sheaf $\mathcal{F}$ on X, such that $\mathcal{F}_i = \mathcal{F}|_{U_i}$, and the isomorphisms over $U_i \cap U_j$ are the obvious ones. (Thus we can "glue sheaves together", using limited patching information.)

10. Suppose $\mathcal{F}$ and $\mathcal{G}$ are two sheaves on X. Let $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})$ be the collection of data

$$\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})(U) := \mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U).$$

Show that this is a sheaf. (This is called the "sheaf $\underline{\mathrm{Hom}}$". If $\mathcal{F}$ and $\mathcal{G}$ are sheaves of sets, $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})$ is a sheaf of sets. If $\mathcal{G}$ is a sheaf of abelian groups, then $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})$ is a sheaf of abelian groups.) I've decided to call this $\underline{\mathrm{Hom}}$ rather than $\mathcal{H}\mathrm{om}$ because of the convention that "underlining often denotes sheaf". (Of course, the calligraphic font also often denotes sheaf.)

11. A sheaf $\mathcal{F}$ is said to be *flasque* if for every $U \subset V$, the restriction map $\mathrm{res}_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$ is surjective. In other words, every section over U extends to a section over V. This is a very strong condition, but it comes up surprisingly often.
(a) Show that $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is exact, and $\mathcal{F}'$ and $\mathcal{F}''$ are flasque, then so is $\mathcal{F}$.
(b) Suppose $f : X \to Y$ is a continuous map, and $\mathcal{F}$ is a flasque sheaf on X. Show that $f_*\mathcal{F}$ is a flasque sheaf on Y.
(If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is exact, and $\mathcal{F}'$ is flasque, then $0 \to \mathcal{F}'(U) \to \mathcal{F}(U) \to \mathcal{F}''(U) \to 0$ is exact, i.e. the global section functor is exact here, even on the right. Similarly, for any continuous map $f : X \to Y$, $0 \to f_*\mathcal{F}' \to f_*\mathcal{F} \to f_*\mathcal{F}'' \to 0$ is exact. I haven't thought about how hard this is yet, so I haven't made this part of the exercise. But it is good to know, and gives a reason to like flasque sheaves.)

Understanding sheaves via stalks

12. Prove that a section of a sheaf is determined by its germs, i.e.

$$\mathcal{F}(U) \to \coprod_{x \in U} \mathcal{F}_x$$

is injective. Hint: you won't use the gluability axiom. So this is true of morphisms of "separated presheaves". (This exercise is important, as you've seen!) Corollary: If a sheaf has all stalks 0, then it is the 0-sheaf.

13. Show that a morphism of sheaves on a topological space X induces a morphism of stalks. More precisely, if $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves on X, describe a natural map $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$.

14. Show that morphisms of sheaves are determined by morphisms of stalks. Hint # 1: you won't use the gluability axiom. Hint # 2: study the following diagram.

(1)
$$\begin{array}{ccc} \mathcal{F}(U) & \longrightarrow & \mathcal{G}(U) \\ \downarrow & & \downarrow \\ \prod_{x \in U} \mathcal{F}_x & \longrightarrow & \prod_{x \in U} \mathcal{G}_x \end{array}$$

15. Show that a morphism of sheaves is an isomorphism if and only if it induces an isomorphism of all stalks. Hint: Use (1). Injectivity of $\mathcal{F}(U) \to \mathcal{G}(U)$ uses the previous exercise. Surjectivity requires gluability. (I largely did this in class, so you should try this mainly if you want to make sure you are clear on the concept.)

16. Show that problems 12, 14, and 15 are false for presheaves in general. (Hint: take a 2-point space with the discrete topology, i.e. every subset is open.)

17. Show that for any morphism of presheaves $\phi : \mathcal{F} \to \mathcal{G}$, we get a natural induced morphism of sheaves $\phi^{sh} : \mathcal{F}^{sh} \to \mathcal{G}^{sh}$.

18. Show that the stalks of $\mathcal{F}^{sh}$ are the same as ("are naturally isomorphic to") the stalks of $\mathcal{F}$. Hint: Use the concrete description of the stalks.

Sheaves on a nice base

19. Suppose $\{B_i\}$ is a "nice base" for the topology of X.
(a) Verify that a morphism of sheaves is determined by the induced morphism of sheaves on the base.
(b) Show that a morphism of sheaves on the base (i.e. such that the diagram

$$\begin{array}{ccc} \Gamma(B_i, \mathcal{F}) & \longrightarrow & \Gamma(B_i, \mathcal{G}) \\ \downarrow & & \downarrow \\ \Gamma(B_j, \mathcal{F}) & \longrightarrow & \Gamma(B_j, \mathcal{G}) \end{array}$$

commutes for all $B_j \hookrightarrow B_i$) gives a morphism of the induced sheaves.

The inverse image sheaf

Suppose we have a continuous map $f : X \to Y$. If $\mathcal{F}$ is a sheaf on X, we have defined the pushforward $f_*\mathcal{F}$, which is a sheaf on Y. There is also a notion of inverse image. If $\mathcal{G}$ is a sheaf on Y, then there is a sheaf on X, denoted $f^{-1}\mathcal{G}$. This gives a covariant functor from sheaves on Y to sheaves on X. For example, if we have a morphism of sheaves on Y, we'll get an induced morphism of their inverse image sheaves on X.

Here is a concrete but unmotivated (and frankly unpleasant) definition: temporarily define $f^{-1}\mathcal{G}^{\mathrm{pre}}(U) = \lim_{\to V \supset f(U)} \mathcal{G}(V)$. (Recall explicit description of direct limit: sections are sections on open sets containing $f(U)$, with an equivalence relation.)

20. Show that this defines a presheaf on X.

Now define the *inverse image sheaf* $f^{-1}\mathcal{G} := (f^{-1}\mathcal{G}^{\mathrm{pre}})^{\mathrm{sh}}$.

21. Show that the stalks of $f^{-1}\mathcal{G}$ are the same as the stalks of $\mathcal{G}$. More precisely, if $f(x) = y$, describe a natural isomorphism $\mathcal{G}_y \cong (f^{-1}\mathcal{G})_x$. (Hint: use the concrete description of the stalk, as a direct limit.)

22. Show that f^{-1} is an exact functor from sheaves of abelian groups on Y to sheaves of abelian groups on X. (Hint: exactness can be checked on stalks, and by the previous exercise, stalks are the same.) The identical argument will show that f^{-1} is an exact functor from sheaves of $\mathcal{O}_Y$-modules on Y to sheaves of $f^{-1}\mathcal{O}_Y$-modules on X, but don't bother writing that down.

Here is a categorical definition of inverse image: it is left-adjoint to f_*. More precisely, suppose $f : X \to Y$ is a continuous map (= morphism) of topological spaces, and $\mathcal{F}$ is a sheaf of sets on X, and $\mathcal{G}$ is a sheaf of sets on Y. There is a natural bijection between $\mathrm{Hom}(f^{-1}(\mathcal{G}), \mathcal{F})$ and $\mathrm{Hom}(\mathcal{G}, f_*\mathcal{F})$. (The same argument will apply for sheaves of abelian groups etc.)

23. Show that the explicit definition of inverse image satisfies this universal property. (Just describe the bijection. One should also check that this bijection is natural, i.e. that for any $\mathcal{F}_1 \to \mathcal{F}_2$, the diagram

$$\begin{array}{ccc} \mathrm{Hom}(f^{-1}(\mathcal{G}), \mathcal{F}_2) & \longrightarrow & \mathrm{Hom}(\mathcal{G}, f_*\mathcal{F}_2) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(f^{-1}(\mathcal{G}), \mathcal{F}_1) & \longrightarrow & \mathrm{Hom}(\mathcal{G}, f_*\mathcal{F}_1) \end{array}$$

commutes, and something similar for the "left argument", but don't worry about that.) This problem requires some elbow grease.

A small exercise on a small affine scheme

24. Describe the set $\mathrm{Spec}\, k[x]/x^2$. This seems like a very boring example, but it will grow up to be very important indeed! (This is not hard.)

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