FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASSES 51 AND 52

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CONTENTS

1. SMOOTH, ÉTALE, UNRAMIFIED

We will next describe analogues of some important notions in differential geometry — the following particular types of maps of manifolds. They naturally form a family of three.

- *Submersions* are maps that induce surjections of tangent spaces everywhere. They are useful in the notion of a fibration.
- *Covering spaces* are maps that induce isomorphisms of tangent spaces, or equivalently, are local isomorphisms.
- *Immersions* are maps that induce injections of tangent spaces.

Warning repeated from earlier: "immersion" is often used in algebraic geometry with a different meaning. We won't use this word in an algebro-geometric context (without an adjective such as "open" or "closed") in order to avoid confusion. I drew pictures of the three. (A fourth notion is related to these three: a map of manifolds is an *embedding* if it is an immersion that is an inclusion of sets, where the source has the subspace topology. This is analogous to *locally closed immersion* in algebraic geometry.)

We will define algebraic analogues of these three notions: smooth, étale, and unramified. In the case of nonsingular varieties over an algebraically closed field, we could take the differential geometric definition. We would like to define these notions more generally. Indeed, one of the points of algebraic geometry is to generalize "smooth" notions to singular situations. Also, we'll want to make arguments by "working over" the generic point, and also over nonreduced subschemes. We may even want to do things over non-algebraically closed fields, or over the integers.

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Our definitions will be combinations of notions we've already seen, and thus we'll see that they have many good properties. We'll see (§2.1) that in the category of nonsingular varieties over algebraically closed fields, we recover the differential geometric definition. Our three definitions won't be so obviously a natural triplet, but I'll mention the definition given in EGA (§4.1), and in this context once again the definitions are very similar.

Let's first consider some examples of things we want to be analogues of "covering space" and "submersion", and see if they help us make good definitions.

We'll start with something we would want to be a covering space. Consider the parabola $x = y^2$ projecting to the x-axis, over the complex numbers. (This example has come up again and again!) We might reasonably want this to be a covering space away from the origin. We might also want the notion of covering space to be an open condition: the locus where a morphism is a covering space should be open on the source. This is true for the differential geometric definition. (More generally, we might want this notion to be preserved by base change.) But then this should be a "covering space" over the generic point, and here we get a non-trivial residue field extension ($\mathbb{C}(y)/\mathbb{C}(y^2)$), not an isomorphism. Thus we are forced to consider (the Spec's of) certain finite extensions of fields to be covering spaces. (We'll see soon that we just want separable extensions.)

Note also in this example there are no (non-empty) Zariski-open subsets $U \subset X$ and $V \subset V$ where the map sends U into V isomorphically. This will later lead to the notion of the étale topology, which is a bizarre sort of topology (not even a topology in the usual sense, but a "Grothendieck topology").

1.1. Here is an issue with smoothness: we would certainly want the fibers to be smooth, so reasonably we would want the fibers to be nonsingular. But we know that nonsingularity over a field does not behave well over a base change (consider $\operatorname{Spec} k(t)[u]/(u^p - t) \to \operatorname{Spec} k(t)$ and base change by $\operatorname{Spec} k(t)[v]/(v^p - t) \to \operatorname{Spec} k(t)$, where $\operatorname{char} k = p$). We can patch that by noting that nonsingularity behaves well over algebraically closed fields, and hence we could require that all the geometric fibers are nonsingular. But that isn't quite enough. For example, a horrible map from a scheme X to a curve Y that maps a different nonsingular variety to a each point Y (X is an infinite disjoint union of these) should not be considered a submersion in any reasonable sense. Also, we might reasonably not want to consider $\operatorname{Spec} k \to \operatorname{Spec} k[\epsilon]/\epsilon^2$ to be a submersion (for example, this isn't surjective on tangent spaces, and more generally the picture "doesn't look like a fibration"). (I drew pictures of these two pathologies.) Both problems are failures of $\pi : X \to Y$ to be a nice, "continuous" family. Whenever we are looking for some vague notion of "niceness" we know that "flatness" will be in the definition. (This is the reason we waited so long before introducing the notion of smoothness — we needed to develop flatness first!)

One last issue: we will require the geometric fibers to be varieties, so we can think of them as "smooth" in the old-fashioned intuitive sense. We could impose this by requiring our morphisms to be locally of finite type, or (a stronger condition) locally of finite presentation.

I should have defined "locally of finite presentation" back when we defined "locally of finite type" and the many other notions satisfying the affine covering lemma. It isn't any harder. A morphism of affine schemes $\operatorname{Spec} A \to \operatorname{Spec} B$ is *locally of finite presentation* if it corresponds to $B \to B[x_1, \dots, x_n]/(f_1, \dots, f_r)$ — A should be finitely generated over B, and also have a finite number of relations. This notion satisfies the hypotheses of the affine covering lemma. A morphism of schemes $\pi : X \to Y$ is *locally of finite presentation* if every map of affine open sets $\operatorname{Spec} A \to \operatorname{Spec} B$ induced by π is locally of finite presentation. If you work only with locally Noetherian schemes, then these two notions are the same.

I haven't thought through why Grothendieck went with the stricter condition of "locally of finite presentation" in his definition of smooth etc., rather than "locally of finite type".

Finally, we define our three notions!

1.2. Definition. A morphism $\pi : X \to Y$ is *smooth of relative dimension* n provided that it is locally of finite presentation and flat of relative dimension n, and $\Omega_{X/Y}$ is locally free of rank n.

A morphism $\pi : X \to Y$ is *étale* provided that it is locally of finite presentation and flat, and $\Omega_{X/Y} = 0$.

A morphism $\pi : X \to Y$ is *unramified* provided that it is locally of finite presentation, and $\Omega_{X/Y} = 0$.

1.3. Examples.

- $\mathbb{A}^n_Y \to Y$, $\mathbb{P}^n_Y \to Y$ are smooth morphisms of relative dimension n.
- Locally finitely presented open immersions are étale.
- *Unramified.* Locally finitely presented locally closed immersions are unramified.

1.4. Quick observations and comments.

1.5. All three notions are local on the target, and local on the source, and are preserved by base change. That's because all of the terms arising in the definition have these properties. *Exercise.* Show that all three notions are open conditions. State this rigorously and prove it. (Hint: Given $\pi : X \to Y$, then there is a largest open subset of X where π is smooth of relative dimension n, etc.)

1.6. Note that π is étale if and only if π is smooth and unramified, if and only if π is flat and unramified.

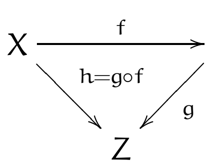
1.7. *Jacobian criterion.* The smooth and étale definitions are perfectly set up to use a Jacobian criterion. *Exercise.* Show that $\operatorname{Spec} B[x_1, \dots, x_n]/(f_1, \dots, f_r) \to \operatorname{Spec} B$ is smooth of relative dimension n (resp. étale) if it is flat of relative dimension n (resp. flat) and the corank of Jacobian matrix is n (resp. the Jacobian matrix is full rank).

1.8. *Exercise: smoothness etc. over an algebraically closed field.* Show that if k is an algebraically closed field, $X \to \operatorname{Spec} k$ is smooth of relative dimension n if and only if X is a disjoint union of nonsingular k-varieties of dimension n. (Hint: use the Jacobian criterion.) Show that $X \to \operatorname{Spec} k$ is étale if and only if it is unramified if and only if X is a union of points isomorphic to $\operatorname{Spec} k$. More generally, if k is a field (not necessarily algebraically closed), show that $X \to \operatorname{Spec} k$ is étale if and only if it is unramified if and only if X is the disjoint union of Spec's of finite separable extensions of k.

1.9. A morphism $\pi : X \to Y$ is *smooth* if it is locally of finite presentation and flat, and in an open neighborhood of every point $x \in X$ in which π is of constant relative dimension, $\Omega_{X/Y}$ is locally free of that relative dimension. (I should have shown earlier that the locus where a locally of finite presentation morphism is flat of a given relative dimension is open, but I may not have. We indeed showed the fact without the "relative dimension" statement, and the argument is essentially the same with this condition added.) (*Exercise.* Show that π is smooth if X can be written as a disjoint union $X = \coprod_{n \geq 0} X_n$ where $\pi|_{X_n}$ is smooth of relative dimension n.) This notion isn't really as "clean" as "smooth of relative dimension n", but people often use the naked adjective "smooth" for simplicity.

Exercise. Show that étale is the same as smooth of relative dimension 0. In other words, show that étale implies relative dimension 0. (Hint: if there is a point $x \in X$ where π has positive relative dimension, show that $\Omega_{X/Y}$ is not 0 at x. You may want to base change, to consider just the fiber above $\pi(x)$.)

1.10. Note that unramified doesn't have a flatness hypothesis, and indeed we didn't expect it, as we would want the inclusion of the origin into $\mathbb{A}^1$ to be unramified. Thus seemingly pathological things of the sort we excluded from the notion of "smooth" and "unramified" morphisms are unramified. For example, if $X = \coprod_{z \in \mathbb{C}} \operatorname{Spec} \mathbb{C}$, then the morphism $X \to \mathbb{A}^1_{\mathbb{C}}$ sending the point corresponding to z to the point $z \in \mathbb{A}^1_{\mathbb{C}}$ is unramified. Such is life.

Exercise. Suppose 

are locally finitely presented morphisms.

(a) Show that if h is unramified, then so is f. (Hint: property P exercise.)
(b) Suppose g is étale. Show that f is smooth (resp. étale, unramified) if and only if h is. (Hint: Observe that $\Omega_{X/Y} \to \Omega_{X/Y}$ is an isomorphism from the relative cotangent sequence, see 2.3 for a reminder.)

Regularity vs. smoothness. Suppose $\operatorname{char} k = p$, and consider the morphism $\operatorname{Spec} k(u) \to \operatorname{Spec} k(u^p)$. Then the source is nonsingular, but the morphism is not étale (or smooth, or unramified).

In fact, if k is not algebraically closed, "nonsingular" isn't a great notion, as we saw in the fall when we had to work hard to develop the theory of nonsingularity. Instead, "smooth (of some dimension)" over a field is much better. You should almost go back in your notes and throw out our discussion of nonsingularity. But don't — there were a couple of key concepts that have been useful: discrete valuation rings (nonsingularity in codimension 1) and nonsingularity at closed points of a variety (nonsingularity in top codimension).

2. Harder facts

I want to segregate three facts which require more effort, to emphasize that the earlier facts are automatic given what we know.

2.1. Connection to differential-geometric notion of smoothness.

The following exercise makes the connection to the differential-geometric notion of smoothness. Unfortunately, we will need this fact in the next section on generic smoothness.

2.2. *Trickier Exercise.* Suppose $\pi : X \to Y$ is a morphism of smooth (pure-dimensional) varieties over a field k. Let $n = \dim X - \dim Y$. Suppose that for each closed point $x \in X$, the induced map on the Zariski tangent space $T_f : T_x \to T_y$ is surjective. Show that f is smooth of relative dimension n. (Hint: The trickiest thing is to show flatness. Use the (second) local criterion for flatness.)

I think this is the easiest of the three "harder" facts, and it isn't so bad.

For pedants: I think the same argument works over a more arbitrary base. In other words, suppose in the following diagram of pure-dimensional Noetherian schemes, Y is reduced.

$$\begin{array}{ccc} X & \longrightarrow & Y \\ & {\scriptstyle smooth}\searrow \quad \swarrow{\scriptstyle smooth} & \\ & Z & \end{array}$$

Let $n = \dim X - \dim Y$. Suppose that for each closed point $x \in X$, the induced map on the Zariski tangent space $T_f : T_x \to T_y$ is surjective. Show that f is smooth of relative dimension n. I think the same argument works, with a twist at the end using Exercise 1.10(b). Please correct me if I'm wrong!

2.3. The relative cotangent sequence is left-exact in good circumstances.

Recall the relative cotangent sequence. Suppose $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms of schemes. Then there is an exact sequence of quasicoherent sheaves on X

$$f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0.$$

We have been always keeping in mind that if you see a right-exact sequence, you should expect that this is the tail end of a long exact sequence. In this case, you should expect that the next term to the left (the "H_1 term") should depend just on X/Y, and not on Z, because the last term on the right does. Indeed this is the case: these "homology" groups are called André-Quillen homology groups. You might also hope then that in some mysteriously "good" circumstances, this first "H_1" on the left should vanish, and hence the relative cotangent sequence should be exact on the left. Indeed that is the case, as is hinted by the following exercise.

2.4. *Exercise on differentials.* If $X \to Y$ is a smooth morphism, show that the relative cotangent sequence is exact on the left as well.

This exercise is the reason this discussion is in the "harder" section — the rest is easy. Can someone provide a clean proof of this fact?

2.5. *Unimportant exercise.* Predict a circumstance in which the relative conormal sequence is left-exact.

2.6. *Corollary.* Suppose f is étale. Then the pullback of differentials $f^*\Omega_{Y/Z} \to \Omega_{X/Z}$ is an isomorphism. (This should be very believable to you from the picture you should have in your head!)

2.7. *Exercise.* Show that all three notions are preserved by composition. (More precisely, in the smooth case, smooth of relative dimension m composed with smooth of relative dimension n is smooth of relative dimension $n + m$.) You'll need Exercise 2.4 in the smooth case.

2.8. *Easy exercise.* Show that all three notions are closed under products. (More precisely, in the case of smoothness: If $X, Y \to Z$ are smooth of relative dimension m and n respectively, then $X \times_Z Y \to Z$ is smooth of relative dimension $m + n$.) (Hint: This is a consequence of base change and composition, as we have discussed earlier. Consider $X \times_Z Y \to Y \to Z$.)

2.9. *Exercise: smoothness implies surjection of tangent sheaves.* Continuing the terminology of the above, Suppose $X \to Y$ is a smooth morphism of Z-schemes. Show that $0 \to T_{X/Y} \to T_{X/Z} \to f^*T_{Y/Z} \to 0$ is an exact sequence of sheaves, and in particular, $T_{X/Z} \to f^*T_{Y/Z}$ is surjective, paralleling the notion of submersion in differential geometry. (Recall $T_{X/Y} = \underline{\mathrm{Hom}}(\Omega_{X/Y}, \mathcal{O}_X)$ and similarly for $T_{X/Z}$, $T_{Y/Z}$.)

2.10. Characterization of smooth and etale in terms of fibers.

By Exercise 1.8, we know what the fibers look like for étale and unramified morphisms; and what the geometric fibers look like for smooth morphisms. There is a good characterization of these notions in terms of the geometric fibers, and this is a convenient way of thinking about the three definitions.

2.11. *Exercise: characterization of etale and unramified morphisms in terms of fibers.* Suppose $\pi : X \to Y$ is a morphism locally of finite presentation. Prove that π is étale if and only if it is flat, and the geometric fibers (above $\operatorname{Spec} \overline{k} \to Y$, say) are unions of Spec's of fields (with discrete topology), each a finite separable extension of the field $\overline{k}$. Prove that π is unramified if and only if the geometric fibers (above $\operatorname{Spec} \overline{k} \to Y$, say) are unions of Spec's of fields (with discrete topology), each a finite separable extension of the field $\overline{k}$. (Hint: a finite type sheaf that is 0 at all points must be the 0-sheaf.)

There is an analogous statement for smooth morphisms, that is harder. (That's why this discussion is in the "harder" section.)

2.12. *Harder exercise.* Suppose $\pi : X \to Y$ is locally of finite presentation. Show that π is smooth of relative dimension n if and only if π is flat, and the geometric fibers are disjoint unions of n-dimensional nonsingular varieties (over the appropriate field).

3. Generic smoothness in characteristic 0

We will next see a number of important results that fall under the rubric of "generic smoothness". All will require working over a field of characteristic 0 in an essential way. So far in this course, we have had to add a few caveats here and there for people encountering positive characteristic. This is probably the first case where positive characteristic people should just skip this section.

Our first result is an algebraic analog of Sard's theorem.

3.1. *Proposition (generic smoothness in the source). — Let k be a field of characteristic 0, and let $\pi : X \to Y$ be a dominant morphism of integral finite-type k-schemes. Then there is a non-empty (=dense) open set $U \subset X$ such that $\pi|_U$ is smooth.*

We've basically seen this argument before, when we showed that a variety has an open subset that is nonsingular.

Proof. Define $n = \dim X - \dim Y$ (the "relative dimension"). Now $FF(X)/FF(Y)$ is a finitely generated field extension of transcendence degree n. It is separably generated by n elements (as we are in characteristic 0). Thus Ω has rank n at the generic point. Its rank is at least n everywhere. By uppersemicontinuity of fiber rank of a coherent sheaf, it is rank n for every point in a dense open set. Recall that on a reduced scheme, constant rank implies locally free of that rank (Class 15, Exercise 5.2); hence Ω is locally free of rank n on that set. Also, by openness of flatness, it is flat on a dense open set. Let U be the intersection of these two open sets. $\square$

For pedants: In class, I retreated to this statement above. However, I think the following holds. Suppose $\pi : X \to Y$ is a dominant finite type morphism of integral schemes, where $\operatorname{char} FF(Y) = 0$ (and hence $\operatorname{char} FF(X) = 0$ from $FF(Y) \hookrightarrow FF(X)$). Then there is a non-empty open set $U \subset X$ such that $\pi|_U$ is smooth.

The proof above needs the following tweak. Define $n = \dim X - \dim Y$. Let η be the generic point of Y, and let X_η be fiber of π above η; it is non-empty by the dominant hypothesis. Then X_η is a finite type scheme over $FF(Y)$. I claim $\dim X_\eta = n$. Indeed, π is flat near X_η (everything is flat over a field, and flatness is an open condition), and we've shown for a flat morphism the dimension of the fiber is the dimension of the source minus the dimension of the target. Then proceed as above.

Please let me know if I've made a mistake!

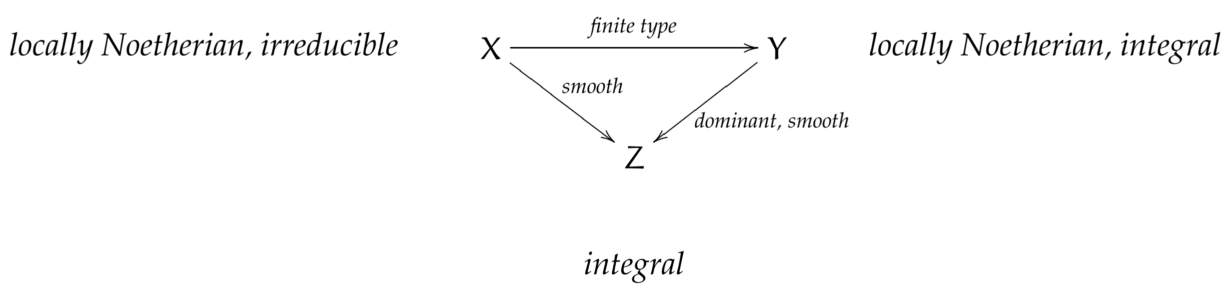
3.2. In §1.1, we saw an example where this result fails in positive characteristic, involving an inseparable extension of fields. Here is another example, over an *algebraically closed* field of characteristic p: $\mathbb{A}^1_k = \operatorname{Spec} k[t] \to \operatorname{Spec} k[u] = \mathbb{A}^1_k$, given by $u \mapsto t^p$. The earlier example (§1.1) is what is going on at the generic point.

If the source of π is smooth over a field, the situation is even nicer.

3.3. *Theorem (generic smoothness in the target). — Suppose $f : X \to Y$ is a morphism of k-varieties, where $\operatorname{char} k = 0$, and X is smooth over k. Then there is a dense open subset of Y such that $f|_{f^{-1}(U)}$ is a smooth morphism.*

(Note: $f^{-1}(U)$ may be empty! Indeed, if f is not dominant, we will have to take such a U.)

For pedants: I think the following generalization holds, assuming that my earlier notes to pedants aren't bogus. Generalize the above hypotheses to the following morphisms of $\mathbb{Q}$-schemes. (Requiring a scheme to be defined over $\mathbb{Q}$ is precisely the same as requiring it to "live in characteristic 0", i.e. the morphism to $\operatorname{Spec} \mathbb{Z}$ has image precisely $[(0)]$.)



To prove this, we'll use a neat trick.

3.4. *Lemma. — Suppose $\pi : X \to Y$ is a morphism of schemes that are finite type over k, where $\operatorname{char} k = 0$. Define*

$$X_r = \{\textit{closed points } x \in X : \textit{rank } T_{\pi,x} \leq r\}.$$

Then $\dim f(X_r) \leq r$*. (Note that* X_r *is a closed subset; it is cut out by determinantal equations. Hence by Chevalley's theorem, its image is constructible, and we can take its dimension.)*

For pedants: I think the only hypotheses we need are that π *is a finite type morphism of locally Noetherian schemes over* $\mathbb{Q}$*. The proof seems to work as is, after an initial reduction to verifying it on an arbitrary affine open subset of* Y*.*

Here is an example of the lemma, to help you find it believable. Suppose X is a nonsingular surface, and Y is a nonsingular curve. Then for each $x \in x$, the tangent map $T_{\pi,x} : T_x \to T_{\pi(x)}$ is a map from a two-dimensional vector space to a one-dimensional vector space, and thus has rank 1 or 0. I then drew some pictures. If π is dominant, then we have a picture like this [omitted]. The tangent map has rank 0 at this one point. The image is indeed rank 0. The tangent map has rank at most 1 everywhere. The image indeed has rank 1.

Now imagine that π contracted X to a point. Then the tangent map has rank 0 everywhere, and indeed the image has dimension 0.

Proof of lemma. We can replace by X by an irreducible component of X_r, and Y by the closure of that component's image of X in Y. (The resulting map will have all of X contained in X_r. This boils down to the following linear algebra observation: if a linear map $\rho : V_1 \to V_2$ has rank at most r, and V_i' is a subspace of V_i, with ρ sending V_1' to V_2', then the restriction of ρ to V_1' has rank at most that of ρ itself.) Thus we have a dominant morphism $f : X \to Y$, and we wish to show that $\dim Y \leq r$. By generic smoothness on the source (Proposition 3.1), there is a nonempty open subset $U \subset X$ such that $f : U \to Y$ is smooth. But then for any $x \in X$, the tangent map $T_{x,X} \to T_{\pi(x),Y}$ is surjective (by smoothness), and has rank at most r, so $\dim Y = \dim_{\pi(x)} Y \leq \dim T_{\pi(x),Y} \leq r$. □

There's not much left to prove the theorem.

Proof of Theorem 3.3. Reduce to the case Y smooth over k (by restricting to a smaller open set, using generic smoothness of Y, Proposition 3.1). Say $n = \dim Y$. $\dim \overline{f(X_{n-1})} \leq n-1$ by the lemma, so remove this as well. Then the rank of T_f is at least r for each closed point of X. But as Y is nonsingular of dimension r, we have that T_f is surjective for every closed point of X, hence surjective. Thus f is smooth by Hard Exercise 2.2. □

3.5. The Kleiman-Bertini theorem. The Kleiman-Bertini theorem is elementary to prove, and extremely useful, for example in enumerative geometry.

Throughout this discussion, we'll work in the category of k-varieties, where k is an algebraically closed field of characteristic 0. The definitions and results generalize easily to the non-algebraically closed case, and I'll discuss this parenthetically.

3.6. Suppose G is a group variety. Then I claim that G is smooth over k. Reason: It is generically smooth (so it has a dense open set U that is smooth), and G acts transitively on itself (so we can cover G with translates of U).

We can generalize this. We say that a G-action $a : G \times X \to X$ on a variety X is *transitive* if it is transitive on closed points. (If k is not algebraically closed, we replace this by saying that it is transitive on $\overline{k}$-valued points. In other words, we base change to the algebraic closure, and ask if the resulting action is transitive. Note that in characteristic 0, reduced = geometrically reduced, so G and X both remain reduced upon base change to $\overline{k}$.)

In other words, if U is a non-empty open subset of X, then we can cover X with translates of X. (Translation: $G \times U \to X$ is surjective.) Such an X (with a transitive G-action) is called a *homogeneous space* for G.

3.7. *Exercise.* Paralleling §3.6, show that a homogeneous space X is smooth over k.

3.8. *The Kleiman-Bertini theorem. — Suppose* X *is homogeneous space for group variety* G *(over an algebraically closed field* k *of characteristic* 0*). Suppose* $f : Y \to X$ *and* $g : Z \to X$ *be morphisms from smooth* k*-varieties* Y, Z*. Then there is a nonempty open subset* $V \subset G$ *such that for every* $\sigma \in V(k)$, $Y \times_X Z$ *defined by*

$$\begin{array}{ccc} Y \times_X Z & \longrightarrow & Z \\ \downarrow & & \downarrow g \\ Y & \xrightarrow{\sigma \circ f} & X \end{array}$$

(i.e. Y *is "translated by* σ*") is smooth over* k *of dimension exactly* $\dim Y + \dim Z - \dim X$*. Better: there is an open subset of* $V \subset G$ *such that*

$$(G \times_k Y) \times_X Z \to G \qquad (1)$$

is a smooth morphism of relative dimension $\dim Y + \dim Z - \dim X$.

(The statement and proof will carry through even if k is not algebraically closed.)

The first time you hear this, you should think of the special case where $Y \to X$ and $Z \to X$ are closed immersions (hence "smooth subvarieties"). In this case, the Kleiman-Bertini theorem says that the second subvariety will meet a "general translate" of the first transversely.

Proof. It is more pleasant to describe this proof "backwards", by considering how we would prove it ourselves. We will end up using generic smoothness twice, as well as many facts we now know and love.

In order to show that the morphism (1) is generically smooth on the target, it would suffice to apply Theorem 3.3), so we wish to show that $(G \times_k Y) \times_X Z$ is a smooth k-variety. Now Z is smooth over k, so it suffices to show that $(G \times_k Y) \times_X Z \to Z$ is a smooth morphism (as the composition of two smooth morphisms is smooth). But this is obtained by base changed from $G \times_k Y \to X$, so it suffices to show that this latter morphism is smooth (as smoothness is preserved by base change).

This is a G-equivariant morphism $G \times_k Y \xrightarrow{a \circ f} X$. (By "G-equivariant", we mean that G action on both sides respects the morphism.) By generic smoothness of the target (Theorem 3.3), this is smooth over a dense open subset X. But then by transitivity of the G action, this morphism is smooth (everywhere). (*Exercise:* verify the relative dimension statement.) □

3.9. *Corollary (Bertini's theorem, improved version). Suppose* X *is a smooth* k*-variety, where* k *is algebraically closed of characteristic* 0*. Let* δ *be a finite-dimensional base-point-free linear system, i.e. a finite vector space of sections of some invertible sheaf* $\mathcal{L}$*. Then almost every element of* δ*, considered as a closed subscheme of* X*, is nonsingular. (More explicitly: each element* $s \in H^0(X, \mathcal{L})$ *gives a closed subscheme of* X*. For a general* s*, considered as a point of* $\mathbb{P}H^0(X, \mathcal{L})$*, the closed subscheme is smooth over* k*.)*

(Again, the statement and proof will carry through even if k is not algebraically closed.)

This is a good improvement on Bertini's theorem. For example, we don't actually need $\mathcal{L}$ to be very ample, or X to be projective.

3.10. *Exercise.* Prove this!

3.11. *Easy Exercise.* Interpret the old version of Bertini's theorem (over a characteristic 0 field) as a corollary of this statement.

Note that this fails in positive characteristic, as shown by the one-dimensional linear system $\{pP : P \in \mathbb{P}^1\}$. This is essentially Example 3.2.

4. Formal interpretations

For those of you who like complete local rings, or who want to make the connection to complex analytic geometry, here are some useful reformulations, which I won't prove.

Suppose $(B, \mathfrak{n}) \to (A, \mathfrak{m})$ is a map of Noetherian local rings, inducing an isomorphism of residue fields, and a morphism of completions at the maximal ideals $\hat{B} \to \hat{A}$ (the "hat" terminology arose first in class 13, immediately after the statement of Theorem 2.2). Then the induced map of schemes $\operatorname{Spec} A \to \operatorname{Spec} B$ is:

- *étale* if $\hat{B} \to \hat{A}$ is a bijection.
- *smooth* if $\hat{B} \to \hat{A}$ is isomorphic to $\hat{B} \to \hat{B}[[x_1, \dots, x_n]]$. In other words, formally, smoothness involves adding some free variables. (In case I've forgotten to say this before: "Formally" means "in the completion".)
- *unramified* if $\hat{B} \to \hat{A}$ is surjective.

4.1. Formally unramified, smooth, and étale. EGA has defines these three notions differently. The definitions there make clear that these three definitions form a family, in a way that is quite similar to the differential-geometric definition. (You should largely ignore what follows, unless you find later in life that you really care. I won't prove anything.) We say that $\pi : X \to Y$ is *formally smooth* (resp. *formally unramified*, *formally étale*) if for all affine schemes Z, and every closed subscheme Z_0 defined by a nilpotent ideal, and every morphism $Z \to Y$, the canonical map $\mathrm{Hom}_Y(Z, X) \to \mathrm{Hom}_Y(Z_0, X)$ is surjective (resp. injective, bijective). This is summarized in the following diagram, which is reminiscent of the valuative criteria for separatedness and properness.

$$\begin{array}{ccc} \operatorname{Spec} Z_0 & \longrightarrow & X \\ {\scriptstyle \text{nilpotent ideal}}\ \big\downarrow & \overset{?}{\nearrow} & \big\downarrow\ \pi \\ \operatorname{Spec} Z & \longrightarrow & Y \end{array}$$

(Exercise: show that this is the same as the definition we would get by replacing "nilpotent" by "square-zero". This is sometimes an easier formulation to work with.)

EGA defines smooth as morphisms that are formally smooth and locally of finite presentation (and similarly for the unramified and étale).

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