FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASSES 45 AND 46

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CONTENTS

This week: Grauert's theorem and the Cohomology and base change theorem, and applications. The Rigidity Lemma. Proof of Grauert's theorem. Dimensions behave well for flat morphisms. Associated points go to associated points.

1. COHOMOLOGY AND BASE CHANGE THEOREMS

We're in the midst of discussing a family of theorems involving the following situation. Suppose $\mathcal{F}$ is a coherent sheaf on X, $\pi : X \to Y$ projective, Y (hence X) Noetherian, and $\mathcal{F}$ flat over Y.

Here are two related questions. Is $R^p\pi_*\mathcal{F}$ locally free? Is $\phi^p : R^p\pi_*\mathcal{F}\otimes k(y) \to H^p(X_y, \mathcal{F}_y)$ an isomorphism?

We have shown a key intermediate result, that if Y is affine, say $Y = \operatorname{Spec} B$, then we can compute the pushforwards of $\mathcal{F}$ by a complex of locally free modules

$$0 \to M^0 \to M^1 \to \cdots \to M^n \to 0$$

where in fact M^p is free for $p > 1$. Moreover, this computes pushforwards "universally": after a base change, this remains true.

We have already shown the constancy of Euler characteristic, and the semicontinuity theorem. I'm now going to discuss two big theorems, Grauert's theorem and the Cohomology and base change theorem, that are in some sense the scariest in Hartshorne, coming at the end of Chapter III (along with the semicontinuity theorem). I hope you

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agree that semicontinuity isn't that scary (given the key fact). I'd like to discuss applications of these two theorems to show you why you care; then given time I'll give proofs. I've found the statements worth remembering, even though they are a little confusing.

Note that if $R^p\pi_*\mathcal{F}$ is locally free and ϕ^p is an isomorphism, then the right side is locally constant. The following is a partial converse.

1.1. *Grauert's Theorem. — If Y is reduced, then h^p locally constant implies $R^p\pi_*\mathcal{F}$ is locally free and ϕ^p is an isomorphism.*

1.2. *Cohomology and base change theorem. — Assume ϕ^p is surjective. Then the following hold.*

(a) *ϕ^p is an isomorphism, and the same is true nearby. [Note: The hypothesis is trivially satisfied in the common case $H^p = 0$. If $H^p = 0$ at a point, then it is true nearby by semicontinuity.]*
(b) *ϕ^{p-1} is surjective (=isomorphic) if and only if $R^p\pi_*\mathcal{F}$ is locally free. [This in turn implies that h^p is locally constant.]*

Notice that (a) is about just what happens over the reduced scheme, but (b) has a neat twist: you can check things over the reduced scheme, and it has implications over the scheme as a whole!

Here are a couple of consequences.

1.3. *Exercise.* Suppose $H^p(X_y, \mathcal{F}_y) = 0$ for all $y \in Y$. Show that ϕ^{p-1} is an isomorphism for all $y \in Y$. (Hint: cohomology and base change (b).)

1.4. *Exercise.* Suppose $R^p\pi_*\mathcal{F} = 0$ for $p \geq p_0$. Show that $H^p(X_y, \mathcal{F}_y) = 0$ for all $y \in Y$, $k \geq k_0$. (Same hint. You can also do this directly from the key theorem above.)

2. When the pushforward of the functions on X are the functions on Y

Many fun applications happen when a certain hypothesis holds, which I'll now describe.

We say that π satisfies (*) if it is projective, and the natural morphism $\mathcal{O}_Y \to \pi_*\mathcal{O}_X$ is an isomorphism. Here are two statements that will give you a feel for this notion. First:

2.1. *Important Exercise.* Suppose π is a projective flat family, each of whose fibers are (nonempty) integral schemes, or more generally whose fibers satisfy $h^0(X_y) = 1$. Then (*) holds. (Hint: consider

$$\mathcal{O}_Y \otimes k(y) \longrightarrow (\pi_*\mathcal{O}_X) \otimes k(y) \xrightarrow{\phi^0} H^0(X_y, \mathcal{O}_{X_y}) \cong k(y) \ .$$

The composition is surjective, hence ϕ^0 is surjective, hence it is an isomorphism (by the Cohomology and base change theorem 1.2 (a)). Then thanks to the Cohomology and base change theorem 1.2 (b), $\pi_* \mathcal{O}_X$ is locally free, thus of rank 1. If I have a map of invertible sheaves $\mathcal{O}_Y \to \pi_* \mathcal{O}_X$ that is an isomorphism on closed points, it is an isomorphism (everywhere) by Nakayama.)

Note in the previous exercise: we are obtaining things not just about closed points!

Second: we will later prove a surprisingly hard result, that given any projective (proper) morphism of Noetherian schemes satisfying (*) (without any flatness hypotheses!), the fibers are all connected ("Zariski's connectedness lemma").

2.2. *Exercise (the Hodge bundle; important in Gromov-Witten theory).* Suppose $\pi : X \to Y$ is a projective flat family, all of whose geometric fibers are connected reduced curves of arithmetic genus g. Show that $R^1\pi_* \mathcal{O}_X$ is a locally free sheaf of rank g. This is called the *Hodge bundle*. [Hint: use cohomology and base change (b) twice, once with $p = 2$, and once with $p = 1$.]

Here is the question we'll address in this section. Given an invertible sheaf $\mathcal{L}$ on X, we wonder when it is the pullback of an invertible sheaf $\mathcal{M}$ on Y. Certainly it is necessary for it to be trivial on the fibers. We'll see that (*) holds, then this basically suffices. Here is the idea: given $\mathcal{L}$, how can we recover $\mathcal{M}$? Thanks to the next exercise, it must be $\pi_* \mathcal{L}$.

2.3. *Exercise.* Suppose $\pi : X \to Y$ satisfies (*). Show that if $\mathcal{M}$ is any invertible sheaf on Y, then the natural morphism $\mathcal{M} \to \pi_* \pi^* \mathcal{M}$ is an isomorphism. In particular, we can recover $\mathcal{M}$ from $\pi^* \mathcal{M}$ by pushing forward. (Hint: projection formula.)

2.4. *Proposition. — Suppose $\pi : X \to Y$ is a morphism of locally Noetherian integral schemes with geometrically integral fibers (hence by Exercise 2.1 satisfying (*)). Suppose also that Y is reduced, and $\mathcal{L}$ is an invertible sheaf on X that is trivial on the fibers of π (i.e. $\mathcal{L}_y$ is a trivial invertible sheaf on X_y). Then $\pi_* \mathcal{L}$ is an invertible sheaf on Y (call it $\mathcal{M}$), and $\mathcal{L} = \pi^* \mathcal{M}$.*

Proof. To show that there exists such an invertible sheaf $\mathcal{M}$ on Y with $\pi^* \mathcal{M} \cong \mathcal{L}$, it suffices to show that $\pi_* \mathcal{L}$ is an invertible sheaf (call it $\mathcal{M}$) and the natural homomorphism $\pi^* \mathcal{M} \to \mathcal{L}$ is an isomorphism.

Now by Grauert's theorem 1.1, $\pi_* \mathcal{L}$ is locally free of rank 1 (again, call it $\mathcal{M}$), and $\mathcal{M} \otimes_{\mathcal{O}_Y} k(y) \to H^0(X_y, \mathcal{L}_y)$ is an isomorphism. We have a natural map of invertible sheaves $\pi^* \mathcal{M} = \pi^* \pi_* \mathcal{L} \to \mathcal{L}$. To show that it is an isomorphism, we need only show that it is surjective, i.e. show that it is surjective on the fibers, which is done. $\square$

Here are some consequences.

A first trivial consequence: if you have two invertible sheaves on X that agree on the fibers of π, then they differ by a pullback of an invertible sheaf on Y.

2.5. *Exercise.* Suppose X is an integral Noetherian scheme. Show that $\operatorname{Pic}(X \times \mathbb{P}^1) \cong \operatorname{Pic} X \times \mathbb{Z}$. (Side remark: If X is non-reduced, this is still true, see Hartshorne Exercise III.12.6(b). It need only be connected of finite type over k. Presumably locally Noetherian suffices.) Extend this to $X \times \mathbb{P}^n$. Extend this to any $\mathbb{P}^n$-bundle over X.

2.6. *Exercise.* Suppose $X \to Y$ is the projectivization of a vector bundle $\mathcal{F}$ over a reduced locally Noetherian scheme (i.e. $X = \underline{\operatorname{Proj}} \operatorname{Sym}^* \mathcal{F}$). Then I think we've already shown in an exercise that it is also the projectivization of $\mathcal{F} \otimes \mathcal{L}$. If Y is reduced and locally Noetherian, show that these are the only ways in which it is the projectivization of a vector bundle. (Hint: note that you can recover $\mathcal{F}$ by pushing forward $\mathcal{O}(1)$.)

2.7. *Exercise.* Suppose $\pi : X \to Y$ is a projective flat morphism over a Noetherian integral scheme, all of whose geometric fibers are isomorphic to $\mathbb{P}^n$ (over the appropriate field). Show that this is a projective bundle if and only if there is an invertible sheaf on X that restricts to $\mathcal{O}(1)$ on all the fibers. (One direction is clear: if it is a projective bundle, then it has a projective $\mathcal{O}(1)$. In the other direction, the candidate vector bundle is $\pi_* \mathcal{O}(1)$. Show that it is indeed a locally free sheaf of the desired rank. Show that its projectivization is indeed $\pi : X \to Y$.)

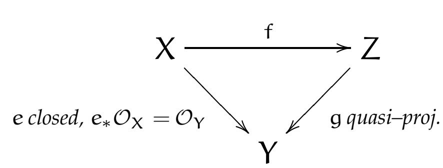
2.8. *Exercise (An example of a Picard scheme).* Show that the Picard scheme of $\mathbb{P}^1_k$ over k is isomorphic to $\mathbb{Z}$.

2.9. *Harder but worthwhile Exercise (An example of a Picard scheme).* Show that if E is an elliptic curve over k (a geometrically integral and nonsingular genus 1 curve with a marked k-point), then $\operatorname{Pic} E$ is isomorphic to $E \times \mathbb{Z}$. Hint: Choose a marked point p. (You'll note that this isn't canonical.) Describe the candidate universal invertible sheaf on $E \times \mathbb{Z}$. Given an invertible sheaf on $E \times X$, where X is an arbitrary Noetherian scheme, describe the morphism $X \to E \times \mathbb{Z}$.

3. The rigidity lemma

The rigidity lemma is another useful fact about morphisms $\pi : X \to Y$ such that $\pi_* \mathcal{O}_X$ (condition (*) of the previous section). It is quite powerful, and quite cheap to prove, so we may as well do it now. (During class, the hypotheses kept on dropping until there was almost nothing left!)

3.1. *Rigidity lemma (first version). — Suppose we have a commutative diagram*



where Y is locally Noetherian, where f takes X_y for some $y \in Y$. Then there is a neighborhood $U \subset Y$ of y on which this is true. Better: over U, f factors through the projection to Y, i.e. the *following diagram commutes for some choice of* h*:*

$$\begin{array}{ccc} X|_U & \xrightarrow{f} & Z|_U \\ & {\scriptstyle e}\searrow \quad \nearrow{\scriptstyle h} & \\ & U & \end{array}$$

.

Proof. This proof is very reminiscent of an earlier result, when we showed that a projective morphisms with finite fibers is a finite morphism.

We can take g to be projective. We can take Y to be an affine neighborhood of y. Then $Z \hookrightarrow \mathbb{P}^n_Y$ for some n. Choose a hyperplane of $\mathbb{P}^n_y$ missing $f(X_y)$, and extend it to a hyperplane H of $\mathbb{P}^n_Y$. (If $Y = \operatorname{Spec} B$, and $y = [\mathfrak{n}]$, then we are extending a linear equation with coefficients in $B/\mathfrak{n}$ to an equation with coefficients in B.) Pull back this hyperplane to X; the preimage is a closed subset. The image of this closed subset in Y is also a closed set $K \subset Y$, as e is a closed map. But $y \notin K$, so let $U = Y - K$. Over U, $f(X_y)$ misses our hyperplane H. Thus the map $X_y \to \mathbb{P}^n_U$ factors through $X_y \to \mathbb{A}^n_U$. Thus the map is given by n functions on $X|_U$. But $e_*\mathcal{O}_X \cong \mathcal{O}_Y$, so these are precisely the pullbacks of functions on U, so we are done. □

3.2. *Rigidity lemma (second version). — Same thing, with the condition on* g *changed from "projective" to simply "finite type".*

Proof. Shrink Y so that it is affine. Choose an open affine subset Z' of Z containing the $f(X_y)$. Then the complement the pullback of $K = Z - Z'$ to X is a closed subset of X whose image in Y is thus closed (as again e is a closed map), and misses y. We shrink Y further such that $f(X)$ lies in Z'. But $Z' \to Y$ is quasiprojective, so we can apply the previous version. □

Here is another mild strengthening.

3.3. *Rigidity lemma (third version). — If* X *is reduced and* g *is separated, and* Y *is connected, and there is a section* $Y \to X$*, then we can take* $U = Y$*.*

Proof. We have two morphisms $X \to Z$: f and $f \circ s \circ e$ which agree on the open set U. But we've shown earlier that any two morphisms from a reduced scheme to a separated scheme agreeing on a dense open set are the same. □

Here are some nifty consequences.

3.4. *Corollary (abelian varieties are abelian). — Suppose* A *is a projective integral group variety (an abelian variety) over a field* k*. Then the multiplication map* $m : A \times A \to A$ *is commutative.*

Proof. Consider the commutator map $c : A \times A \to A$ that corresponds to $(x, y) \mapsto xyx^{-1}y^{-1}$. We wish to show that this map sends $A \times A$ to the identity in A. Consider $A \times A$ as a family over the first factor. Then over $x = e$, c maps the fiber to e. Thus by the rigidity lemma (third version), the map c is a function only of the first factor. But then $c(x, y) = c(x, e) = e$. ☐

3.5. *Exercise.* By a similar argument show that any map $f : A \to A'$ from one abelian variety to another is a group homomorphism followed by a translation. (Hint: reduce quickly to the case where f sends the identity to the identity. Then show that "$f(x + y) - f(x) - f(y) = e$".)

4. Proof of Grauert's theorem

I'll prove Grauert, but not Cohomology and Base Change. It would be wonderful if Cohomology and Base Change followed by just mucking around with maps of free modules over a ring.

4.1. *Exercise++.* Find such an argument.

We'll need a preliminary result.

4.2. *Lemma.* — *Suppose $Y = \operatorname{Spec} B$ is a reduced Noetherian scheme, and $f : M \to N$ is a homomorphism of coherent free (hence projective, flat) B-modules. If $\dim_{k(y)} \operatorname{im}(f \otimes k(y))$ is locally constant, then there are splittings $M = M_1 \oplus M_2$ and $N = N_1 \oplus N_2$ with f killing M_1, and sending M_2 isomorphically to N_1.*

Proof. Note that $f(M) \otimes k \cong f(M \otimes k)$ from that surjection. From $0 \to f(M) \to N \to N/f(M) \to 0$ we have

$$\begin{array}{ccccccc} f(M) \otimes k & \longrightarrow & N \otimes k & \longrightarrow & N/f(M) \otimes k & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ f(M \otimes k) & \longrightarrow & N \otimes k & \longrightarrow & N \otimes k/f(M \otimes k) & \longrightarrow & 0 \end{array}$$

from which $(N/f(M)) \otimes k \cong (N \otimes k)/f(M \otimes k)$. Now the one on the right has locally constant rank, so the one on the left does too, hence is locally free, and flat, and projective. Hence $0 \to f(M) \to N \to N/f(M) \to 0$ splits, so let $N_2 = N/f(M)$, $N_1 = f(M)$. Also, N and $N/f(M)$ are flat and coherent, hence so is $f(M)$.

We now play the same game with

$$0 \to \ker f \to M \to f(M) \to 0.$$

$f(M)$ is projective, hence this splits. Let $\ker f = M_1$. ☐

Now let's prove Grauert's theorem 1.1. We can use this lemma to rewrite

$$M^{p-1} \xrightarrow{d^{p-1}} M^p \xrightarrow{d^p} M^{p+1}$$

as $Z^{p-1} \oplus K^{p-1} \longrightarrow B^p \oplus H^p \oplus K^p \longrightarrow B^{p+1} \oplus K^{p+1}$ where d^{p-1} sends K^{p-1} isomorphically onto B^p (and is otherwise 0), and d^p sends K^p isomorphically onto B_{p+1}. Here H^p is a projective module, so we have local freeness. Thus when we tensor with some other ring, this structure is preserved as well; hence we have isomorphism. □

5. Dimensions behave well for flat morphisms

There are a few easier statements about flatness that I could have said much earlier.

Here's a basic statement about how dimensions behave in flat families.

5.1. *Proposition. — Suppose $f : X \to Y$ is a flat morphism of schemes all of whose stalks are localizations of finite type k-algebras, with $f(x) = y$. (For example, X and Y could be finite type k-schemes.) Then the dimension of X_y at x plus the dimension of Y and y is the dimension X at x.*

In other words, there can't be any components contained in a fibers; and you can't have any dimension-jumping.

In class, I first incorrectly stated this with the weaker hypotheses that X and Y are just locally Noetherian. Kirsten pointed out that I used the fact that height = codimension, which is not true for local Noetherian rings in general. However, we have shown it for local rings of finite type k-schemes. Joe suggested that one could work around this problem.

Proof. This is a question about local rings, so we can consider $\operatorname{Spec} \mathcal{O}_{X,x} \to \operatorname{Spec} \mathcal{O}_{Y,y}$. We may assume that Y is reduced. We prove the result by induction on $\dim Y$. If $\dim Y = 0$, the result is immediate, as $X_y = X$ and $\dim_y Y = 0$.

Now for $\dim Y > 0$, I claim there is an element $t \in \mathfrak{m}$ that is not a zero-divisor, i.e. is not contained in any associated prime, i.e. (as Y is reduced) is not contained in any minimal prime. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be the (finite number of) minimal primes. If $\mathfrak{m} \subset \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n$, then in the first quarter we showed (in an exercise) that $\mathfrak{m} \subset \mathfrak{p}_i$ for some i. But as $\mathfrak{m}$ is maximal, and $\mathfrak{p}_i$ is minimal, we must have $\mathfrak{m} = \mathfrak{p}_i$, and $\dim Y = 0$.

Now by flatness t is not a zero-divisor of $\mathcal{O}_{X,x}$. (Recall that non-zero-divisors pull back to non-zero-divisors.) $\dim \mathcal{O}_{Y,y}/t = \dim \mathcal{O}_{Y,y} - 1$ by Krull's principal ideal theorem (here we use the fact that codimension = height), and $\dim \mathcal{O}_{X,x}/t = \dim \mathcal{O}_{X,x} - 1$ similarly. □.

5.2. *Corollary. — Suppose $f : X \to Y$ is a flat finite-type morphism of locally Noetherian schemes, and Y is irreducible. Then the following are equivalent.*

- *Every irreducible component of* X *has dimension* $\dim Y + n$.
- *For any point* $y \in Y$ *(not necessarily closed!), every irreducible component of the fiber* X_y *has dimension* n.

5.3. *Exercise.* Prove this.

Important definition: If these conditions hold, we say that π is *flat of relative dimension* n. This definition will come up when we define *smooth of relative dimension* n.

5.4. *Exercise.*
(a) Suppose $\pi : X \to Y$ is a finite-type morphism of locally Noetherian schemes, and Y is irreducible. Show that the locus where π is flat of relative dimension n is an open condition.
(b) Suppose $\pi : X \to Y$ is a *flat* finite-type morphism of locally Noetherian schemes, and Y is irreducible. Show that X can be written as the disjoint union of schemes $X_0 \cup X_1 \cup \cdots$ where $\pi|_{X_n} : X_n \to Y$ is flat of relative dimension n.

5.5. *Important Exercise.* Use a variant of the proof of Proposition 5.1 to show that if $f : X \to Y$ is a flat morphism of finite type k-schemes (or localizations thereof), then any associated point of X must map to an associated point of Y. (I find this an important point when visualizing flatness!)

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