FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASSES 41 AND 42

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CONTENTS

Today: Flatness; Tor**; ideal-theoretic characterization of flatness; for coherent modules over a Noetherian local ring flat = free, flatness over a nonsingular curve.**

1. INTRODUCTION TO FLATNESS

We come next to the important concept of flatness. This topic is also not a hard topic, and we could have dealt with it as soon as we had discussed quasicoherent sheaves and morphisms. But it is an intuitively unexpected one, and the algebra and geometry are not obviously connected, so we've left it for relatively late. It is answer to many of your geometric prayers, but you just haven't realized it yet.

The notion of flatness apparently was first defined in Serre's landmark "GAGA" paper.

Here are some of the reasons it is a good concept. We would like to make sense of the notion of "fibration" in the algebraic category (i.e. in algebraic geometry, as opposed to differential geometry), and it turns out that flatness is essential to this definition. It turns out that flat is the right algebraic version of a "nice" or "continuous" family, and this notion is more general than you might think. For example, the double cover $\mathbb{A}^1 \to \mathbb{A}^1$ over an algebraically closed field given by $y \mapsto x^2$ is a flat family, which we interpret as two points coming together to a fat point. The fact that the degree of this map always is 2 is a symptom of how this family is well-behaved. Another key example is that of a family of smooth curves degenerating to a nodal curve, that I sketched on the board in class. One can prove things about smooth curves by first proving them about the nodal curve, and then showing that the result behaves well in flat families. In general, we'll see that certain things behave well in nice families, such as cohomology groups (and even better Euler characteristics) of fibers.

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There is a second flavor of prayer that is answered. It would be very nice if tensor product (of quasicoherent sheaves, or of modules over a ring) were an exact functor, and certain statements of results and proofs we have seen would be cleaner if this were true. Those modules for which tensoring is always exact are flat (this will be the definition!), and hence for flat modules (or quasicoherent sheaves, or soon, morphisms) we'll be able to get some very useful statements. A flip side of that is that exact sequences of *flat* modules remain exact when tensored with *any* other module.

In this section, we'll discuss flat morphisms. When introducing a new notion, I prefer to start with a number of geometric examples, and figure out the algebra on the fly. In this case, because there is enough algebra, I'll instead discuss the algebra at some length and then later explain why you care geometrically. This will require more patience than usual on your part.

2. Algebraic definition and easy facts

Many facts about flatness are easy or immediate, and a few are tricky. I'm going to try to make clear which is which, to help you remember the easy facts and the idea of proof for the harder facts.

The definition of a *flat* A*-module* is very simple. Recall that if

$$0 \to N' \to N \to N'' \to 0 \tag{1}$$

is a short exact sequence of A-modules, and M is another A-module, then

$$M \otimes_A N' \to M \otimes_A N \to M \otimes_A N'' \to 0$$

is exact. In other words, $M \otimes_A$ is a right-exact functor. We say that M *is a flat* A*-module* if $M \otimes_A$ is an exact functor, i.e. if for all exact sequences (1),

$$0 \to M \otimes_A N' \to M \otimes_A N \to M \otimes_A N'' \to 0$$

is exact as well.

Exercise. If $N' \to N \to N''$ is exact and M is a flat A-module, show that $M \otimes_A N' \to M \otimes_A N \to M \otimes_A N''$ is exact. Hence *any* exact sequence of A-modules remains exact upon tensoring with M. (We've seen things like this before, so this should be fairly straightforward.)

We say that *a ring homomorphism* $B \to A$ *is flat* if A is flat as a B-module. (We don't care about the algebra structure of A.)

Here are two key examples of flat ring homomorphisms:

(i) free modules A-modules are clearly flat.
(ii) Localizations are flat: Suppose S is a multiplicative subset of B. Then $B \to S^{-1}B$ is a flat ring morphism.

Exercise. Verify (ii). We have used this before: localization is an exact functor.

Here is a useful way of recognizing when a module is *not* flat. Flat modules are torsion-free. More precisely, if x is a non-zero-divisor of A, and M is a flat A-module, then $M \xrightarrow{\times x} M$ is injective. Reason: apply the exact functor $M \otimes_A$ to the exact sequence $0 \longrightarrow A \xrightarrow{\times x} A$.

We make some quick but important observations:

2.1. *Proposition (flatness is a stalk/prime-local property). — An A-module M is flat if and only if $M_{\mathfrak{p}}$ is a flat $A_{\mathfrak{p}}$-module for all primes $\mathfrak{p}$.*

Proof. Suppose first that M is flat. Given any exact sequence of $A_{\mathfrak{p}}$-modules (1),

$$0 \to M \otimes_A N' \to M \otimes_A N \to M \otimes_A N'' \to 0$$

is exact too. But $M \otimes_A N$ is canonically isomorphic to $M \otimes_{A_{\mathfrak{p}}} N_{\mathfrak{p}}$ (exercise: show this!), so $M_{\mathfrak{p}}$ is a flat $A_{\mathfrak{p}}$-module.

Suppose next that M is *not* flat. Then there is some short exact sequence (1) that upon tensoring with M becomes

$$0 \to K \to M \otimes_A N' \to M \otimes_A N \to M \otimes_A N'' \to 0 \tag{2}$$

where $K \neq 0$ is the kernel of $M \otimes_A N' \to M \otimes_A N$. Then as $K \neq 0$, K has non-empty support, so there is some prime $\mathfrak{p}$ such that $K_{\mathfrak{p}} \neq 0$. Then

$$0 \to N'_{\mathfrak{p}} \to N_{\mathfrak{p}} \to N''_{\mathfrak{p}} \to 0 \tag{3}$$

is a short exact sequence of $A_{\mathfrak{p}}$-modules (recall that localization is exact — see (ii) before the statement of the Proposition), but is no longer exact upon tensoring (over $A_{\mathfrak{p}}$) with $M_{\mathfrak{p}}$ (as

$$0 \to K_{\mathfrak{p}} \to M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} N'_{\mathfrak{p}} \to M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} N_{\mathfrak{p}} \to M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} N''_{\mathfrak{p}} \to 0 \tag{4}$$

is exact). (Here we use that localization commutes with tensor product.) □

2.2. *Proposition (flatness is preserved by change of base ring). — If M flat A-module, $A \to B$ is a homomorphism, then $M \otimes_A B$ is a flat B-module.*

Proof. Exercise. □

2.3. *Proposition (transitivity of flatness). — If B is a flat A-algebra, and M is B-flat, then it is also A-flat.*

Proof. Exercise. (Hint: consider the natural isomorphism $(M \otimes_A B) \otimes_B \cdot \cong M \otimes_B (B \otimes_A \cdot)$.) □

The extension of this notion to schemes is straightforward.

2.4. *Definition: flat quasicoherent sheaf.* We say that a quasicoherent sheaf $\mathcal{F}$ on a scheme X is flat (over X) if for all $x \in X$, $\mathcal{F}_x$ is a flat $\mathcal{O}_{X,x}$-module. In light of Proposition 2.1, we can check this notion on affine open cover of X.

2.5. *Definition: flat morphism.* Similarly, we say that a morphism of schemes $\pi : X \to Y$ is flat if for all $x \in X$, $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{Y,\pi(x)}$-module. Again, we can check locally, on maps of affine schemes.

We can combine these two definitions into a single definition.

2.6. *Definition: flat quasicoherent sheaf over some base.* Suppose $\pi : X \to Y$ is a morphism of schemes, and $\mathcal{F}$ is a quasicoherent sheaf on X. We say that $\mathcal{F}$ *is flat over* Y if for all $x \in X$, $\mathcal{F}_x$ is a flat $\mathcal{O}_{Y,\pi(x)}$-module.

Definitions 2.4 and 2.5 correspond to the cases $X = Y$ and $\mathcal{F} = \mathcal{O}_X$ respectively.

This definition can be extended without change to the category of ringed spaces, but we won't need this.

All of the Propositions above carry over naturally. For example, flatness is preserved by base change. (More explicitly: suppose $\pi : X \to Y$ is a morphism, and $\mathcal{F}$ is a quasicoherent sheaf on X, flat over Y. If $Y' \to Y$ is any morphism, and $p : X \times_Y Y' \to X$ is the projection, then $p^*\mathcal{F}$ is flat over Y'.) Also, flatness is transitive. (More explicitly: suppose $\pi : X \to Y$ and $\mathcal{F}$ is a quasicoherent sheaf on X, flat over Y. Suppose also that $\psi : Y \to Z$ is a flat morphism. Then $\mathcal{F}$ is flat over Z.)

We also have other statements easily. For example: open immersions are flat.

2.7. *Exercise.* If X is a scheme, and η is the generic point for an irreducible component, show that the natural morphism $\operatorname{Spec} \mathcal{O}_{X,\eta} \to X$ is flat. (Hint: localization is flat.)

We earlier proved the following important fact, although we did not have the language of flatness at the time.

2.8. *Theorem (cohomology commutes with flat base change). — Suppose*

$$\begin{CD} X' @>{g'}>> X \\ @V{f'}VV @VV{f}V \\ Y' @>{g}>> Y \end{CD}$$

is a fiber diagram, and f (and thus f′) is quasicompact and separated (so higher pushforwards exist). Suppose also that g is flat, and $\mathcal{F}$ is a quasicoherent sheaf on X. Then the natural morphisms $g^* R^i f_* \mathcal{F} \to R^i f'_*(g^*\mathcal{F})$ are isomorphisms.

A special case that is often useful is the case where Y' is the generic point of a component of Y. In other words, in light of Exercise 2.7, the stalk of the higher pushforward of

$\mathcal{F}$ at the generic point is the cohomology of $\mathcal{F}$ on the fiber over the generic point. This is a first example of something important: understanding cohomology of (quasicoherent sheaves on) fibers in terms of higher pushforwards. (We would certainly hope that higher pushforwards would tell us something about higher cohomology of fibers, but this is certainly not a priori clear!)

(I might dig up the lecture reference later, but I'll tell you now where proved it: where we described this natural morphism, I had a comment that if we had exactness of tensor product, them morphisms would be an isomorphism.)

We will spend the rest of our discussion on flatness as follows. First, we will ask ourselves: what are the flat modules over particularly nice rings? More generally, how can you check for flatness? And how should you picture it geometrically? We will then prove additional facts about flatness, and using flatness, answering the essential question: "why do we care?"

2.9. *Faithful flatness.* The notion of *faithful flatness* is handy, although we probably won't use it. We say that an extension of rings $B \to A$ is *faithfully flat* if for every A-module M, M is A-flat if and only if $M \otimes_A B$ is B-flat. We say that a morphism of schemes $X \to Y$ is *faithfully flat* if it is flat and surjective. These notions are the "same", as shown by the following exercise.

Exercise. Show that $B \to A$ is faithfully flat if and only if $\operatorname{Spec} A \to \operatorname{Spec} B$ is faithfully flat.

3. The "Tor" functors, and a "cohomological" criterion for flatness

In order to prove more facts about flatness, it is handy to have the notion of Tor. (Tor is short for "torsion". The reason for this name is that the 0th and/or 1st Tor-group measures common torsion in abelian groups (aka $\mathbb{Z}$-modules).) If you have never seen this notion before, you may want to just remember its properties, which are natural. But I'd like to prove everything anyway — it is surprisingly easy.

The idea behind Tor is as follows. Whenever we see a right-exact functor, we always hope that it is the end of a long-exact sequence. Informally, given a short exact sequence (1), we are hoping to see a long exact sequence

$$\cdots \longrightarrow \operatorname{Tor}_i^A(M,N') \longrightarrow \operatorname{Tor}_i^A(M,N) \longrightarrow \operatorname{Tor}_i^A(M,N'') \longrightarrow \cdots$$
$$\longrightarrow \operatorname{Tor}_1^A(M,N') \longrightarrow \operatorname{Tor}_1^A(M,N) \longrightarrow \operatorname{Tor}_1^A(M,N'')$$
$$\longrightarrow M \otimes_A N' \longrightarrow M \otimes_A N \longrightarrow M \otimes_A N'' \longrightarrow 0. \tag{5}$$

More precisely, we are hoping for *covariant functors* $\operatorname{Tor}_i^A(\cdot, N)$ from A-modules to A-modules (giving 2/3 of the morphisms in that long exact sequence), with $\operatorname{Tor}_0^A(M,N) \equiv$

$M \otimes_A N$, and natural δ morphisms $\mathrm{Tor}^A_{i+1}(M, N'') \to \mathrm{Tor}^A_i(M, N')$ for every short exact sequence (1) giving the long exact sequence. (In case you care, "natural" means: given a morphism of short exact sequences, the natural square you would write down involving the δ-morphism must commute. I'm not going to state this explicitly.)

It turns out to be not too hard to make this work, and this will later motivate derived functors. I'll now define $\mathrm{Tor}^A_i(M, N)$. Take any resolution $\mathcal{R}$ of N by free modules:

$$\cdots \longrightarrow A^{\oplus n_2} \longrightarrow A^{\oplus n_1} \longrightarrow A^{\oplus n_0} \longrightarrow N \to 0.$$

More precisely, build this resolution from right to left. Start by choosing generators of N as an A-module, giving us $A^{\oplus n_0} \to N \to 0$. Then choose generators of the kernel, and so on. Note that we are not requiring the n_i to be finite, although if N is a finitely-generated module and A is Noetherian (or more generally if N is coherent and A is coherent over itself), we can choose the n_i to be finite. Truncate the resolution, by stripping off the last term. Then tensor with M (which may lose exactness!). Let $\mathrm{Tor}_A^i(M, N)_{\mathcal{R}}$ be the homology of this complex at the ith stage ($i \geq 0$). The subscript $\mathcal{R}$ reminds us that our construction depends on the resolution, although we will soon see that it is independent of the resolution.

We make some quick observations.

- $\mathrm{Tor}^A_0(M, N)_{\mathcal{R}} \cong M \otimes_A N$ (and this isomorphism is canonical). Reason: as tensoring is right exact, and $A^{\oplus n_1} \to A^{\oplus n_0} \to N \to 0$ is exact, we have that $M^{\oplus n_1} \to M^{\oplus n_0} \to M \otimes_A N \to 0$ is exact, and hence that the homology of the truncated complex $M^{\oplus n_1} \to M^{\oplus n_0} \to 0$ is $M \otimes_A N$.
- If M is flat, then $\mathrm{Tor}^A_i(M, N)_{\mathcal{R}} = 0$ for all i.

Now given two modules N and N' and resolutions $\mathcal{R}$ and $\mathcal{R}'$ of N and N', we can "lift" any morphism $N \to N'$ to a morphism of the two resolutions:

$$\begin{array}{ccccccccccccc} \cdots & \longrightarrow & A^{\oplus n_i} & \longrightarrow & \cdots & \longrightarrow & A^{\oplus n_1} & \longrightarrow & A^{\oplus n_0} & \longrightarrow & N & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & A^{\oplus n'_i} & \longrightarrow & \cdots & \longrightarrow & A^{\oplus n'_1} & \longrightarrow & A^{\oplus n'_0} & \longrightarrow & N' & \longrightarrow & 0 \end{array}$$

Denote the choice of lifts by $\mathcal{R} \to \mathcal{R}'$. Now truncate both complexes and tensor with M. Maps of complexes induce maps of homology, so we have described maps (a priori depending on $\mathcal{R} \to \mathcal{R}'$)

$$\mathrm{Tor}^A_i(M, N)_{\mathcal{R}} \to \mathrm{Tor}^A_i(M, N')_{\mathcal{R}'}.$$

We say two maps of complexes $f, g : C_* \to C'_*$ are *homotopic* if there is a sequence of maps $w : C_i \to C'_{i+1}$ such that $f - g = dw + wd$. Two homotopic maps give the same map on homology. (Exercise: verify this if you haven't seen this before.)

Crucial Exercise: Show that any two lifts $\mathcal{R} \to \mathcal{R}'$ are homotopic.

We now pull these observations together.

(1) We get a covariant functor from $\mathrm{Tor}^A_i(M, N)_{\mathcal{R}} \to \mathrm{Tor}^A_i(M, N')_{\mathcal{R}'}$ (independent of the lift $\mathcal{R} \to \mathcal{R}'$).

(2) Hence for any two resolutions $\mathcal{R}$ and $\mathcal{R}'$ we get a canonical isomorphism $\mathrm{Tor}_i^A(M,N)_{\mathcal{R}} \cong \mathrm{Tor}_i^1(M,N)_{\mathcal{R}'}$. Here's why. Choose lifts $\mathcal{R} \to \mathcal{R}'$ and $\mathcal{R}' \to \mathcal{R}$. The composition $\mathcal{R} \to \mathcal{R}' \to \mathcal{R}$ is homotopic to the identity (as it is a lift of the identity map $N \to N$). Thus if $f_{\mathcal{R}\to\mathcal{R}'} : \mathrm{Tor}_i^A(M,N)_{\mathcal{R}} \to \mathrm{Tor}_i^1(M,N)_{\mathcal{R}'}$ is the map induced by $\mathcal{R} \to \mathcal{R}'$, and similarly $f_{\mathcal{R}'\to\mathcal{R}}$ is the map induced by $\mathcal{R} \to \mathcal{R}'$, then $f_{\mathcal{R}'\to\mathcal{R}} \circ f_{\mathcal{R}\to\mathcal{R}'}$ is the identity, and similarly $f_{\mathcal{R}\to\mathcal{R}'} \circ f_{\mathcal{R}'\to\mathcal{R}}$ is the identity.

(3) Hence the covariant functor doesn't depend on the resolutions!

Finally:
(4) For any short exact sequence (1) we get a long exact sequence of Tor's (5). Here's why: given a short exact sequence (1), choose resolutions of N' and N''. Then use these to get a resolution for N in the obvious way (see below; the map $A^{\oplus(n_0' \to n_0'')} \to N$ is the composition $A^{\oplus n_0'} \to N' \to N$ along with any lift of $A^{n_0''} \to N''$ to N) so that we have a short exact sequence of resolutions

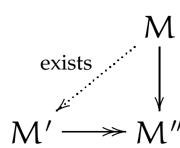
$$
\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & A^{\oplus n_1'} & \longrightarrow & A^{\oplus n_0'} & \longrightarrow & N' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & A^{\oplus(n_1'+n_1'')} & \longrightarrow & A^{\oplus(n_0'+n_0'')} & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & A^{\oplus n_1''} & \longrightarrow & A^{\oplus n_0''} & \longrightarrow & N'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Then truncate (removing the column (1)), tensor with M (obtaining a short exact sequence of complexes) and take cohomology, yielding a long exact sequence.

We have thus established the foundations of Tor!

Note that if N is a free module, then $\mathrm{Tor}_i^A(M,N) = 0$ for all M and all $i > 0$, as N has itself as a resolution.

3.1. *Remark: Projective resolutions.* We used very little about free modules in the above construction; in fact we used only that free modules are *projective*, i.e. those modules M such that for any surjection $M' \to M''$, it is possible to lift any morphism $M \to M''$ to $M \to M'$. This is summarized in the following diagram.



Equivalently, $\mathrm{Hom}(M,\cdot)$ is an *exact functor* ($\mathrm{Hom}(N,\cdot)$ is always left-exact for any N). (More generally, we can define the notion of a *projective object in any abelian category*.)

Hence (i) we can compute$\mathrm{Tor}_i^A(M,N)$ by taking any projective resolution of N, and (ii) $\mathrm{Tor}_i^A(M,N) = 0$ for any projective A-module N.

3.2. *Remark: Generalizing this construction.* The above description was low-tech, but immediately generalizes drastically. All we are using is that $M \otimes_A$ is a right-exact functor. In general, if F is *any* right-exact covariant functor from the category of A-modules to any abelian category, this construction will define a sequence of functors $L_i F$ (called left-derived functors of F) such that $L_0 F = F$ and the L_i's give a long-exact sequence. We can make this more general still. We say that an abelian category *has enough projectives* if for any object N there is a surjection onto it from a projective object. Then if F is any right-exact functor from an abelian category with enough projectives to any abelian category, then F has left-derived functors.

3.3. *Exercise.* The notion of an *injective object* in an abelian category is dual to the notion of a projective object. Define derived functors for (i) covariant left-exact functors (these are called right-derived functors), (ii) contravariant left-exact functors (also right-derived functors), and (iii) contravariant right-exact functors (these are called left-derived functors), making explicit the necessary assumptions of the category having enough injectives or projectives.

Here are two quick practice exercises, giving useful properties of Tor.

Important exercise. If B is A-flat, then we get isomorphism $B \otimes \mathrm{Tor}_i^A(M,N) \cong \mathrm{Tor}_i^B(B \otimes M, B \otimes N)$. (This is tricky rather than hard; it has a clever one-line answer. Here is a fancier fact that experts may want to try: if B is not A-flat, we don't get an isomorphism; instead we get a spectral sequence.)

Exercise- (not too important, but good practice if you haven't played with Tor *before).* If x is not a 0-divisor, show that $\mathrm{Tor}_i^A(A/x, M)$ is 0 for $i > 1$, and for $i = 0$, get M/xM, and for $i = 1$, get $(M : x)$ (those things sent to 0 upon multiplication by x).

3.4. "Symmetry" of Tor. The natural isomorphism $M \otimes N \to N \otimes M$ extends to the following.

3.5. *Theorem. — There is a natural isomorphism* $\mathrm{Tor}_i(M,N) \cong \mathrm{Tor}(N,M)$.

Proof. Take two resolutions of M and N:

$$\cdots \to A^{\oplus m_1} \to A^{\oplus m_0} \to M \to 0$$

and

$$\cdots \to A^{\oplus n_1} \to A^{\oplus n_0} \to N \to 0.$$

Consider the double complex obtained by tensoring their truncations.

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & A^{\oplus(m_2+n_2)} & \longrightarrow & A^{\oplus(m_1+n_2)} & \longrightarrow & A^{\oplus(m_0+n_2)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & A^{\oplus(m_2+n_1)} & \longrightarrow & A^{\oplus(m_1+n_1)} & \longrightarrow & A^{\oplus(m_0+n_1)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & A^{\oplus(m_2+n_0)} & \longrightarrow & A^{\oplus(m_1+n_0)} & \longrightarrow & A^{\oplus(m_0+n_0)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}
$$

Apply our spectral sequence machinery. We compute the homology of this complex in two ways.

We start by using the vertical arrows. Notice that the ith column is precisely the truncated resolution of N, tensored with $A^{\oplus m_i}$. Thus the homology in the vertical direction in the ith column is 0 except in the bottom element of the column, where it is $N^{\oplus m_i}$. We next take homology in the horizontal direction. In the only non-zero row (the bottom row), we see precisely the complex computing $\operatorname{Tor}_i(N, M)$. After using these second arrows, the spectral sequence has converged. Thus the ith homology of the double complex is (naturally isomorphic to) $\operatorname{Tor}_i(N, M)$.

Similarly, if we began with the arrows in the horizontal direction, we would conclude that the ith homology of the double complex is $\operatorname{Tor}_i(M, N)$. □

This gives us a quick but very useful result. Recall that if $0 \to N' \to N \to N'' \to 0$ is exact, then so is the complex obtained by tensoring with M *if* M *is flat*. (Indeed that is the definition of flatness!) But in general we have an exact sequence

$$\operatorname{Tor}_1^A(M, N'') \to M \otimes_A N' \to M \otimes_A N \to M \otimes_A N'' \to 0$$

Hence we conclude:

3.6. *Proposition. — If* $0 \to N' \to N \to N'' \to 0$ *is exact, and* N'' *is flat, then* $0 \to M \otimes_A N' \to M \otimes_A N \to M \otimes_A N'' \to 0$ *is exact.*

Note that we would have cared about this result long before learning about Tor. This gives some motivation for learning about Tor. Presumably one can also show this directly by some sort of diagram chase. (Is there an easy proof?)

One important consequence is the following. Suppose we have a short exact sequence of sheaves on Y, and the rightmost element is flat (e.g. locally free). Then if we pull this exact sequence back to X, it remains exact. (I think we may have used this.)

3.7. An ideal-theoretic criterion for flatness. We come now to a useful fact. Observe that $\mathrm{Tor}_1(M,N) = 0$ for all N implies that M is flat; this in turn implies that $\mathrm{Tor}_i(M,N) = 0$ for all $i > 0$.

The following is a very useful variant on this.

3.8. *Key theorem. — M is flat if and only if $\mathrm{Tor}_1^A(M, A/I) = 0$ for all ideals I.*

(The interested reader can tweak the proof below a little to show that it suffices to consider *finitely generated ideals* I, but we won't use this fact.)

Proof. [The M's and N's are messed up in this proof.] We have already observed that if N is flat, then $\mathrm{Tor}_1^A(M, R/I) = 0$ for all I. So we assume that $\mathrm{Tor}_1^A(M, A/I) = 0$, and hope to prove that $\mathrm{Tor}_1^A(M,N) = 0$ for all A-modules N, and hence that M is flat.

By induction on the number of generators of N, we can prove that $\mathrm{Tor}_1^A(M,N) = 0$ for all *finitely generated* modules N. (The base case is our assumption, and the inductive step is as follows: if N is generated by $a_1, \dots, a_n$, then let N' be the submodule generated by $a_1, \dots, a_{n-1}$, so $0 \to N' \to N \to A/I \to 0$ is exact, where I is some ideal. Then the long exact sequence for Tor gives us $0 = \mathrm{Tor}_1^A(M, N') \to \mathrm{Tor}_1^A(M,N) \to \mathrm{Tor}_1^A(M, A/I) = 0$.)

We conclude by noting that N is the union (i.e. direct limit) of its finitely generated submodules. As $\otimes$ commutes with direct limits, Tor_1 commutes with direct limits as well. (This requires some argument!)

Here is a sketch of an alternate conclusion. We wish to show that for any exact $0 \to N' \to N$, $0 \to M \otimes N' \to M \otimes N$ is also exact. Suppose $\sum m_i \otimes n_i' \mapsto 0$ in $M \otimes N$. Then that equality involves only finitely many elements of N. Work instead in the submodule generated by these elements of N. Within these submodules, we see that $\sum m_i \otimes n_i' = 0$. Thus this equality holds inside $M \otimes N'$ as well.

(I may try to write up a cleaner argument. Joe pointed out that the cleanest thing to do is to show that injectivity commutes with direct limits.) □

This has some cheap but important consequences.

Recall (or reprove) that flatness over a domain implies torsion-free.

3.9. *Corollary to Theorem 3.8. — Flatness over principal ideal domain is the* **same** *as torsion-free.*

This follows directly from the proposition.

3.10. *Important Exercise (flatness over the dual numbers).* This fact is important in deformation theory and elsewhere. Show that M is flat over $k[t]/t^2$ if and only if the natural map $M/tM \to tM$ is an isomorphism.

3.11. Flatness in exact sequences.

Suppose $0 \to M' \to M \to M'' \to 0$ is an exact sequence of A-modules.

3.12. *Proposition. — If M and M'' are both flat, then so is M'. If M' and M'' are both flat, then so is M.*

Proof. We use the characterization of flatness that N is flat if and only if $\mathrm{Tor}_i(N, N') = 0$ for all $i > 0$, N'. The result follows immediately from the long exact sequence for Tor. □

3.13. *Unimportant remark.* This begs the question: if M' and M are both flat, is M'' flat? (The argument above breaks down.) The answer is no: over $k[t]$, consider $0 \to tk[t] \to k[t] \to k[t]/t \to 0$ (geometrically: the closed subscheme exact sequence for a point on $\mathbb{A}^1$). The module on the right has torsion, and hence is not flat. The other two modules are free, hence flat.

3.14. *Easy exercise.* (We will use this shortly.) If $0 \to M_0 \to M_1 \to \cdots \to M_n \to 0$ is an exact sequence, and M_i is flat for $i > 0$, show that M_0 is flat too. (Hint: break the exact sequence into short exact sequences.)

We now come to the next result about flatness that will cause us to think hard.

3.15. *Important Theorem (for coherent modules over Noetherian local rings, flat equals free). — Suppose $(A, \mathfrak{m})$ is a local ring, and M is a coherent A-module (e.g. if A is Noetherian, then M is finitely generated). Then M is flat if and only if it is free.*

(It is true more generally, although we won't use those facts: apparently we can replace coherent with finitely presented, which only non-Noetherian people care about; or we can give up coherent completely if A is Artinian, although I haven't defined this notion. Reference: Mumford p. 296. I may try to clean the proof up to work in these cases.)

Proof. Clearly we are going to be using Nakayama's lemma. Now $M/\mathfrak{m}M$ is a finite-dimensional vector space over the field $A/\mathfrak{m}$. Choose a basis, and lift it to elements $m_1, \dots, m_n \in M$. Then consider $A^n \to M$ given by $e_i \mapsto m_i$. We'll show this is an isomorphism. This is surjective by Nakayama's lemma: the image is all of M modulo the maximal ideal, hence is everything. Let K be the kernel, which is finitely generated by coherence:

$$0 \to K \to A^n \to M \to 0.$$

Tensor this with $A/\mathfrak{m}$. As M is flat, the result is still exact (Proposition 3.6):

$$0 \to K/\mathfrak{m}K \to (A/\mathfrak{m})^n \to M/\mathfrak{m}M \to 0.$$

But $(A/\mathfrak{m})^n \to M/\mathfrak{m}M$ is an isomorphism, so $K/\mathfrak{m}K = 0$. As K is finitely generated, $K = 0$. □

Here is an immediate corollary (or really just a geometric interpretation).

3.16. *Corollary.* — *Suppose $\mathcal{F}$ is coherent over a locally Noetherian scheme* X. *Then $\mathcal{F}$ is flat over* X *if and only if it is locally free.*

(Reason: we have shown that local-freeness can be checked at the stalks.)

This is a useful fact. Here's a consequence that we prove earlier by other means: if $C \to C'$ is a surjective map of nonsingular irreducible projective curves, then $\pi_* \mathcal{O}_C$ is locally free.

In general, this gives us a useful criterion for flatness: Suppose $X \to Y$ finite, and Y integral. Then f is flat if and only if $\dim_{FF(Y)} f_*(\mathcal{O}_X)_y \otimes FF(Y)$ is constant. So the normalization of a node is not flat (I drew a picture here).

3.17. A useful special case: flatness over nonsingular curves. When are morphisms to nonsingular curves flat? Local rings of nonsingular curves are discrete valuation rings, which are principal ideal domains, so for them flat = torsion-free (Prop. 3.9). Thus, any map from a scheme to a nonsingular curve where all associated points go to a generic point is flat. (I drew several pictures of this.)

Here's a version we've seen before: a map from an irreducible curve to a nonsingular curve.

Here is another important consequence, which we can informally state as: we can take flat limits over one-parameter families. More precisely: suppose A is a discrete valuation ring, and let 0 be the closed point of $\operatorname{Spec} A$ and η the generic point. Suppose X is a scheme over A, and Y is a scheme over $X|_\eta$. Let Y' be the scheme-theoretic closure of Y in X. Then Y' is flat over A. Then $Y'|_0$ is often called the *flat limit* of Y.

(Suppose A is a discrete valuation ring, and let η be the generic point of $\operatorname{Spec} A$. Suppose X is proper over A, and Y is a closed subscheme of X_η. *Exercise:* Show that there is only one closed subscheme Y' of X, proper over A, such that $Y'|_\eta = Y$, and Y' is flat over A. Aside for experts: For those of you who know what the Hilbert scheme is, by taking the case of X as projective space, this shows that the Hilbert scheme is proper, using the valuative criterion for properness.)

3.18. *Exercise (an interesting explicit example of a flat limit).* (Here the base is $\mathbb{A}^1$, not a discrete valuation ring. You can either restrict to the discrete valuation ring that is the stalk near 0, or generalize the above discussion appropriately.) Let $X = \mathbb{A}^3 \times \mathbb{A}^1 \to Y = \mathbb{A}^1$ over a field k, where the coordinates on $\mathbb{A}^3$ are x, y, and z, and the coordinates on $\mathbb{A}^1$ are t. Define X away from $t = 0$ as the union of the two lines $y = z = 0$ (the x-axis) and $x = z - t = 0$ (the y-axis translated by t). Find the flat limit at $t = 0$. (Hint: it is *not* the union of the two axes, although it includes it. The flat limit is non-reduced.)

3.19. Stray but important remark: flat morphisms are (usually) open. I'm discussing this here because I have no idea otherwise where to put it.

3.20. *Exercise.* Prove that flat and locally finite type morphisms of locally Noetherian schemes are open. (Hint: reduce to the affine case. Use Chevalley's theorem to show that the image is constructible. Reduce to a target that is the spectrum of a local ring. Show that the generic point is hit.)

3.21. I ended by stating an important consequence of flatness: flat plus projective implies constant Euler characteristic. I'll state this properly in next Tuesday's notes, where I will also give consequences and a proof.

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