FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 38

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CONTENTS

Last day I introduced differentials on affine schemes, for a morphism $B \to A$. The differential was an A-module, as well as a homomorphism of B-modules, $d : A \to \Omega_{A/B}$. The A-module $\Omega_{A/B}$ is generated by da, and d satisfies 3 rules: additivity, Leibniz rule, and $db = 0$ (or $d1 = 0$). It satisfies a universal property: any derivation $A \to M$ uniquely factors through an A-module homomorphism $\Omega_{A/B} \to M$.

1. A THIRD DEFINITION OF Ω, SUITABLE FOR EASY GLOBALIZATION

1.1. Third definition. We now want to globalize this definition for an arbitrary morphism of schemes $f : X \to Y$. We could do this "affine by affine"; we just need to make sure that the above notion behaves well with respect to "change of affine sets". Thus a relative differential on X would be the data of, for every affine $U \subset X$, a differential of the form $\sum a_i \, db_i$, and on the intersection of two affine open sets $U \cap U'$, with representatives $\sum a_i \, db_i$ on U and $\sum a'_i \, db'_i$ on the second, an equality on the overlap. Instead, we'll take a different tack. We'll get what intuitively seems to be a very weird definition! I'll give the definition, then give you some intuition, and then get back to the definition.

Suppose $f : X \to Y$ be any morphism of schemes. Recall that $\delta : X \to X \times_Y X$ is a locally closed immersion (Class 9, p. 5). Thus there is an open subscheme $U \subset X \times_Y X$ for which $\delta : X \to U$ is a closed immersion, cut out by a quasicoherent sheaf of ideals $\mathcal{I}$. Then $\mathcal{I}/\mathcal{I}^2$ is a quasicoherent sheaf naturally supported on X (affine-locally this is the statement that I/I^2 is naturally an A/I-module). We call this the *conormal sheaf to* X (or somewhat more precisely,to the locally closed immersion). (For the motivation for this name, see last day's notes.) We denote it by $\mathcal{N}^\vee_{X/X \times_Y X}$. Then we will *define* $\Omega_{X/Y}$ as this conormal sheaf.

(Small technical point for pedants: what does $\mathcal{I}^2$ mean? In general, if $\mathcal{I}$ and $\mathcal{J}$ are quasicoherent ideal sheaves on a scheme Z, what does $\mathcal{I}\mathcal{J}$ mean? Of course it means that on each affine, we take the product of the two corresponding ideals. To make sure this

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is well-defined, we need only check that if A is a ring, and $f \in A$, and $I, J \subset A$ are two ideals, then $(IJ)_f = I_f J_f$ in A_f.)

Brief aside on (co)normal sheaves to locally closed immersions. For any locally closed immersion $W \to Z$, we can define the *conormal sheaf* $\mathcal{N}^\vee_{W/Z}$, a quasicoherent sheaf on W, similarly, and the *normal sheaf* as its dual $\mathcal{N}_{W/Z} := \underline{\mathrm{Hom}}(\mathcal{N}^\vee, \mathcal{O}_W)$. This is somewhat imperfect notation, as it suggests that the dual of $\mathcal{N}$ is always $\mathcal{N}^\vee$. This is not always true, as for A-modules, the natural morphism from a module to its double-dual is not always an isomorphism. (Modules for which this is true are called *reflexive*, but we won't use this notion.)

1.2. *Exercise: normal bundles to effective Cartier divisors.* Suppose $D \subset X$ is an effective Cartier divisor. Show that the conormal sheaf $\mathcal{N}^\vee_{D/X}$ is $\mathcal{O}(-D)|_D$ (and in particular is an invertible sheaf), and hence that the normal sheaf is $\mathcal{O}(D)|_D$. It may be surprising that the normal sheaf should be locally free if $X \cong \mathbb{A}^2$ and D is the union of the two axes (and more generally if X is nonsingular but D is singular), because you may be used to thinking that the normal bundle is isomorphic to a "tubular neighborhood".

Let's get back to talking about differentials.

We now define the d operator $d : \mathcal{O}_X \to \Omega_{X/Y}$. Let $\pi_1, \pi_2 : X \times_Y X \to X$ be the two projections. Then define $d : \mathcal{O}_X \to \Omega_{X/Y}$ on the open set U as follows: $df = \pi_2^* f - \pi_1^* f$. (*Warning:* this is not a morphism of quasicoherent sheaves, although it *is* $\mathcal{O}_Y$-linear.) We'll soon see that this is indeed a derivation, and at the same time see that our new notion of differentials agrees with our old definition on affine open sets, and hence globalizes the definition.

Before we do, let me try to convince you that this is a reasonable definition to make. (This paragraph is informal, and is in no way mathematically rigorous.) Say for example that Y is a point, and X is something smooth. Then the tangent space to $X \times X$ is $T_X \oplus T_X$: $T_{X \times X} = T_X \oplus T_X$. Restrict this to the diagonal Δ, and look at the normal bundle exact sequence:

$$0 \to T_\Delta \to T_{X \times X}|_\Delta \to N_{\Delta, X} \to 0.$$

Now the left morphism sends v to (v, v), so the cokernel can be interpreted as $(v, -v)$. Thus $N_{\Delta,X}$ is isomorphic to T_X. Thus we can turn this on its head: we know how to find the normal bundle (or more precisely the conormal sheaf), and we can use this to define the tangent bundle (or more precisely the cotangent sheaf). (Experts may want to ponder the above paragraph when Y is more general, but where $X \to Y$ is "nice". You may wish to think in the category of manifolds, and let $X \to Y$ be a submersion.)

Let's now see how this works for the special case $\operatorname{Spec} A \to \operatorname{Spec} B$. Then the diagonal $\operatorname{Spec} A \hookrightarrow \operatorname{Spec} A \otimes_B A$ corresponds to the ideal I of $A \otimes_B A$ that is the cokernel of the ring map

$$\sum x_i \otimes y_i \to \sum x_i y_i.$$

The derivation is $d : A \to A \otimes_B A$, $a \mapsto da := 1 \otimes a - a \otimes 1$ (taken modulo I^2). (I shouldn't really call this "d" until I've verified that it agrees with our earlier definition, but bear with me.)

Let's check that this satisfies the 3 conditions, i.e. that it is a derivation. Two are immediate: it is linear, vanishes on elements of b. Let's check the Leibniz rule:

$$\begin{aligned} d(aa') - a\, da' - a'\, da &= 1 \otimes aa' - aa' \otimes 1 - a \otimes a' + aa' \otimes 1 - a' \otimes a + a'a \otimes 1 \\ &= -a \otimes a' - a' \otimes a + a'a \otimes 1 + 1 \otimes aa' \\ &= (1 \otimes a - a \otimes 1)(1 \otimes a' - a' \otimes 1) \\ &\in I^2. \end{aligned}$$

Thus by the universal property of $\Omega_{A/B}$, we get a natural morphism $\Omega_{A/B} \to I/I^2$ of A-modules.

1.3. *Theorem. — The natural morphism $f : \Omega_{A/B} \to I/I^2$ induced by the universal property of $\Omega_{A/B}$ is an isomorphism.*

Proof. We'll show this as follows. (i) We'll show that f is surjective, and (ii) we will describe $g : I/I^2 \to \Omega_{A/B}$ such that $g \circ f : \Omega_{A/B} \to \Omega_{A/B}$ is the identity. Both of these steps will be very short. Then we'll be done, as to show $f \circ g$ is the identity, we need only show (by surjectivity of g) that $(f \circ g)(f(a)) = f(a)$, which is true (by (ii) $g \circ f = id$).

(i) For surjectivity, we wish to show that I is generated (modulo I^2) by $a \otimes 1 - 1 \otimes a$ as a runs over the elements of A. This has a one sentence explanation: If $\sum x_i \otimes y_i \in I$, i.e. $\sum x_i y_i = 0$ in A, then $\sum_i x_i \otimes y_i = \sum_i x_i (1 \otimes y_i - y_i \otimes 1)$.

(ii) Define $g : I/I^2 \to \Omega_{A/B}$ by $x \otimes y \mapsto x\, dy$. We need to check that this is well-defined, i.e. that elements of I^2 are sent to 0, i.e. we need that

$$\left(\sum x_i \otimes y_i\right)\left(\sum x'_j \otimes y'_j\right) = \sum_{i,j} x_i x'_j \otimes y_i y'_j \mapsto 0$$

where $\sum_i x_i y_i = \sum x'_j y'_j = 0$. But by the Leibniz rule,

$$\begin{aligned} \sum_{i,j} x_i x'_j\, d(y_i y'_j) &= \sum_{i,j} x_i x'_j y_i\, dy'_j + \sum_{i,j} x_i x'_j y'_j\, dy_i \\ &= \left(\sum_i x_i y_i\right)\left(\sum_j x'_j\, dy'_j\right) + \left(\sum_i x_i\, dy_i\right)\left(\sum_j x'_j y'_j\right) \\ &= 0. \end{aligned}$$

Then $f \circ g$ is indeed the identity, as

$$da \xrightarrow{g} 1 \otimes a - a \otimes 1 \xrightarrow{f} 1\, da - a\, d1 = da$$

as desired. □

We can now use our understanding of how Ω works on affine open sets to state some global results.

1.4. *Exercise.* Suppose $f : X \to Y$ is locally of finite type, and X is locally Noetherian. Show that $\Omega_{X/Y}$ is a coherent sheaf on X.

The relative cotangent exact sequence and the conormal exact sequence for schemes now directly follow.

1.5. *Theorem. — (Relative cotangent exact sequence) Suppose* $X \xrightarrow{f} Y \xrightarrow{g} Z$ *be morphisms of schemes. Then there is an exact sequence of quasicoherent sheaves on* X

$$f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0.$$

(Conormal exact sequence) Suppose $f : X \to Y$ *morphism of schemes,* $Z \hookrightarrow X$ *closed subscheme of* X*, with ideal sheaf* $\mathcal{I}$*. Then there is an exact sequence of sheaves on* Z*:*

$$\mathcal{I}/\mathcal{I}^2 \xrightarrow{\delta} \Omega_{X/Y} \otimes \mathcal{O}_Z \longrightarrow \Omega_{Z/Y} \longrightarrow 0.$$

Similarly, the sheaf of relative differentials pull back, and behave well under base change.

1.6. *Theorem (pullback of differentials). — (a) If*

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d] & X \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd}$$

is a commutative diagram of schemes, there is a natural homomorphism of quasicoherent sheaves on X' $g^*\Omega_{X/Y} \to \Omega_{X'/Y'}$*. An important special case is* $Y = Y'$*.*
(b) (Ω behaves well under base change) If furthermore the above diagram is a tensor diagram (i.e. $X' \cong X \otimes_Y Y'$*) then* $g^*\Omega_{X/Y} \to \Omega_{X'/Y'}$ *is an isomorphism.*

This follows immediately from an Exercise in last day's notes. Part (a) implicitly came up in our earlier discussion of the Riemann-Hurwitz formula.

As a particular case of part (b), the fiber of the sheaf of relative differentials is indeed the sheaf of differentials of the fiber. *Thus this notion indeed glues together the differentials on each fiber.*

2. EXAMPLES

2.1. The projective line. As an important first example, let's consider $\mathbb{P}^1_k$, with the usual projective coordinates x_0 and x_1. As usual, the first patch corresponds to $x_0 \neq 0$, and is of the form $\operatorname{Spec} k[x_{1/0}]$ where $x_{1/0} = x_1/x_0$. The second patch corresponds to $x_1 \neq 0$, and is of the form $\operatorname{Spec} k[x_{0/1}]$ where $x_{0/1} = x_0/x_1$.

Both patches are isomorphic to $\mathbb{A}^1_k$, and $\Omega_{\mathbb{A}^1_k} = \mathcal{O}_{\mathbb{A}^1_k}$. (More precisely, $\Omega_{k[x]/k} = k[x]\ dx$.) Thus $\Omega_{\mathbb{P}^1_k}$ is an invertible sheaf (a line bundle). Now we have classified the invertible sheaves on $\mathbb{P}^1_k$ — they are each of the form $\mathcal{O}(m)$. So which invertible sheaf is $\Omega_{\mathbb{P}^1/k}$?

Let's take a section, $dx_{1/0}$ on the first patch. It has no zeros or poles there, so let's check what happens on the other patch. As $x_{1/0} = 1/x_{0/1}$, we have $dx_{1/0} = -(1/x_{0/1}^2)\ dx_{0/1}$. Thus this section has a double pole where $x_{0/1} = 0$. Hence $\Omega_{\mathbb{P}^1_k/k} \cong \mathcal{O}(-2)$.

Note that the above argument did not depend on k being a field, and indeed we could replace k with any ring A (or indeed with any base scheme).

2.2. A plane curve. Consider next the plane curve $y^2 = x^3 - x$ in $\mathbb{A}^2_k$, where the characteristic of k is not 2. Then the differentials are generated by dx and dy, subject to the constraint that

$$2y\ dy = (3x^2 - 1)\ dx.$$

Thus in the locus where $y \neq 0$, dx is a generator (as dy can be expressed in terms of dx). Similarly, in the locus where $3x^2 - 1 \neq 0$, dy is a generator. These two loci cover the entire curve, as solving $y = 0$ gives $x^3 - x = 0$, i.e. $x = 0$ or ± 1, and in each of these cases $3x^2 - 1 \neq 0$.

Now consider the differential dx. Where does it vanish? Answer: precisely where $y = 0$. You should find this believable from the picture (which I gave in class).

2.3. Exercise: differentials on hyperelliptic curves. Consider the double cover $f : C \to \mathbb{P}^1_k$ branched over $2g+2$ distinct points. (We saw earlier that this curve has genus g.) Then $\Omega_{C/k}$ is again an invertible sheaf. What is its degree? (Hint: let x be a coordinate on one of the coordinate patches of $\mathbb{P}^1_k$. Consider $f^* dx$ on C, and count poles and zeros.) In class I gave a sketch showing that you should expect the answer to be $2g - 2$.

2.4. Exercise: differentials on nonsingular plane curves. Suppose C is a nonsingular plane curve of degree d in $\mathbb{P}^2_k$, where k is algebraically closed. By considering coordinate patches, find the degree of $\Omega_{C/k}$. Make any reasonable simplifying assumption (so that you believe that your result still holds for "most" curves).

Because Ω behaves well under pullback, note that the assumption that k is algebraically closed may be quickly excised:

2.5. *Exercise.* Suppose that C is a nonsingular projective curve over k such that $\Omega_{C/k}$ is an invertible sheaf. (We'll see that for nonsingular curves, the sheaf of differentials is always locally free. But we don't yet know that.) Let $C_{\overline{k}} = C \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$. Show that $\Omega_{C_{\overline{k}}/\overline{k}}$ is locally free, and that

$$\deg \Omega_{C_{\overline{k}}/\overline{k}} = \deg \Omega_{C/k}.$$

2.6. Projective space. We next examine the differentials of projective space $\mathbb{P}^n_k$. As projective space is covered by affine open sets of the form $\mathbb{A}^n$, on which the differential form a rank n locally free sheaf, $\Omega_{\mathbb{P}^n_k/k}$ is also a rank n locally free sheaf.

2.7. *Theorem (the Euler exact sequence). — The sheaf of differentials $\Omega_{\mathbb{P}^n_k/k}$ satisfies the following exact sequence*

$$0 \to \Omega_{\mathbb{P}^n_A} \to \mathcal{O}_{\mathbb{P}^n_A}(-1)^{\oplus(n+1)} \to \mathcal{O}_{\mathbb{P}^n_A} \to 0.$$

This is handy, because you can get a hold of Ω in a concrete way. Next day I will give an explicit example, to give you some practice.

I discussed some philosophy behind this theorem. Next day, I'll give a proof, and repeat the philosophy.

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