FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 37

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Welcome back to the third quarter! The theme for this quarter, insofar as there is one, will be "useful ideas to know". We'll start with differentials for the first three lectures.

I prefer to start any topic with a number of examples, but in this case I'm going to spend a fair amount of time discussing technicalities, and then get to a number of examples. Here is the main message I want you to get. Differentials are an intuitive geometric notion, and we're going to figure out the right description of them algebraically. I find the algebraic manifestation a little non-intuitive, so I always like to tie it to the geometry. So please don't tune out of the statements. Also, I want you to notice that although the algebraic statements are odd, none of the proofs are hard or long.

This topic could have been done as soon as we knew about morphisms and quasicoherent sheaves.

1. MOTIVATION AND GAME PLAN

Suppose X is a "smooth" k-variety. We hope to define a tangent bundle. We'll see that the right way to do this will easily apply in much more general circumstances.

• We'll see that cotangent is more "natural" for schemes than tangent bundle. This is similar to the fact that the Zariski *cotangent* space is more natural than the *tangent space* (i.e. if A is a ring and $\mathfrak{m}$ is a maximal ideal, then $\mathfrak{m}/\mathfrak{m}^2$ is "more natural" than $(\mathfrak{m}/\mathfrak{m}^2)^\vee$. So we'll define the cotangent sheaf first.

• Our construction will work for general X, even if X is not "smooth" (or even at all nice, e.g. finite type). The cotangent sheaf won't be locally free, but it will still be a quasicoherent sheaf.

• Better yet, this construction will work "relatively". For *any* $X \to Y$, we'll define $\Omega_{X/Y}$, a quasicoherent sheaf on X, the sheaf of *relative differentials*. This will specialize to the earlier

Date: Tuesday, April 4, 2006. Minor update June 25, 2007.

case by taking $Y = \operatorname{Spec} k$. The idea is that this glues together the cotangent sheaves of the fibers of the family. (I drew an intuitive picture in the "smooth" case. I introduced the phrase "vertical (co)tangent vectors".)

2. THE AFFINE CASE: THREE DEFINITIONS

We'll first study the affine case. Suppose A is a B-algebra, so we have a morphism of rings $\phi : B \to A$ and a morphism of schemes $\operatorname{Spec} A \to \operatorname{Spec} B$. I will define an A-module $\Omega_{A/B}$ in three ways. This is called the *module of relative differentials* or the *module of Kähler differentials*. The module of differentials will be defined to be this module, as well as a map $d : A \to \Omega_{A/B}$ satisfying three properties.
(i) additivity. $da + da' = d(a + a')$
(ii) Leibniz. $d(aa') = a\, da' + a' da$
(iii) triviality on pullbacks. $db = 0$ for $b \in \phi(B)$.

As motivation, think of the case $B = k$. So for example, $da^n = na^{n-1} da$, and more generally, if f is a polynomial in one variable, $df(a) = f'(a)\ da$ (where f' is defined formally: if $f = \sum c_i x^i$ then $f' = \sum c_i i x^{i-1}$).

I'll give you three definitions of this sheaf in the affine case (i.e. this module). The first is a concrete hands-on definition. The second is by universal property. And the third will globalize well, and will allow us to define $\Omega_{X/Y}$ conveniently in general.

The first two definitions are analogous to what we have seen for tensor product. Recall that there are two common definitions of $\otimes$. The first is in terms of formal symbols satisfying some rules. This is handy for showing certain things, e.g. if $M \to M'$ is surjective, then so is $M \otimes N \to M' \otimes N$. The second is by universal property.

2.1. First definition of differentials: explicit description.

We define $\Omega_{A/B}$ to be finite A-linear combinations of symbols "da" for $a \in A$, subject to the three rules (i)–(iii) above. For example, take $A = k[x, y]$, $B = k$. Then a sample differential is $3x^2\ dy + 4dx \in \Omega_{A/B}$. We have identities such as $d(3xy^2) = 3y^2\ dx + 6xy\ dy$.

Key fact. Note that if A is generated over B (as an algebra) by $x_i \in A$ (where i lies in some index set, possibly infinite), subject to some relations r_j (where j lies in some index set, and each is a polynomial in some finite number of the x_i), then the A-module $\Omega_{A/B}$ is generated by the dx_i, subject to the relations (i)—(iii) and $dr_j = 0$. In short, we needn't take every single element of A; we can take a generating set. And we needn't take every single relation among these generating elements; we can take generators of the relations.

2.2. *Exercise.* Verify the above key fact.

In particular:

2.3. *Proposition. — If* A *is a finitely generated* B*-algebra, then* $\Omega_{A/B}$ *is a finite type (i.e. finitely generated)* A*-module. If* A *is a finitely presented* B*-algebra, then* $\Omega_{A/B}$ *is a finitely presented* A*-module.*

("Finitely presented" algebra means finite number of generators (=finite type) and finite number of relations. If A is Noetherian, then the two hypotheses are the same, so most of you will not care.)

Let's now see some examples. Among these examples are three particularly important kinds of ring maps that we often consider: adding free variables; localizing; and taking quotients. If we know how to deal with these, we know (at least in theory) how to deal with any ring map.

2.4. Example: taking a quotient. If $A = B/I$, then $\Omega_{A/B} = 0$ basically immediately: $da = 0$ for all $a \in A$, as each such a is the image of an element of B. This should be believable; in this case, there are no "vertical tangent vectors".

2.5. Example: adding variables. If $A = B[x_1, \dots, x_n]$, then $\Omega_{A/B} = A dx_1 \oplus \cdots \oplus A dx_n$. (Note that this argument applies even if we add an arbitrarily infinite number of indeterminates.) The intuitive geometry behind this makes the answer very reasonable. The cotangent bundle should indeed be trivial of rank n.

2.6. Example: two variables and one relation. If $B = \mathbb{C}$, and $A = \mathbb{C}[x, y]/(y^2 - x^3)$, then $\Omega_{A/B} = \mathbb{C}\ dx \oplus \mathbb{C}\ dy/(2y\ dy - 3x^2\ dx)$.

2.7. Example: localization. If S is a multiplicative set of B, and $A = S^{-1}B$, then $\Omega_{A/B} = 0$. Reason: Note that the quotient rule holds. (If $b = as$, then $db = a\ ds + s\ da$, which can be rearranged to give $da = (s\ db - b\ ds)/s^2$.) Thus if $a = b/s$, then $da = (s\ db - b\ ds)/s^2 = 0$. (If $A = B_f$ for example, this is intuitively believable; then $\operatorname{Spec} A$ is an open subset of $\operatorname{Spec} B$, so there should be no "vertical cotangent vectors".)

2.8. Exercise: localization (stronger form). If S is a multiplicative set of A, show that there is a natural isomorphism $\Omega_{S^{-1}A/B} \cong S^{-1}\Omega_{A/B}$. (Again, this should be believable from the intuitive picture of "vertical cotangent vectors".) If T is a multiplicative set of B, show that there is a natural isomorphism $\Omega_{S^{-1}A/T^{-1}B} \cong S^{-1}\Omega_{A/B}$ where S is the multiplicative set of A that is the image of the multiplicative set $T \subset B$.

2.9. *Exercise.* (a) *(pullback of differentials)* If

$$\begin{array}{ccc} A' & \longleftarrow & A \\ \uparrow & & \uparrow \\ B' & \longleftarrow & B \end{array}$$

is a commutative diagram, show that there is a natural homomorphism of A'-modules $A' \otimes_A \Omega_{A/B} \to \Omega_{A'/B'}$. An important special case is $B = B'$.

(b) *(differentials behave well with respect to base extension, affine case)* If furthermore the above diagram is a tensor diagram (i.e. $A' \cong B' \otimes_B A$) then show that $A' \otimes_A \Omega_{A/B} \to \Omega_{A'/B'}$ is an isomorphism.

2.10. *Exercise.* Suppose k is a field, and K is a separable algebraic extension of k. Show that $\Omega_{K/k} = 0$.

2.11. *Exercise (Jacobian description of $\Omega_{A/B}$).* — Suppose $A = B[x_1, \dots, x_n]/(f_1, \dots, f_r)$. Then $\Omega_{A/B} = \{\oplus_i B dx_i\}/\{df_j = 0\}$ maybe interpreted as the cokernel of the Jacobian matrix $J : A^{\oplus r} \to A^{\oplus n}$.

I now want to tell you two handy (geometrically motivated) exact sequences. The arguments are a bit tricky. They are useful, but a little less useful than the foundation facts above.

2.12. *Theorem (the relative cotangent sequence, affine version).* — *Suppose* $C \to B \to A$ *are ring homomorphisms. Then there is a natural exact sequence of* A*-modules*

$$A \otimes_B \Omega_{B/C} \to \Omega_{A/C} \to \Omega_{A/B} \to 0.$$

Before proving this, I drew a picture motivating the statement. I drew pictures of two maps of schemes, $\operatorname{Spec} A \xrightarrow{f} \operatorname{Spec} B \xrightarrow{g} \operatorname{Spec} C$, where $\operatorname{Spec} C$ was a point, $\operatorname{Spec} B$ was $\mathbb{A}^1$ (or a "smooth curve"), and $\operatorname{Spec} A$ was $\mathbb{A}^2$ (or a "smooth surface"). The tangent space to a point upstairs has a subspace that is the tangent space to the vertical fiber. The cokernel is the pullback of the tangent space to the image point in $\operatorname{Spec} B$. Thus we have an exact sequence $0 \to T_{\operatorname{Spec} A/\operatorname{Spec} B} \to T_{\operatorname{Spec} A/\operatorname{Spec} C} \to f^* T_{\operatorname{Spec} B/\operatorname{Spec} C} \to 0$. We want the corresponding sequence of cotangent vectors, so we dualize. We end up with precisely the statement of the Theorem, except we also have left-exactness. This discrepancy is because the statement of the theorem is more general; we'll see that in the "smooth" case, we'll indeed have left-exactness.

Proof. (Before we start, note that surjectivity is clear, from $da \mapsto da$. The composition over the middle term is clearly 0: $db \to db \to 0$.) We wish to identify $\Omega_{A/B}$ as the cokernel of $A \otimes_B \Omega_{B/C} \to \Omega_{A/C}$. Now $\Omega_{A/B}$ is exactly the same as $\Omega_{A/C}$, except we have extra relations: $db = 0$ for $b \in B$. These are precisely the images of $1 \otimes db$ on the left. □

2.13. *Theorem (Conormal exact sequence, affine version).* — *Suppose* B *is a* C*-algebra,* I *is an ideal of* B*, and* $A = B/I$*. Then there is a natural exact sequence of* A*-modules*

$$I/I^2 \xrightarrow{\delta: i \mapsto 1 \otimes di} A \otimes_B \Omega_{B/C} \xrightarrow{a \otimes db \mapsto a\, db} \Omega_{A/C} \longrightarrow 0.$$

Before getting to the proof, some discussion is necessary. (The discussion is trickier than the proof itself!)

The map δ is a bit subtle, so I'll get into its details before discussing the geometry. For any $i \in I$, $\delta i = 1 \otimes di$. Note first that this is well-defined: If $i, i' \in I$, $i \equiv i' \pmod{I^2}$, say $i - i' = i''i'''$ where $i'', i''' \in I$, then $\delta i - \delta i' = 1 \otimes (i''\, di''' + i'''\, di'') \in I\Omega_{B/C}$ is 0 in $A \otimes_B \Omega_{B/C} = (B/I) \otimes_B \Omega_{B/C}$. Next note that I/I^2 indeed is an $A = (B/I)$-module. Finally, note that the map $I/I^2 \to A \otimes_B \Omega_{B/C}$ is indeed a homomorphism of A-modules: If $a \in A$, $b \in I$, then $ab \mapsto 1 \otimes d(ab) = 1 \otimes (a\, db + b\, da) = 1 \otimes (a\, db) = a(1 \otimes db)$.

Having dispatched that formalism, let me get back to the geometry. I drew a picture where $\operatorname{Spec} C$ is a point, $\operatorname{Spec} B$ is a plane, and $\operatorname{Spec} A$ is something smooth in it. Let j be the inclusion. Then we have $0 \to T_{\operatorname{Spec} A/\operatorname{Spec} C} \to j^* T_{\operatorname{Spec} B/\operatorname{Spec} C} \to N_{\operatorname{Spec} B/\operatorname{Spec} C} \to 0$. Dualizing it, we get $0 \to N^\vee_{A/B} \to A \otimes \Omega_{B/C} \to \Omega_{A/C} \to 0$. This exact sequence reminds me of several things above and beyond the theorem. First of all, I/I^2 will later be the conormal bundle — hence the name of the theorem. Second, in good circumstances, the conormal exact sequence of Theorem 2.13 will be injective on the left.

2.14. *Aside: Why should I/I^2 be the conormal bundle?.* We'll *define* I/I^2 to be the conormal bundle later, so I'll try to give you an idea as to why this is reasonable. You believe now that $\mathfrak{m}/\mathfrak{m}^2$ should be the cotangent space to a point in $\mathbb{A}^n$. In other words, $(x_1, \dots, x_n)/(x_1, \dots, x_n)^2$ is the cotangent space to $\vec{0}$ in $\mathbb{A}^n$. Translation: it is the conormal space to the point $\vec{0} \in \mathbb{A}^n$. Then you might believe that in $\mathbb{A}^{n+m}$, $(x_1, \dots, x_n)/(x_1, \dots, x_n)^2$ is the conormal bundle to the coordinate n-plane $\mathbb{A}^m \subset \mathbb{A}^{n+m}$.

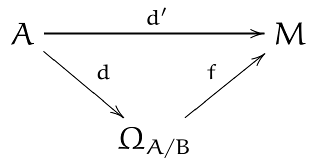
Let's finally prove the conormal exact sequence.

Proof of the conormal exact sequence (affine version) 2.13. We need to identify the cokernel of $\delta : I/I^2 \to A \otimes_B \Omega_{B/C}$ with $\Omega_{A/C}$. Consider $A \otimes_B \Omega_{B/C}$. As an A-module, it is generated by db ($b \in B$), subject to three relations: $dc = 0$ for $c \in \phi(C)$ (where $\phi : C \to B$ describes B as a C-algebra), additivity, and the Leibniz rule. Given any relation *in* B, d of that relation is 0.

Now $\Omega_{A/C}$ is defined similarly, except there are more relations *in* A; these are precisely the elements of $i \in B$. Thus we obtain $\Omega_{A/C}$ by starting out with $A \otimes_B \Omega_{B/C}$, and adding the additional relations di where $i \in I$. But this is precisely the image of δ! ☐

2.15. Second definition: universal property. Here is a second definition that we'll use at least once, and is certainly important philosophically. Suppose A is a B-algebra, and M is a A-module. An *B-linear derivation of A into M* is a map $d : A \to M$ of B-modules (*not necessarily A-modules*) satisfying the Leibniz rule: $d(fg) = f\, dg + g\, df$. As an example, suppose $B = k$, and $A = k[x]$, and $M = A$. Then an example of a k-linear derivation is d/dx. As a second example, if $B = k$, $A = k[x]$, and $M = k$. Then an example of a k-linear derivation is $d/dx|_0$.

Then $d : A \to \Omega_{A/B}$ is defined by the following universal property: any other B-linear derivation $d' : A \to M$ factors uniquely through d:



Here f is a map of A-modules. (Note again that d and d' are not! They are only B-linear.) By universal property nonsense, if it exists, it is unique up to unique isomorphism. The candidate I described earlier clearly satisfies this universal property (in particular, it is a derivation!), hence this is it. [Thus Ω is the "unversal derivation". I should rewrite this paragraph at some point.]

The next result will give you more evidence that this deserves to be called the (relative) cotangent bundle.

2.16. *Proposition. Suppose* B *is a* k*-algebra, with residue field* k*. Then the natural map* $\delta : \mathfrak{m}/\mathfrak{m}^2 \to \Omega_{B/k} \otimes_B k$ *is an isomorphism.*

I skipped this proof in class, but promised it in the notes.

Proof. By the conormal exact sequence 2.13 with $I = \mathfrak{m}$ and $A = C = k$, δ is a surjection (as $\Omega_{k/k} = 0$), so we need to show that it is injection, or equivalently that $\mathrm{Hom}_k(\Omega_{B/k} \otimes_B k, k) \to \mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k)$ is a surjection. But any element on the right is indeed a derivation from B to k (an earlier exercise from back in the dark ages on the Zariski tangent space), which is precisely an element of $\mathrm{Hom}_B(\Omega_{B/k}, k)$ (by the universal property of $\Omega_{B/k}$), which is canonically isomorphic to $\mathrm{Hom}_k(\Omega_{B/k} \otimes_B k, k)$ as desired. □

Remark. As a corollary, this (in combination with the Jacobian exercise 2.11 above) gives a second proof of an exercise from the first quarter, showing the Jacobian criterion for nonsingular varieties over an algebraically closed field.

Aside. If you wish, you can use the universal property to show that $\Omega_{A/B}$ behaves well with respect to localization. For example, if S is a multiplicative set of A, then there is a natural isomorphism $\Omega_{S^{-1}A/B} \cong S^{-1}\Omega_{A/B}$. This can be used to give a different solution to Exercise 2.8. It can also be used to give a second definition of $\Omega_{X/Y}$ for a morphism of schemes $X \to Y$ (different from the one given below): we define it as a quasicoherent sheaf, by describing how it behaves on affine open sets, and showing that it behaves well with respect to distinguished localization.

Next day, I'll give a third definition which will globalize well, and we'll see that we already understand differentials for morphisms of schemes.

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