

FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 34

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Last day: The Leray spectral sequence. Beginning fun with curves: Ω_C , and the Riemann-Hurwitz formula.

Today: More fun with curves: Serre duality, criterion for closed immersion, series of useful remarks, curves of genus 0 and 2

1. LAST DAY

Last day we began to talk about curves over a field k . Our standing assumptions will be that a curve C is projective, geometrically integral and nonsingular over a field k .

(People happy to work over algebraically closed fields can continue to ignore the adverb “geometrically”.)

I’m in the process of telling you a few facts that we will prove next quarter. We will use these facts to prove lots of things about curves.

Last day I defined Ω_C , sheaf of differentials on C . I really should have called it $\Omega_{C/k}$, to make clear that this sheaf on C depends on the structure morphism $C \rightarrow k$. I stated that $\Omega_{C/k}$ is an invertible sheaf, and told you that we will soon see that has degree $\deg \Omega_C = 2g_C - 2$. I stated that differentials pullback under covers $f : C \rightarrow C'$ (i.e. that there is a morphism $f^* \Omega_{C'/k} \rightarrow \Omega_{C/k}$), and if we are in characteristic 0, then this yields an

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inclusion of invertible sheaves, which yields $0 \rightarrow f^*\Omega_{C'} \rightarrow \Omega_C \rightarrow \mathcal{R} \rightarrow 0$, where \mathcal{R} corresponds to the *ramification divisor* on C , which keeps track of the branching of $C \rightarrow C'$. From this I claimed that we will deduce the *Riemann-Hurwitz formula*

$$\boxed{2g_C - 2 = d(2g_{C'} - 2) + \deg \mathcal{R}}$$

1.1. Serre duality. (We are not requiring k to be algebraically closed.) In general, nonsingular varieties will have a special invertible sheaf \mathcal{K}_X which is the determinant of Ω_X . This invertible sheaf is called the *canonical bundle*, and will later be defined in much greater generality. In our case, $X = C$ is a curve, so $\mathcal{K}_C = \Omega_C$, and from here on in, we'll use \mathcal{K}_C instead of Ω_C . The reason it is called the dualizing sheaf is because it arises in Serre duality. Serre duality states that $H^1(C, \mathcal{K}) \cong k$, or more precisely that there is a *trace morphism* $\boxed{H^1(C, \mathcal{K}) \rightarrow k}$ that is an isomorphism. (Example: if $C = \mathbb{P}^1$, then we indeed have $h^1(\mathbb{P}^1, \mathcal{O}(-2)) = 1$.)

Further, for any coherent sheaf \mathcal{F} , the natural map

$$\boxed{H^0(C, \mathcal{F}) \otimes_k H^1(C, \mathcal{K} \otimes \mathcal{F}^\vee) \rightarrow H^1(C, \mathcal{K})}$$

is a perfect pairing. Thus in particular, $h^0(C, \mathcal{F}) = h^1(C, \mathcal{K} \otimes \mathcal{F}^\vee)$. Recall we defined the arithmetic genus of a curve to be $h^1(C, \mathcal{O}_C)$. Then $h^0(C, \mathcal{K}) = g$ as well.

Recall that Riemann-Roch for an invertible sheaf \mathcal{L} states that

$$h^0(C, \mathcal{L}) - h^1(C, \mathcal{L}) = \deg \mathcal{L} - g + 1.$$

Applying this to $\mathcal{L} = \mathcal{K}$, we get

$$\deg \mathcal{K} = h^0(C, \mathcal{K}) - h^1(C, \mathcal{K}) + g - 1 = h^1(C, \mathcal{O}) - h^0(C, \mathcal{O}) + g - 1 = g - 1 + g - 1 = 2g - 2$$

as promised earlier.

1.2. A criterion for when a morphism is a closed immersion. We'll also need a criterion for when something is a closed immersion. To help set it up, let's observe some facts about closed immersions. Suppose $f : X \rightarrow Y$ is a closed immersion. Then f is projective, and it is injective on points. This is not enough to ensure that it is a closed immersion, as the example of the normalization of the cusp shows (Figure 1). Another example is the Frobenius morphism from \mathbb{A}^1 to \mathbb{A}^1 , given by $k[t] \rightarrow k[u]$, $u \rightarrow t^p$, where k has characteristic p .

The additional information you need is that the tangent map is an isomorphism at all closed points. (Exercise: show this is false in those two examples.)

1.3. Theorem. — *Suppose k is an algebraically closed field, and $f : X \rightarrow Y$ is a projective morphism of finite-type k -schemes that is injective on closed points and injective on tangent vectors of closed points. Then f is a closed immersion.*

The example of $\text{Spec } \mathbb{C} \rightarrow \text{Spec } \mathbb{R}$ shows that we need the hypothesis that k is algebraically closed.

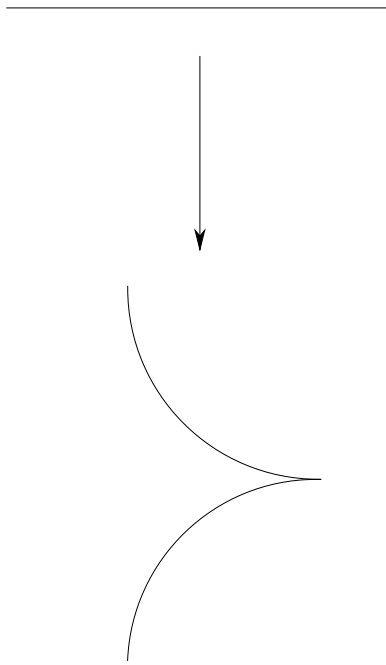


FIGURE 1. Projective morphisms that are injective on points need not be closed immersions

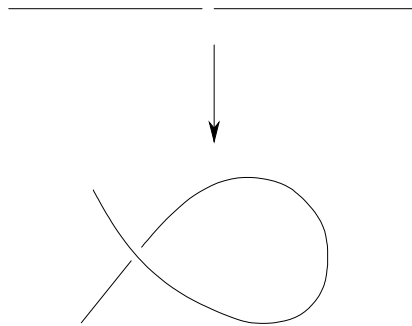


FIGURE 2. We need the projective hypothesis in Theorem 1.3

We need the hypothesis of projective morphism, as shown by the following example (which was described at the blackboard, see Figure 2). We map \mathbb{A}^1 to the plane, so that its image is a curve with one node. We then consider the morphism we get by discarding one of the preimages of the node. Then this morphism is an injection on points, and is also injective on tangent vectors, but it is not a closed immersion. (In the world of differential geometry, this fails to be an embedding because the map doesn't give a homeomorphism onto its image.)

Suppose $f(p) = q$, where p and q are closed points. We will use the hypothesis that X and Y are k -schemes where k is algebraically closed at only one point of the argument: that the map induces an isomorphism of residue fields at p and q .

(For those of you who are allergic to algebraically closed fields: still pay attention, as we'll use this to prove things about curves over k where k is *not* necessarily algebraically closed.)

This is the hardest result of today. We will kill the problem in old-school French style: death by a thousand cuts.

Proof. We may assume that Y is affine, say $\text{Spec } B$.

I next claim that f has finite fibers, not just finite fibers above closed points: the fiber dimension for projective morphisms is upper-semicontinuous (Class 32 Exercise 2.3), so the locus where the fiber dimension is at least 1 is a closed subset, so if it is non-empty, it must contain a closed point of Y . Thus the fiber over any point is a dimension 0 finite type scheme over that point, hence a finite set.

Hence f is a projective morphism with finite fibers, thus affine, and even finite (Class 32 Corollary 2.4).

Thus X is affine too, say $\text{Spec } A$, and f corresponds to a ring morphism $B \rightarrow A$. We wish to show that this is a surjection of rings, or (equivalently) of B -modules. We will show that for any maximal ideal \mathfrak{n} of B , $B_{\mathfrak{n}} \rightarrow A_{\mathfrak{n}}$ is a surjection of $B_{\mathfrak{n}}$ -modules. (This will show that $B \rightarrow A$ is a surjection. Here is why: if K is the cokernel, so $B \rightarrow A \rightarrow K \rightarrow 0$, then we wish to show that $K = 0$. Now A is a finitely generated B -module, so K is as well, being a homomorphic image of A . Thus $\text{Supp } K$ is a closed set. If $K \neq 0$, then $\text{Supp } K$ is non-empty, and hence contains a closed point $[\mathfrak{n}]$. Then $K_{\mathfrak{n}} \neq 0$, so from the exact sequence $B_{\mathfrak{n}} \rightarrow A_{\mathfrak{n}} \rightarrow K_{\mathfrak{n}} \rightarrow 0$, $B_{\mathfrak{n}} \rightarrow A_{\mathfrak{n}}$ is not a surjection.)

If $A_{\mathfrak{n}} = 0$, then clearly $B_{\mathfrak{n}}$ surjects onto $A_{\mathfrak{n}}$, so assume otherwise. I claim that $A_{\mathfrak{n}} = A \otimes_B B_{\mathfrak{n}}$ is a local ring. Proof: $\text{Spec } A_{\mathfrak{n}} \rightarrow \text{Spec } B_{\mathfrak{n}}$ is a finite morphism (as it is obtained by base change from $\text{Spec } A \rightarrow \text{Spec } B$), so we can use the going-up theorem. $A_{\mathfrak{n}} \neq 0$, so $A_{\mathfrak{n}}$ has a prime ideal. Any point p of $\text{Spec } A_{\mathfrak{n}}$ maps to some point of $\text{Spec } B_{\mathfrak{n}}$, which has $[\mathfrak{n}]$ in its closure. Thus there is a point q in the closure of p that maps to $[\mathfrak{n}]$. But there is only one point of $\text{Spec } A_{\mathfrak{n}}$ mapping to $[\mathfrak{n}]$, which we denote $[m]$. Thus we have shown that m contains all other prime ideals of $\text{Spec } A_{\mathfrak{n}}$, so $A_{\mathfrak{n}}$ is a local ring.

Injectivity of tangent vectors *means* surjectivity of cotangent vectors, i.e. $\mathfrak{n}/\mathfrak{n}^2 \rightarrow \mathfrak{m}/\mathfrak{m}^2$ is a surjection, i.e. $\mathfrak{n} \rightarrow \mathfrak{m}/\mathfrak{m}^2$ is a surjection. Claim: $\mathfrak{n}A_{\mathfrak{n}} = \mathfrak{m}A_{\mathfrak{n}}$. Reason: By Nakayama's lemma for the local ring $A_{\mathfrak{n}}$ and the $A_{\mathfrak{n}}$ -module $\mathfrak{m}A_{\mathfrak{n}}$, we conclude that $\mathfrak{n}A_{\mathfrak{n}} = \mathfrak{m}A_{\mathfrak{n}}$.

Next apply Nakayama's Lemma to the $B_{\mathfrak{n}}$ -module $A_{\mathfrak{n}}$. The element $1 \in A_{\mathfrak{n}}$ gives a generator for $A_{\mathfrak{n}}/\mathfrak{n}A_{\mathfrak{n}} = A_{\mathfrak{n}}/\mathfrak{m}A_{\mathfrak{n}}$, which equals $B_{\mathfrak{n}}/\mathfrak{n}B_{\mathfrak{n}}$ (as both equal k), so we conclude that 1 also generates $A_{\mathfrak{n}}$ as a $B_{\mathfrak{n}}$ -module as desired. \square

1.4. Exercise. Use this to show that the d th Veronese morphism from \mathbb{P}_k^n , corresponding to the complete linear series (see Class 22) $|\mathcal{O}_{\mathbb{P}_k^n}(d)|$, is a closed immersion. Do the same for the Segre morphism from $\mathbb{P}_k^m \times_{\text{Spec } k} \mathbb{P}_k^n$. (This is just for practice for using this criterion.)

This is a weaker result than we had before; we've earlier checked this over an arbitrary base ring, and we are now checking it only over algebraically closed fields.)

2. A SERIES OF USEFUL REMARKS

Suppose now that \mathcal{L} is an invertible sheaf on a curve C (which as always in this discussion is projective, geometrically integral and nonsingular, over a field k which is not necessarily algebraically closed). I'll give a series of small useful remarks that we will soon use to great effect.

2.1. $h^0(C, \mathcal{L}) = 0$ if $\deg \mathcal{L} < 0$. Reason: if there is a non-zero section, then the degree of \mathcal{L} can be interpreted as the number of zeros minus the number of poles. But there are no poles, so this would have to be non-negative. A slight refinement gives:

2.2. $h^0(C, \mathcal{L}) = 0$ or 1 if $\deg \mathcal{L} = 0$. This is because if there is a section, then the degree of \mathcal{L} is the number of zeros minus the number of poles. Then as there are no poles, there can be no zeros. Thus the section (call it s) vanishes nowhere, and gives a trivialization for the invertible sheaf. (Recall how this works: we have a natural bijection for any open set $\Gamma(U, \mathcal{L}) \leftrightarrow \Gamma(U, \mathcal{O}_U)$, where the map from left to right is $s' \mapsto s'/s$, and the map from right to left is $f \mapsto sf$.) Thus if there is a section, $\mathcal{L} \cong \mathcal{O}$. But we've already checked that for a geometrically integral and nonsingular curve C , $h^0(C, \mathcal{L}) = 1$.

2.3. Suppose p is any closed point of degree 1. (In other words, the residue field of p is k .) Then $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-p)) = 0$ or 1 . Reason: consider $0 \rightarrow \mathcal{O}_C(-p) \rightarrow \mathcal{O}_C \rightarrow \mathcal{O}_p \rightarrow 0$, tensor with \mathcal{L} (this is exact as \mathcal{L} is locally free) to get

$$0 \rightarrow \mathcal{L}(-p) \rightarrow \mathcal{L} \rightarrow \mathcal{L}|_p \rightarrow 0.$$

Then $h^0(C, \mathcal{L}|_p) = 1$, so as the long exact sequence of cohomology starts off

$$0 \rightarrow H^0(C, \mathcal{L}(-p)) \rightarrow H^0(C, \mathcal{L}) \rightarrow H^0(C, \mathcal{L}|_p),$$

we are done.

2.4. Suppose for this remark that k is algebraically closed. (In particular, *all* closed points have degree 1 over k .) Then if $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-p)) = 1$ for *all* closed points p , then \mathcal{L} is base-point-free, and hence induces a morphism from C to projective space. (Note that \mathcal{L} has a finite-dimensional vector space of sections: all cohomology groups of all coherent sheaves on a projective k -scheme are finite-dimensional.) Reason: given any p , our equality shows that there exists a section of \mathcal{L} that does not vanish at p .

2.5. Next, suppose p and q are distinct points of degree 1. Then $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-p - q)) = 0, 1$, or 2 (by repeating the argument of 2.3 twice). If $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-p - q)) = 2$, then necessarily

$$(1) \quad h^0(C, \mathcal{L}) = h^0(C, \mathcal{L}(-p)) + 1 = h^0(C, \mathcal{L}(-q)) + 1 = h^0(C, \mathcal{L}(-p - q)) + 2.$$

I claim that the linear system \mathcal{L} separates points p and q , by which I mean that the corresponding map f to projective space satisfies $f(p) \neq f(q)$. Reason: there is a hyperplane of projective space passing through p but not passing through q , or equivalently, there is a section of \mathcal{L} vanishing at p but not vanishing at q . This is because of the last equality in (1).

2.6. By the same argument as above, if p is a point of degree 1, then $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-2p)) = 0, 1, \text{ or } 2$. I claim that if this is 2, then map corresponds to \mathcal{L} (which is already seen to be base-point-free from the above) separates the tangent vectors at p . To show this, I need to show that the cotangent map is *surjective*. To show surjectivity onto a one-dimensional vector space, I just need to show that the map is non-zero. So I need to give a function on the target vanishing at the image of p that pulls back to a function that vanishes at p to order 1 but not 2. In other words, I want a section of \mathcal{L} vanishing at p to order 1 but not 2. But that is the content of the statement $h^0(C, \mathcal{L}(-p)) - h^0(C, \mathcal{L}(-2p)) = 1$.

2.7. Combining some of our previous comments: suppose C is a curve over an *algebraically closed* field k , and \mathcal{L} is an invertible sheaf such that for *all* closed points p and q , *not necessarily distinct*, $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-p - q)) = 2$, then \mathcal{L} gives a closed immersion into projective space, as it separates points and tangent vectors, by Theorem 1.3.

2.8. We now bring in Serre duality. I claim that $\deg \mathcal{L} > 2g - 2$ implies

$$h^0(C, \mathcal{L}) = \deg \mathcal{L} - g - 1.$$

This is important — remember this! Reason: $h^1(C, \mathcal{L}) = h^0(C, \mathcal{K} \otimes \mathcal{L}^\vee)$; but $\mathcal{K} \otimes \mathcal{L}^\vee$ has negative degree (as \mathcal{K} has degree $2g - 2$), and thus this invertible sheaf has no sections. Thus Riemann-Roch gives us the desired result.

Exercise. Suppose \mathcal{L} is a degree $2g - 2$ invertible sheaf. Show that it has $g - 1$ or g sections, and it has g sections if and only if $\mathcal{L} \cong \mathcal{K}$.

2.9. We now come to our most important conclusion. Thus if k is algebraically closed, then $\deg \mathcal{L} \geq 2g$ implies that \mathcal{L} is basepoint free (and hence determines a morphism to projective space). Also, $\deg \mathcal{L} \geq 2g + 1$ implies that this is in fact a closed immersion. Remember this! [k need not be algebraically closed.]

2.10. I now claim (for the people who like fields that are not algebraically closed) that *the previous remark holds true even if k is not algebraically closed*. Here is why: suppose C is our curve, and $C_{\bar{k}} := C \otimes_k \bar{k}$ is the base change to the algebraic closure (which we are assuming is connected and nonsingular), with $\pi : C_{\bar{k}} \rightarrow C$ (which is an affine morphism, as it is obtained by base change from the affine morphism $\text{Spec } \bar{k} \rightarrow \text{Spec } k$). Then $H^0(C, \mathcal{L}) \otimes_k \bar{k} \cong H^0(C_{\bar{k}}, \pi^* \mathcal{L})$ for reasons I explained last day (see the first exercise on the class 33 notes,

and also on problem set 15).

$$\begin{array}{ccc} C_{\bar{k}} & \xrightarrow{\pi} & C \\ \downarrow & & \downarrow \\ \text{Spec } \bar{k} & \longrightarrow & \text{Spec } k \end{array}$$

Let s_0, \dots, s_n be a basis for the k -vector space $H^0(C, \mathcal{L})$; they give a basis for the \bar{k} -vector space $H^0(C_{\bar{k}}, \pi^* \mathcal{L})$. If \mathcal{L} has degree at least $2g$, then these sections have no common zeros on $C_{\bar{k}}$; but this means that they have no common zeros on C . If \mathcal{L} has degree at least $2g + 1$, then these sections give a closed immersion $C_{\bar{k}} \hookrightarrow \mathbb{P}_{\bar{k}}^n$. Then I claim that $f : C \rightarrow \mathbb{P}_k^n$ (given by the same sections) is also a closed immersion. Reason: we can check this on each affine open subset $U = \text{Spec } A \subset \mathbb{P}_k^n$. Now f has finite fibers, and is projective, hence is a finite morphism (and in particular affine). Let $\text{Spec } B = f^{-1}(U)$. We wonder if $A \rightarrow B$ is a surjection of rings. But we know that this is true upon base changing by \bar{k} : $A \otimes_k \bar{k} \rightarrow B \otimes_k \bar{k}$ is surjective. So we are done.

We're now ready to take these facts and go to the races.

3. GENUS 0

3.1. Claim. — *Suppose C is genus 0, and C has a k -valued point. Then $C \cong \mathbb{P}_k^1$.*

Of course C automatically has a k -point if k is algebraically closed. Thus we see that all genus 0 (integral, nonsingular) curves over an algebraically closed field are isomorphic to \mathbb{P}^1 .

If k is not algebraically closed, then C needn't have a k -valued point: witness $x^2 + y^2 + z^2 = 0$ in $\mathbb{P}_{\mathbb{R}}^2$. We have already observed that this curve is *not* isomorphic to $\mathbb{P}_{\mathbb{R}}^1$, because it doesn't have an \mathbb{R} -valued point.

Proof. Let p be the point, and consider $\mathcal{L} = \mathcal{O}(p)$. Then $\deg \mathcal{L} = 1$, so we can apply what we know above: first of all, $h^0(C, \mathcal{L}) = 2$, and second of all, these two sections give a closed immersion into \mathbb{P}_k^1 . But the only closed immersion of a curve into \mathbb{P}_k^1 is the isomorphism! \square

As a fun bonus, we see that the weird real curve $x^2 + y^2 + z^2 = 0$ in $\mathbb{P}_{\mathbb{R}}^2$ has no *divisors* of degree 1 over \mathbb{R} ; otherwise, we could just apply the above argument to the corresponding line bundle.

Our weird curve shows us that over a non-algebraically closed field, there can be genus 0 curves that are not isomorphic to \mathbb{P}_k^1 . The next result lets us get our hands on them as well.

3.2. Claim. — *All genus 0 curves can be described as conics in \mathbb{P}_k^2 .*

Proof. Any genus 0 curve has a degree -2 line bundle — the canonical bundle \mathcal{K} . Thus any genus 0 curve has a degree 2 line bundle: $\mathcal{L} = \mathcal{K}^\vee$. We apply our machinery to this bundle: $h^0(C, \mathcal{L}) = 3 \geq 2g + 1$, so this line bundle gives a closed immersion into \mathbb{P}^2 . [This proof is not complete if $k = \bar{k}$, as the criterion we are using requires this hypothesis. Exercise: Use §2.10 to give a complete proof.] \square

3.3. Exercise. Suppose C is a genus 0 curve (projective, geometrically integral and non-singular). Show that C has a point of degree at most 2.

We will use the following result later.

3.4. Claim. — Suppose C is not isomorphic to \mathbb{P}_k^1 (with no restrictions on the genus of C), and \mathcal{L} is an invertible sheaf of degree 1. Then $h^0(C, \mathcal{L}) < 2$.

Proof. Otherwise, let s_1 and s_2 be two (independent) sections. As the divisor of zeros of s_i is the degree of \mathcal{L} , each vanishes at a single point p_i (to order 1). But $p_1 \neq p_2$ (or else s_1/s_2 has no poles or zeros, i.e. is a constant function, i.e. s_1 and s_2 are dependent). Thus we get a map $C \rightarrow \mathbb{P}^1$ which is basepoint free. This is a finite degree 1 map of nonsingular curves, which induces a degree 1 extension of function fields, i.e. an isomorphism of function fields, which means that the curves are isomorphic. But we assumed that C is not isomorphic to \mathbb{P}_k^1 . \square

4. GENUS ≥ 2

It might make most sense to jump to genus 1 at this point, but the theory of elliptic curves is especially rich and beautiful, so I'll leave it for the end.

In general, the curves have quite different behaviors (topologically, arithmetically, geometrically) depending on whether $g = 0$, $g = 1$, or $g > 2$. This trichotomy extends to varieties of higher dimension. I gave a very brief discussion of this trichotomy for curves. For example, arithmetically, genus 0 curves can have lots and lots of points, genus 1 curves can have lots of points, and by Faltings' Theorem (Mordell's Conjecture) any curve of genus at least 2 has at most finitely many points. (Thus we knew before Wiles that $x^n + y^n = z^n$ in \mathbb{P}^2 has at most finitely many solutions for $n \geq 4$, as such curves have genus $\binom{n-1}{2} > 1$.) Geometrically, Riemann surfaces of genus 0 are positively curved, Riemann surfaces of genus 1 are flat, and Riemann surfaces of genus $g > 1$ are negatively curved. We will soon see that curves of genus at least 2 have finite automorphism groups, while curves of genus 1 have some automorphisms (a one-dimensional family), and (we've seen earlier) curves of genus 1 (over an algebraically closed field) have a three-dimensional automorphism group.

4.1. Genus 2. Fix a curve C of genus 2. Then \mathcal{K} is degree 2, and has 2 sections. I claim that \mathcal{K} is base-point-free. Otherwise, if p is a base point, then $\mathcal{K}(-p)$ is a degree 1 invertible sheaf with 2 sections, and we just showed (Claim 3.4) that this is impossible. Thus we

have a double cover of \mathbb{P}^1 . Conversely, any double cover $C \rightarrow \mathbb{P}^1$ arises from a degree 2 invertible sheaf with at least 2 sections, so by one of our useful facts, if $g(C) = 2$, this invertible sheaf must be the canonical bundle (as the only degree 2 invertible sheaf on a genus 2 curve with at least 2 sections is \mathcal{K}_C). Hence we have a natural bijection between genus 2 curves and genus 2 double covers of \mathbb{P}^1 .

We now specialize to the case where $k = \bar{k}$, and the characteristic of k is 0. (All we will need, once we actually prove the Riemann-Hurwitz formula, is that the characteristic be distinct from 2.) Then the Riemann-Hurwitz formula shows that the cover is branched over 6 points. We will see next day that a double cover is determined by its branch points. Hence genus 2 curves are in bijection with unordered sextuples of points on \mathbb{P}^1 . There is thus a 3-dimensional family of genus 2 curves — we have found them all!

(This is still a little imprecise; we would like to say that the moduli space of genus 2 curves is of dimension 3, but we haven't defined what we mean by moduli space!)

More generally, we may see next week (admittedly informally) that if $g > 1$, the curves of genus g “form a family” of dimension $3g - 3$. (If we knew the meaning of “moduli space”, we would say that the dimension of the moduli space of genus g curves \mathcal{M}_g is $3g - 3$.) What goes wrong in genus 0 and 1? The following table (as yet unproved by us!) might help.

genus	dimension of family of curves	dimension of automorphism group of curve
0	0	3
1	1	1
2	3	0
3	6	0
4	9	0
5	12	0
\vdots	\vdots	\vdots

You can probably see the pattern. This is a little like the behavior of the Hilbert function: the dimension of the moduli space is “eventually polynomial”, so there is something that is better-behaved that is an alternating sum, and once the genus is sufficiently high, the “error term” becomes zero. The interesting question then becomes: why is the “right” notion the second column of the table minus the third? (In fact the second column is $h^1(C, T_C)$, where T_C is the tangent bundle — not yet defined — and the third column is $h^0(C, T_C)$. All other cohomology groups of the tangent bundle vanish by dimensional vanishing.)

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