FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 26

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CONTENTS

Last day: separatedness, definition of variety.

Today: proper morphisms.

I said a little more about separatedness of moduli spaces, for those familiar such objects. Suppose we are interested in a moduli space of a certain kind of object. That means that there is a scheme M with a "universal family" of such objects over M, such that there is a bijection between families of such objects over an arbitrary scheme S, and morphisms $S \to B$. (One direction of this map is as follows: given a morphism $S \to B$, we get a family of objects over S by pulling back the universal family over B.) The separatedness of the moduli space (over the base field, for example, if there is one) can be interpreted as follows. Fix a valuation ring A (or even discrete valuation ring, if our moduli space of of finite type) with fraction field K. We interpret Spec intuitively as a germ of a curve, and we interpret $\operatorname{Spec} K$ as the germ minus the "origin" (an analogue of a small punctured disk). Then we have a family of objects over $\operatorname{Spec} K$ (or over the punctured disk), or equivalently a map $\operatorname{Spec} K \to M$, and the moduli space is separated if there is *at most one way* to fill in the family over the origin, i.e. a family over $\operatorname{Spec} A$.

1. PROPER MORPHISMS

I'll now tell you about a new property of morphisms, the notion of *properness*. You can think about this in several ways.

Recall that a map of topological spaces (also known as a continuous map!) is said to be proper if the preimage of compact sets is compact. Properness of morphisms is an analogous property. For example, proper varieties over $\mathbb{C}$ will be the same as compact in the "usual" topology.

Alternatively, we will see that projective morphisms are proper — this is the hardest thing we will prove — so you can see this as nice property satisfied by projective morphisms, and hence as a generalization of projective morphisms. Indeed, in some sense,

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essentially all interesting properties of projective morphisms that don't explicitly involve $\mathcal{O}(1)$ turn out to be properties of proper morphisms. The key tool in showing such results is Chow's Lemma, which I will state but not prove. Like separatedness, there is a valuative criterion for properness.

Definition. We say a map of topological spaces (i.e. a continuous map) $f : X \to Y$ is *closed* if for each closed subset $S \subset X$, $f(S)$ is also closed. (This is the definition used elsewhere in mathematics.) We say a morphism of schemes is closed if the underlying continuous map is closed. We say that a morphism of schemes $f : X \to Y$ is *universally closed* if for every morphism $g : Z \to Y$, the induced $Z \times_Y X \to Z$ is closed. In other words, a morphism is universally closed if it remains closed under any base change. (A note on terminology: if P is some property of schemes, then a morphism of schemes is said to be "universally P" if it remains P under any base change.)

A morphism $f : X \to Y$ *is* **proper** *if it is separated, finite type, and universally closed.*

As an example: we expect that $\mathbb{A}^1_{\mathbb{C}} \to \operatorname{Spec} \mathbb{C}$ is not proper, because the complex manifold corresponding to $\mathbb{A}^1_{\mathbb{C}}$ is not compact. However, note that this map is separated (it is a map of affine schemes), finite type, and closed. So the "universally" is what matters here. What's the base change that turns this into a non-closed map? Consider $\mathbb{A}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$.

1.1. *Exercise.* Show that $\mathbb{A}^1_{\mathbb{C}} \to \operatorname{Spec} \mathbb{C}$ is not proper.

Here are some examples of proper maps.

1.2. Closed immersions are proper: they are clearly separated (as affine morphisms are separated). They are finite type. After base change, they remain closed immersions, and closed immersions are always closed.

More generally, finite morphisms are proper: they are separated (as affine), and finite type. The notion of "finite morphism" behaves well under base change, and we have checked that finite morphisms are always closed (I believe in class 21, using the Going-up theorem).

I mentioned that we are going to show that projective morphisms are proper. In fact, finite morphisms are projective (and closed immersions are finite), so the previous two facts will follow from our fancier fact. I should have explained earlier why finite morphisms are projective, but I'll do so now. Suppose $X \to Y$ is a finite morphism, i.e. $X = \underline{\operatorname{Spec}}\, \mathcal{A}$ where $\mathcal{A}$ is a finite type sheaf of algebras. I will now show that $X = \underline{\operatorname{Proj}}\, \mathcal{S}_*$, where $\mathcal{S}_*$ is a sheaf of graded algebras, satisfying all of our various conditions: $\mathcal{S}_0 = \mathcal{O}_Y$, $\mathcal{S}_*$ is "locally generated" by $\mathcal{S}_1$ as a $\mathcal{S}_0$-algebra (i.e. this is true over every open affine subset of Y). Given the statement, you might be able to guess what $\mathcal{S}_*$ should be. I must tell you what $\mathcal{S}_n$ is, and how to multiply. Take $\mathcal{S}_n = \mathcal{A}$ for $n > 0$, with the "obvious" map.

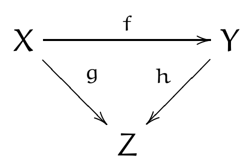
1.3. *Exercise.* Verify that $X = \underline{\operatorname{Proj}}\, \mathcal{S}_*$. What is $\mathcal{O}_{\underline{\operatorname{Proj}}\, \mathcal{S}_*}(1)$?

1.4. Properties of proper morphisms.

1.5. *Proposition. —*

(a) *The notion of "proper morphism" is stable under base change.*
(b) *The notion of "proper morphism" is local on the target (i.e.* $f : X \to Y$ *is proper if and only if for any affine open cover* $U_i \to Y$, $f^{-1}(U_i) \to U_i$ *is proper). Note that the "only if" direction follows from (a) — consider base change by* $U_i \hookrightarrow Y$.
(c) *The notion of "proper morphism" is closed under composition.*
(d) *The product of two proper morphisms is proper (i.e. if* $f : X \to Y$ *and* $g : X' \to Y'$ *are proper, where all morphisms are morphisms of* Z*-schemes) then* $f \times g : X \times_Z X' \to Y \times_Z Y'$ *is proper.*
(e) *Suppose*

(1)



is a commutative diagram, and g *is proper, and* h *is separated. Then* f *is proper.*
(f) *(I don't know if this is useful, but I may as well say it anyway.) Suppose* (1) *is a commutative diagram, and* f *is surjective,* g *is proper, and* h *is separated and finite type. Then* h *is proper.*

Proof. (a) We have already shown that the notions of separatedness and finite type are local on the target. The notion of closedness is local on the target, and hence so is the notion of universal closedness.

(b) The notions of separatedness, finite type, and universal closedness are all preserved by fiber product. (Notice that this is why universal closedness is better than closedness — it is automatically preserved by base change!)

(c) The notions of separatedness, finite type, and universal closedness are all preserved by composition.

(d) Both $X \times_Z Y \to X' \times_Z Y$ and $X' \times_Z Y \to X' \times_Z Y'$ are proper, because the notion is preserved by base change (part (b)). Then their composition is also proper (part (c)).

(e) Closed immersions are proper, so we invoke our magic and weird "property P fact" from last day.

(f) *Exercise.* □

We come to the hardest thing I will prove today.

1.6. *Theorem. — Projective morphisms are proper.*

It is not easy to come up with an example of a morphism that is proper but not projective! I'll give an simple example before long of a proper but not projective surface (over a field), once we have the notion of the fact that line bundles on nice families of curves have constant degree. Once we discuss blow-ups, I'll give Hironaka's example of a proper but not projective *nonsingular* threefold over $\mathbb{C}$.

I'll give part of the proof today, and the rest next day (because I thought I had a simplification that I realized this morning didn't work out).

Proof. Suppose $f : X \to Y$ is projective. Because the notion of properness is local on the base, we may assume that Y is affine, say $\operatorname{Spec} A$. Then $X \hookrightarrow \mathbb{P}^n_A$ for some n. As closed immersions are proper (§1.2), and the composition of two proper morphisms is proper, it suffices to prove that $\mathbb{P}^n_A \to \operatorname{Spec} A$ is proper. However, we have shown that projective morphisms are separated (last day), and finite type, so it suffices to show that $\mathbb{P}^n_A \to \operatorname{Spec} A$ is universally closed.

We will next show that it suffices to show that $\mathbb{P}^n_R \to \operatorname{Spec} R$ is closed for all rings R. Indeed, we need to show that given any base change $X \to \operatorname{Spec} A$, the resulting base changed morphisms $\mathbb{P}^n_X \to X$ is closed. But the notion of being "closed" is local on the base, so we can replace X by an affine cover.

Next day I will complete the proof by showing that $\mathbb{P}^n_A \to \operatorname{Spec} A$ is closed. This is sometimes called the fundamental theorem of elimination theory. Here are some examples to show you that this is a bit subtle.

First, let $A = k[a, b, c, \dots, i]$, and consider the closed subscheme of $\mathbb{P}^2_A$ (taken with coordinates x, y, z) corresponding to $ax + by + cz = 0$, $dx + ey + fz = 0$, $gx + hy + iz = 0$. Then we are looking for the locus in $\operatorname{Spec} A$ where these equations have a non-trivial solution. This indeed corresponds to a Zariski-closed set — where

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 0.$$

As a second example, let $A = k[a_0, a_1, \dots, a_m, b_0, b_1, \dots, b_n]$. Now consider the closed subscheme of $\mathbb{P}^1_A$ (taken with coordinates x and y) corresponding to $a_0 x^m + a_1 x^{m-1} y + \cdots + a_m y^m = 0$ and $b_0 x^n + b_1 x^{m-1} y + \cdots + b_n y^n = 0$. Then we are looking at the locus in $\operatorname{Spec} A$ where these two polynomials have a common root — this is known as the *resultant*. □

I'll end my discussion of properness with some results that I'll not prove and not use.

1.7. Miscellaneous facts.

Here are some enlightening facts.

(a) Proper and affine = finite. (b) Proper and quasifinite = finite.

(We'll show all three of this in the case of projective morphisms.)

As an application: quasifinite morphisms from proper schemes to separated schemes are finite. Here is why: suppose $X \to Y$ is a quasifinite morphism over Z, where X is proper over Z. Then by one of our weird "property P" facts (Proposition 1.24(b) in class 25), $X \to Y$ is proper. Hence by (b) above, it is finite.

Here is an explicit example: consider a morphism $\mathbb{P}^1 \to \mathbb{P}^1$ given by two distinct sections of $\mathcal{O}_{\mathbb{P}^1}(2)$. The fibers are finite, hence this is a finite morphism. (This could also be checked directly.)

Here is a third fact: If $\pi : X \to Y$ is proper, and $\mathcal{F}$ is a coherent sheaf on X, then $\pi_* \mathcal{F}$ is coherent.

In particular, if X is proper over k, $H^0(X, \mathcal{F})$ is finite-dimensional. (This is just the special case of the morphism $X \to k$.)

1.8. Valuative criterion.

There is a valuative criterion for properness too. I've never used it personally, but it *is* useful, both directly, and also philosophically. I'll make statements, and then discuss some philosophy.

1.9. *Theorem (Valuative criterion for properness for morphisms of finite type of Noetherian schemes).* — *Suppose* $f : X \to Y$ *is a morphism of finite type of locally Noetherian schemes. Then* f *is proper if and only if the following condition holds. For any discrete valuation ring* R *with function field* K*, and or any diagram of the form*

(2)
$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & X \\ \big\downarrow & & \big\downarrow f \\ \operatorname{Spec} R & \longrightarrow & Y \end{array}$$

(where the vertical morphism on the left corresponds to the inclusion $R \hookrightarrow K$*), there is* exactly *one morphism* $\operatorname{Spec} R \to X$ *such that the diagram*

(3)
$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & X \\ \big\downarrow & \nearrow & \big\downarrow f \\ \operatorname{Spec} R & \longrightarrow & Y \end{array}$$

commutes.

Recall that the valuative criterion for properness was the same, except that *exact* was replaced by *at most*.

In the case where Y is a field, you can think of this as saying that limits of one-parameters always exist, and are unique.

1.10. *Theorem (Valuative criterion of properness).* — *Suppose* $f : X \to Y$ *is a quasiseparated, finite type (hence quasicompact) morphism. Then* f *is proper if and only if the following condition* *holds. For any valuation ring* R *with function field* K*, and or any diagram of the form* (2)*, there is* exactly *one morphism* $\operatorname{Spec} \mathsf{R} \to \mathsf{X}$ *such that the diagram* (3) *commutes.*

Uses: (1) intuition. (2) moduli idea: exactly one way to fill it in (stable curves). (3) motivates the definition of properness for stacks.

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