FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 21

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CONTENTS

Today: integral extensions, Going-up theorem, Noether Normalization, proof that transcendence degree = Krull dimension, proof of Chevalley's theorem. Invertible sheaves and morphisms to (quasi)projective schemes

Welcome back everyone! This is the second quarter in a three-quarter experimental sequence on algebraic geometry.

We know what schemes are, their properties, quasicoherent sheaves on them, and morphisms between them. This quarter, we're going to talk about fancier concepts: fibered products; normalization; separatedness and the definition of a variety; rational maps; classification of curves; cohomology; differentials; and Riemann-Roch.

I'd like to start with some notions that I now think I should have done in the middle of last quarter. They are some notions that I think are easier than are usually presented.

1. Integral extensions, the going-up theorem, Noether normalization, and a proof of the big dimension theorem (that transcendence degree = Krull dimension)

Recall the maps of sets corresponding to a map of rings. If we have $\phi : B \to A$, we get a map $\operatorname{Spec} A \to \operatorname{Spec} B$ as sets (and indeed as topological spaces, and schemes), which sends $\mathfrak{p} \subset A$ to $\phi^{-1}\mathfrak{p} \subset B$. The notion behaves well under quotients and localization of both the source and target affine scheme.

A ring homomorphism $\phi : B \to A$ is *integral* if every element of A is integral over $\phi(B)$. (Thanks to Justin for pointing out that this notation is not just my invention — it is in Atiyah-Macdonald, p. 60.) In other words, if a is any element of A, then a satisfies some

Date: Tuesday, January 10, 2006. Updated January 28, 2007.

monic polynomial $a^n + \cdots = 0$ where all the coefficients lie in $\phi(B)$. We call it an *integral extension* if ϕ is an inclusion of rings.

1.1. Exercise. The notion of integral morphism is well behaved with respect to localization and quotient of B, and quotient of A (but not localization of A, witness $k[t] \to k[t]$, but $k[t] \to k[t]_{(t)}$). The notion of integral extension is well behaved with respect to localization and quotient of B, but not quotient of A (same example, $k[t] \to k[t]/(t)$).

1.2. Exercise. Show that if B is an integral extension of A, and C is an integral extension of B, then C is an integral extension of A.

1.3. *Proposition. — If A is finitely generated as a B-module, then ϕ is an integral morphism.*

Proof. (If B is Noetherian, this is easiest: suppose $a \in B$. Then A is a Noetherian B-module, and hence the ascending chain of B-submodules of A $(1) \subset (1, a) \subset (1, a, a^2) \subset (1, a, a^2, a^3) \subset \cdots$ eventually stabilizes, say $(1, a, \dots, a^{n-1}) = (1, a, \dots, a^{n-1}, a^n)$. Hence a^n is a B-linear combination of $1, \dots, a^{n-1}$, i.e. is integral over B. So Noetherian-minded readers can stop reading.) We use a trick we've seen before. Choose a finite generating set $m_1, \dots, m_n$ of A as a B-module. Then $am_i = \sum a_{ij} m_j$, where $a_{ij} \in B$. Thus

$$(aI_{n\times n} - [a_{ij}]_{ij}) \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Multiplying this equation by the adjoint of the left side, we get

$$\det(aI_{n\times n} - [a_{ij}]_{ij}) \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

(We saw this trick when discussing Nakayama's lemma.) So $\det(aI - M)$ annihilates A, i.e. $\det(aI - M) = 0$. □

1.4. Exercise (cf. Exercise 1.2). Show that if B is a finite extension of A, and C is a finite extension of B, then C is an finite extension of A. (Recall that if we have a ring homomorphism $A \to B$ such that B is a finitely-generated A-module (*not* necessarily A-algebra) then we say that B is a finite extension of A.)

We now recall the Going-up theorem.

1.5. *Cohen-Seidenberg Going up theorem. — Suppose $\phi : B \to A$ is an integral extension. Then for any prime ideal $\mathfrak{q} \subset B$, there is a prime ideal $\mathfrak{p} \subset A$ such that $\mathfrak{p} \cap B = \mathfrak{q}$.*

Although this is a theorem in algebra, the name reflects its geometric motivation: the theorem asserts that the corresponding morphism of schemes is surjective, and that "above" every prime $\mathfrak{q}$ "downstairs", there is a prime $\mathfrak{q}$ "upstairs". (I drew a picture here.) For this reason, it is often said that $\mathfrak{q}$ is "above" $\mathfrak{p}$ if $\mathfrak{p} \cap B = \mathfrak{q}$. (Joe points out that my speculation on the origin of the name "going up" is wrong.)

As a reality check: note that the morphism $k[t] \to k[t]_{(t)}$ is not integral, so the conclusion of the Going-up theorem 1.5 fails. (I drew a picture again.)

Proof of the Cohen-Seidenberg Going-Up theorem 1.5. This proof is eminently readable, but could be skipped on first reading. We start with an exercise.

1.6. Exercise. Show that the special case where A is a field translates to: if $B \subset A$ is a subring with A integral over B, then B is a field. Prove this. (Hint: all you need to do is show that all nonzero elements in B have inverses in B. Here is the start: If $b \in B$, then $1/b \in A$, and this satisfies some integral equation over B.)

We're ready to prove the Going-Up Theorem 1.5.

We first make a reduction: by localizing at $\mathfrak{q}$, so we can assume that $(B, \mathfrak{q})$ is a local ring.

Then let $\mathfrak{p}$ be any *maximal* ideal of A. We will see that $\mathfrak{p} \cap B = \mathfrak{q}$. Consider the following diagram.

$$\begin{tikzcd} A \arrow[r, two heads] & A/\mathfrak{p} & \text{field} \\ B \arrow[u, hook] \arrow[r, two heads] & B/(B \cap \mathfrak{p}) \arrow[u, hook] \end{tikzcd}$$

By the Exercise above, the lower right is a field too, so $B \cap \mathfrak{p}$ is a maximal ideal, hence $\mathfrak{q}$. ∎

1.7. Important but straightforward exercise (sometimes also called the going-up theorem). Show that if $\mathfrak{q}_1 \subset \mathfrak{q}_2 \subset \cdots \subset \mathfrak{q}_n$ is a chain of prime ideals of B, and $\mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_m$ is a chain of prime ideals of A such that $\mathfrak{p}_i$ "lies over" $\mathfrak{q}_i$ (and $m < n$), then the second chain can be extended to $\mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n$ so that this remains true.

The going-up theorem has an important consequence.

1.8. Important exercise. Show that if $f : \operatorname{Spec} A \to \operatorname{Spec} B$ corresponds to an integral *extension of rings*, then $\dim \operatorname{Spec} A = \dim \operatorname{Spec} B$.

I'd like to walk you through much of this exercise. You can show that a chain downstairs gives a chain upstairs, by the going up theorem, of the same length. Conversely, a chain upstairs gives a chain downstairs. We need to check that no two elements of the chain upstairs goes to the same element of the chain downstairs. That boils down to this: If $\phi : k \to A$ is an integral extension, then $\dim A = 0$. *Proof.* Suppose $\mathfrak{p} \subset \mathfrak{m}$ are two prime ideals of $\mathfrak{p}$. Mod out by $\mathfrak{p}$, so we can assume that A is a domain. I claim that any non-zero element is invertible. Here's why. Say $x \in A$, and $x \neq 0$. Then the minimal monic polynomial for x has non-zero constant term. But then x is invertible (recall coefficients are in a field).

We now introduce another important and ancient result, Noether's Normalization Lemma.

1.9. *Noether Normalization Lemma. — Suppose A is an integral domain, finitely generated over a field k. If $\operatorname{tr.deg.}_k A = n$, then there are elements $x_1, \ldots, x_n \in A$, algebraically independent over k, such that A is a finite (hence integral by Proposition 1.3) extension of $k[x_1, \ldots, x_n]$.*

The geometric content behind this result is that given any integral affine k-scheme X, we can find a surjective finite morphism $X \to \mathbb{A}_k^n$, where n is the transcendence degree of the function field of X (over k).

Proof of Noether normalization. We give Nagata's proof, following Mumford's Red Book (§1.1). Suppose we can write $A = k[y_1, \ldots, y_m]/\mathfrak{p}$, i.e. that A can be chosen to have m generators. Note that $m \geq n$. We show the result by induction on m. The base case $m = n$ is immediate.

Assume now that $m > n$, and that we have proved the result for smaller m. We will find $m-1$ elements $z_1, \ldots, z_{m-1}$ of A such that A is finite over $A' := k[z_1, \ldots, z_{m-1}]$ (by which we mean the subring of A generated by $z_1, \ldots, z_{m-1}$). Then by the inductive hypothesis, A' is finite over some $k[x_1, \ldots, x_n]$, and A is finite over A, so by Exercise 1.4 A is finite over $k[x_1, \ldots, x_n]$.

As $y_1, \ldots, y_m$ are algebraically dependent, there is some non-zero algebraic relation $f(y_1, \ldots, y_m) = 0$ among them (where f is a polynomial in m variables).

Let $z_1 = y_1 - y_m^{r_1}$, $z_2 = y_2 - y_m^{r_2}$, $\ldots$, $z_{m-1} = y_{m-1} - y_m^{r_{m-1}}$, where $r_1, \ldots, r_{m-1}$ are positive integers to be chosen shortly. Then

$$f(z_1 + y_m^{r_1}, z_2 + y_m^{r_2}, \ldots, z_{m-1} + y_m^{r_{m-1}}, y_m) = 0.$$

Then upon expanding this out, each monomial in f (as a polynomial in m variables) will yield a single term in that is a constant times a power of y_m (with no z_i factors). By choosing the r_i so that $0 \ll r_1 \ll r_2 \ll \cdots \ll r_{m-1}$, we can ensure that the powers of y_m appearing are all distinct, and so that in particular there is a leading term y_m^N, and all other terms (including those with z_i-factors) are of smaller degree in y_m. Thus we have described an integral dependence of y_m on $z_1, \ldots, z_{m-1}$ as desired. $\square$

Now we can give a proof of something we used a lot last quarter:

1.10. *Important Theorem about Dimension. — Suppose R is a finitely-generated domain over a field k. Then $\dim \operatorname{Spec} R$ is the transcendence degree of the fraction field $\operatorname{Frac}(R)$ over k.*

We proved this in class 9, but I think this proof is much slicker.

Proof. Suppose X is an integral affine k-scheme. We show that $\dim X$ equals the transcendence degree n of its function field, by induction on n. Fix X, and assume the result is known for all transcendence degrees less than n. The base case $n = -1$ is vacuous.

By Exercise 1.8, $\dim X = \dim \mathbb{A}^n_k$. If $n = 0$, we are done.

We now show that $\dim \mathbb{A}^n_k = n$ for $n > 0$. Clearly $\dim \mathbb{A}^n_k \geq n$, as we can describe a chain of irreducible subsets of length $n + 1$: if $x_1, \ldots, x_n$ are coordinates on $\mathbb{A}^n$, consider the chain of ideals

$$(0) \subset (x_1) \subset \cdots \subset (x_1, \ldots, x_n)$$

in $k[x_1, \ldots, x_n]$. Suppose we have a chain of prime ideals of length at least n:

$$(0) = \mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_m.$$

where $\mathfrak{p}_1$ is a height 1 prime ideal. Then $\mathfrak{p}_1$ is principal (as $k[x_1, \ldots, x_n]$ is a unique factorization domain, cf. Exercises 1 and 4 on problem set 6); say $\mathfrak{p}_1 = (f(x_1, \ldots, x_n))$, where f is an irreducible polynomial. Then $k[x_1, \ldots, x_n]/(f(x_1, \ldots, x_n))$ has transcendence degree $n - 1$, so by induction,

$$\dim k[x_1, \ldots, x_n]/(f) = n - 1.$$

□

2. Images of morphisms

Here are two applications of the going-up theorem, which are quite similar to each other.

2.1. *Exercise.* Show that finite morphisms are *closed*, i.e. the image of any closed subset is closed.

2.2. *Exercise.* Show that integral ring extensions induce a surjective map of spectra.

I now want to use the Noether normalization lemma to prove Chevalley's theorem. Recall that we define a *constructable subset* of a scheme to be a subset which belongs to the smallest family of subsets such that (i) every open set is in the family, (ii) a finite intersection of family members is in the family, and (iii) the complement of a family member is also in the family. So for example the image of $(x, y) \mapsto (x, xy)$ is constructable.

2.3. Exercise. Suppose X is a Noetherian scheme. Show that a subset of X is constructable if and only if it is the finite disjoint union of locally closed subsets.

Last quarter we stated the following.

2.4. *Chevalley's Theorem. — Suppose $f : X \to Y$ is a morphism of finite type of Noetherian schemes. Then the image of any constructable set is constructable.*

We'll now prove this using Noether normalization. (This is remarkable: Noether normalization is about finitely generated algebras over a field. There is no field in the statement of Chevalley's theorem. Hence if you prefer to work over arbitrary rings (or schemes), this shows that you still care about facts about finite type schemes over a field. Also, even if you are interested in finite type schemes over a given field (like $\mathbb{C}$), the field that comes up in the proof of Chevalley's theorem is *not* that field, so even if you prefer to work over $\mathbb{C}$, this argument shows that you still care about working over arbitrary fields, not necessarily algebraically closed.)

We say a morphism $f : X \to Y$ is *dominant* if the image of f meets every dense open subset of Y. (This is sometimes called *dominating*, but we will not use this notation.)

2.5. Exercise. Show that a dominant morphism of integral schemes $X \to Y$ induces an inclusion of function fields in the other direction.

2.6. *Exercise.* If $\phi : A \to B$ is a ring morphism, show that the corresponding morphism of affine schemes $\operatorname{Spec} B \to \operatorname{Spec} A$ is dominant iff ϕ has nilpotent kernel.

2.7. Exercise. Reduce the proof of the Going-up theorem to the following case: suppose $f : X = \operatorname{Spec} A \to Y = \operatorname{Spec} B$ is a dominant morphism, where A and B are domains, and f corresponds to $\phi : B \to B[x_1, \dots, x_n]/I \cong A$. Show that the image of f contains a dense open subset of $\operatorname{Spec} B$.

Proof. We prove the problem posed in the previous exercise. This argument uses Noether normalization 1.9 in an interesting context — even if we are interested in schemes over a field k, this argument will use a larger field, the field $K := \operatorname{Frac}(B)$. Now $A \otimes_B K$ is a localization of A with respect to B^*, so it is a domain, and it is finitely generated over K (by $x_1, \dots, x_n$), so it has finite transcendence degree r over K. Thus by Noether normalization, we can find a subring $K[y_1, \dots, y_r] \subset A \otimes_B K$, so that $A \otimes_B K$ is integrally dependent on $K[y_1, \dots, y_r]$. We can choose the y_i to be in A: each is in $(B^*)^{-1}A$ to begin with, so we can replace each y_i by a suitable K-multiple.

Sadly A is not necessarily integrally dependent on $K[y_1, \dots, y_r]$ (as this would imply that $\operatorname{Spec} A \to \operatorname{Spec} B$ is surjective). However, each x_i satisfies some integral equation

$$x_i^n + f_1(y_1, \dots, y_r)x_i^{n-1} + \cdots + f_n(y_1, \dots, y_r) = 0$$

where f_j are polynomials with coefficients in $K = \operatorname{Frac}(B)$. Let g be the product of the denominators of all the coefficients of all these polynomials (a finite set). Then A_g is integral over B_g, and hence $\operatorname{Spec} A_g \to \operatorname{Spec} B_g$ is surjective; $\operatorname{Spec} B_g$ is our open subset. $\square$

3. Important example: Morphisms to projective (and quasiprojective) schemes, and invertible sheaves

This will tell us why invertible sheaves are crucially important: they tell us about maps to projective space, or more generally to quasiprojective schemes. (And given that we have had a hard time naming any non-quasiprojective schemes, they tell us about maps to essentially all schemes that are interesting to us.)

3.1. *Important theorem. — Maps to $\mathbb{P}^n$ correspond to $n+1$ sections of a line bundle, not all vanishing at any point (= generated by global sections, by an earlier exercise, Class 16 Exercise 4.2, = Problem Set 7, Exercise 28), modulo sections of $\mathcal{O}_X^*$.*

The explanation and proof of the correspondence is in the notes for next day.

Here are some examples.

Example 1. Consider the $n+1$ functions $x_0, \dots, x_n$ on $\mathbb{A}^{n+1}$ (otherwise known as $n+1$ sections of the trivial bundle). They have no common zeros on $\mathbb{A}^n - 0$. Hence they determine a morphism $\mathbb{A}^{n+1} - 0 \to \mathbb{P}^n$. (We've talked about this morphism before. But now we don't have to worry about gluing.)

Example 2: the Veronese morphism. Consider the line bundle $\mathcal{O}_{\mathbb{P}^n}(m)$ on $\mathbb{P}^n$. We've checked that the number of sections of this line bundle are $\binom{n+m}{m}$, and they correspond to homogeneous degree m polynomials in the projective coordinates for $\mathbb{P}^n$. Also, they have no common zeros (as for example the subset of sections $x_0^m, x_1^m, \dots, x_n^m$ have no common zeros). Thus these determine a morphism $\mathbb{P}^n \to \mathbb{P}^{\binom{n+m}{m}-1}$. This is called the *Veronese morphism*. For example, if $n = 2$ and $m = 2$, we get a map $\mathbb{P}^2 \to \mathbb{P}^5$.

This is in fact a closed immersion. Reason: This map corresponds to a surjective map of graded rings. The first ring R_1 has one generator for each of degree m monomial in the x_i. The second ring is not $k[x_0, \dots, x_n]$, as R_1 does not surject onto it. Instead, we take $R_2 = k[x_0, \dots, x_n]_{(m)}$, i.e. we consider only those polynomials all of whose terms have degree divisible by m. Then the natural map $R_1 \to R_2$ is fairly clearly a surjection. Thus the corresponding map of projective schemes is a closed immersion by an earlier exercise.

How can you tell in general if something is a closed immersion, and not just a map? Here is one way.

3.2. *Exercise.* Let $f : X \to \mathbb{P}^n_A$ be a morphism of A-schemes, corresponding to an invertible sheaf $\mathcal{L}$ on X and sections $s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$ as above. Then ϕ is a closed immersion iff (1) each open set $X_i = X_{s_i}$ is affine, and (2) for each i, the map of rings $A[y_0, \dots, y_n] \to \Gamma(X_i, \mathcal{O}_{X_i})$ given by $y_j \mapsto s_j/s_i$ is surjective.

We'll give another method of detecting closed immersions later. The intuition for this will come from differential geometry: the morphism should separate points, and also separate tangent vectors.

Example 3. The rational normal curve. The image of the Veronese morphism when $n = 1$ is called a *rational normal curve of degree* m. Our map is $\mathbb{P}^1 \to \mathbb{P}^m$ given by $[x; y] \to [x^m; x^{m-1}y; \cdots ; xy^{m-1}; y^m]$. When $m = 3$, we get our old friend the *twisted cubic*. When $m = 2$, we get a smooth conic. What happens when $m = 1$?

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