FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 19

CONTENTS

Last day: Associated points; more on normality; invertible sheaves and divisors take 1.

Today: Maps to affine schemes; surjective, open immersion, closed immersion, quasicompact, locally of finite type, finite type, affine morphism, finite, quasifinite. Images of morphisms: constructible sets, and Chevalley's theorem (finite type morphism of Noetherian schemes sends constructibles to constructibles).

Last day, I defined a morphism of schemes $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ as follows.

I first defined the notion of a morphism of ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, which is a continuous map of topological spaces $f : X \to Y$ along with a map of sheaves of rings (on Y) $\mathcal{O}_Y \to f_*\mathcal{O}_X$, or equivalently (by adjointness of inverse image and pushfoward) $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ (a map of sheaves of rings on X). This should be seen as a description of how to pull back functions on Y to get functions on X.

An example is a morphism of affine schemes $\operatorname{Spec} A \to \operatorname{Spec} B$. These correspond to morphisms of rings $B \to A$.

Then a morphism of schemes $X \to Y$ can be defined as a morphism of these ringed spaces, that locally looks like a morphism of affine schemes. In other words, X can be covered by affine open sets, such that for each such $\operatorname{Spec} R$, there is an affine open set $\operatorname{Spec} S$ of Y containing its image, such that the map $\operatorname{Spec} R \to \operatorname{Spec} S$ is of the form described in the primordial example.

We proved this by temporarily introducing a new concept, that of a *locally ringed space*. Then a morphism of schemes $X \to Y$ is just the same as a morphism of locally ringed spaces; we showed this by showing this for affine schemes.

I encouraged you to get practice with this in the following exercise, to make sense of the map $\mathbb{A}^{n+1} - 0 \to \mathbb{P}^n$ "given by" $(x_0, \dots, x_n) \mapsto [x_0; \dots; x_n]$.

We thus have described the *category of schemes*. The notion of an isomorphism of schemes subsumes our earlier definition.

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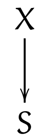
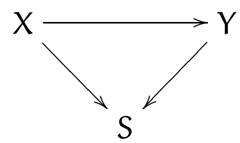
I described the category of k-schemes, or more generally A-schemes where A is a ring. More generally, if S is a scheme, we have the category of S-schemes. The objects are diagrams of the form



and morphisms are commutative diagrams of the form



The category of k-schemes corresponds to the case $S = \operatorname{Spec} k$, and the category of A-schemes correspond to the case $S = \operatorname{Spec} A$.

We now give some examples.

0.1. *Exercise.* Show that morphisms $X \to \operatorname{Spec} A$ are in natural bijection with ring morphisms $A \to \Gamma(X, \mathcal{O}_X)$. (Hint: Show that this is true when X is affine. Use the fact that morphisms glue.)

In particular, there is a canonical morphism from a scheme to Spec of its space of global sections. (Warning: Even if X is a finite-type k-scheme, the ring of global sections might be nasty! In particular, it might not be finitely generated.)

Example: Suppose S_* is a graded ring, with $S_0 = A$. Then we get a natural morphism $\operatorname{Proj} S_* \to \operatorname{Spec} A$. For example, we have a natural map $\mathbb{P}^n_A \to \operatorname{Spec} A$

0.2. *Exercise.* Show that $\operatorname{Spec} \mathbb{Z}$ is the final object in the category of schemes. In other words, if X is any scheme, there exists a unique morphism to $\operatorname{Spec} \mathbb{Z}$. (Hence the category of schemes is isomorphic to the category of $\mathbb{Z}$-schemes.)

0.3. *Exercise.* Show that morphisms $X \to \operatorname{Spec} \mathbb{Z}[t]$ correspond to global sections of the structure sheaf.

This is one of our first explicit examples of an important idea, that of representable functors! This is a very useful idea, whose utility isn't immediately apparent. We have a contravariant functor from schemes to sets, taking a scheme to its set of global sections. We have another contravariant functor from schemes to sets, taking X to $\operatorname{Hom}(X, \operatorname{Spec} \mathbb{Z}[t])$. This is describing an "isomorphism" of the functors. More precisely, we are describing an isomorphism $\Gamma(X, \mathcal{O}_X) \cong \operatorname{Hom}(X, \operatorname{Spec} \mathbb{Z}[t])$ that behaves well with respect to morphisms of schemes: given any morphism $f : X \to Y$, the diagram

$$\begin{array}{ccc} \Gamma(Y, \mathcal{O}_Y) & \xrightarrow{\sim} & \operatorname{Hom}(Y, \operatorname{Spec} \mathbb{Z}[t]) \\ \downarrow f^* & & \downarrow f\circ \\ \Gamma(X, \mathcal{O}_X) & \xrightarrow{\sim} & \operatorname{Hom}(X, \operatorname{Spec} \mathbb{Z}[t]) \end{array}$$

commutes. *Given a contravariant functor from schemes to sets, by Yoneda's lemma, there is only one possible scheme* Y*, up to isomorphism, such that there is a natural isomorphism between this functor and* $\operatorname{Hom}(\cdot, Y)$. If there is such a Y, we say that the functor is *representable*.

Here are a couple of examples of something you've seen to put it in context. (i) The contravariant functor $\operatorname{Hom}(\cdot, Y)$ (i.e. $X \mapsto \operatorname{Hom}(X, Y)$) is representable by Y. (ii) Suppose we have morphisms $X, Y \to Z$. The contravariant functor $\operatorname{Hom}(\cdot, X) \times_{\operatorname{Hom}(\cdot, Z)} \operatorname{Hom}(\cdot, Y)$ is representable if and only if the fibered product $X \times_Z Y$ exists (and indeed then the contravariant functor is represented by $\operatorname{Hom}(\cdot, X \times_Z Y)$). This is purely a translation of the definition of the fibered product — you should verify this yourself.

Remark for experts: The global sections form something better than a set — they form a scheme. You can define the notion of ring scheme, and show that if a contravariant functor from schemes to rings is representable (as a contravariant functor from schemes to sets) by a scheme Y, then Y is guaranteed to be a ring scheme. The same is true for group schemes.

0.4. *Related Exercise.* Show that global sections of $\mathcal{O}_X^*$ correspond naturally to maps to $\operatorname{Spec} \mathbb{Z}[t, t^{-1}]$. ($\operatorname{Spec} \mathbb{Z}[t, t^{-1}]$ is a *group scheme*. We will discuss group schemes more in class 36.)

Morphisms and tangent spaces. Suppose $f : X \to Y$, and $f(p) = q$. Then if we were in the category of manifolds, we would expect a tangent map, from the tangent space of p to the tangent space at q. Indeed that is the case; we have a map of stalks $\mathcal{O}_{Y,q} \to \mathcal{O}_{X,p}$, which sends the maximal ideal of the former $\mathfrak{n}$ to the maximal ideal of the latter $\mathfrak{m}$ (we have checked that this is a "local morphism" when we briefly discussed locally ringed spaces). Thus $\mathfrak{n}^2 \to \mathfrak{m}^2$, from which $\mathfrak{n}/\mathfrak{n}^2 \to \mathfrak{m}/\mathfrak{m}^2$, from which we have a natural map $(\mathfrak{m}/\mathfrak{m}^2)^\vee \to (\mathfrak{n}/\mathfrak{n}^2)^\vee$. This is the map from the tangent space of p to the tangent space at q that we sought.

0.5. *Exercise.* Suppose X is a finite type k-scheme. Describe a natural bijection $\operatorname{Hom}(\operatorname{Spec} k[\epsilon]/\epsilon^2, X)$ to the data of a k-valued point (a point whose residue field is k, necessarily closed) and a tangent vector at that point.

1. Properties of morphisms of schemes

I'm going to define a lot of useful notions.

The notion of **surjective** will have the same meaning as always: $X \to Y$ is surjective if the map of sets is surjective.

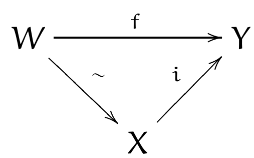
1.1. *Unimportant Exercise.* Show that integral ring extensions induces a surjective map of spectra. (Hint: Recall the Cohen-Seidenberg Going-up Theorem: Suppose $B \subset A$ is an inclusion of rings, with A integrally dependent on B. For any prime $\mathfrak{q} \subset B$, there is a prime $\mathfrak{p} \subset A$ such that $\mathfrak{p} \cap B = \mathfrak{q}$.)

Definition. If U is an open subscheme of Y, then there is a natural morphism $U \to Y$. We say that $f : X \to Y$ is an *open immersion* if f gives an isomorphism from X to an open subscheme of Y. (Really, we want to say that X "is" an open subscheme of Y.) Observe that if f is an open immersion, then $f^{-1}\mathcal{O}_Y \cong \mathcal{O}_X$.

1.2. *Exercise.* Suppose $i : U \to Z$ is an open immersion, and $f : Y \to Z$ is any morphism. Show that $U \times_Z Y$ exists. (Hint: I'll even tell you what it is: $(f^{-1}(U), \mathcal{O}_Y|_{f^{-1}(U)})$.)

1.3. *Easy exercise.* Show that open immersions are monomorphisms.

Suppose X is a closed subscheme of Y. Then there is a natural morphism $i : X \to Y$: on the affine open set $\operatorname{Spec} R$ of Y, where X is "cut out" by the ideal $I \subset R$, this corresponds to the ring map $R \to R/I$. A morphism $f : W \to Y$ is a **closed immersion** if it can be factored as



where $i : X \to Y$ is a closed subscheme. (Really, we want to say that W "is" a closed subscheme of Y.)

(Example: If X is a scheme and X^{red} is its reduction, then there is a natural closed immersion $X^{\text{red}} \to X$.)

1.4. *Proposition (the property of being a closed immersion is affine-local on the target). — Suppose* $f : X \to Y$ *is a morphism of schemes. It suffices to check that* f *is a closed immersion on an affine open cover of* Y.

Reason: The way in which closed subschemes are defined is local on the target.

(In particular, a morphism of affine schemes is a closed immersion if and only if it corresponds to a surjection of rings.)

1.5. Exercise. Suppose $Y \to Z$ is a closed immersion, and $X \to Z$ is any morphism. Show that the fibered product $X \times_Z X$ exists, by explicitly describing it. Show that the projection $X \times_Z Y \to Y$ is a closed immersion. We say that "closed immersions are preserved by base change" or "closed immersions are preserved by fibered product". (Base change is another word for fibered products.)

1.6. Less important exercise. Show that closed immersions are monomorphisms.

Definition. A morphism $X \to Y$ is a *locally closed immersion* if it factors into $X \xrightarrow{f} Z \xrightarrow{g} Y$ where f is a closed immersion and g is an open immersion. Example: $\operatorname{Spec} k[t, t^{-1}] \to \operatorname{Spec} k[x, y]$ where $x \mapsto t$, $y \mapsto 0$. (Unimportant fact: as the composition of monomorphisms are monomorphisms, so locally closed immersions are monomorphisms. Clearly open immersions and closed immersions are locally closed immersions.)

(Interesting question: is this the same as defining locally closed immersions as open immersions of closed immersions? In other words, can the roles of open and closed immersions in the definition be reversed?)

A morphism $f : X \to Y$ is **quasicompact** if for every open affine subset U of Y, $f^{-1}(U)$ is quasicompact.

1.7. *Exercise (quasicompactness is affine-local on the target).* Show that a morphism $f : X \to Y$ is quasicompact if there is cover of Y by open affine sets U_i such that $f^{-1}(U_i)$ is quasicompact. (Hint: easy application of the affine communication lemma!)

1.8. *Exercise.* Show that the composition of two quasicompact morphisms is quasicompact.

A morphism $f : X \to Y$ is **locally of finite type** if for every affine open set $\operatorname{Spec} B$ of Y, $f^{-1}(\operatorname{Spec} B)$ can be covered with open sets $\operatorname{Spec} A_i$ so that the induced morphism $B \to A_i$ expresses A_i as a finitely generated B-algebra.

A morphism is **of finite type** if it is locally of finite type and quasicompact. Translation: for every affine open set $\operatorname{Spec} B$ of Y, $f^{-1}(\operatorname{Spec} B)$ can be covered with *a finite number of* open sets $\operatorname{Spec} A_i$ so that the induced morphism $B \to A_i$ expresses A_i as a finitely generated B-algebra.

1.9. *Exercise (the notions "locally of finite type" and "finite type" are affine-local on the target).* Show that a morphism $f : X \to Y$ is locally of finite type if there is a cover of Y by open affine sets $\operatorname{Spec} R_i$ such that $f^{-1}(\operatorname{Spec} R_i)$ is locally of finite type over R_i.

1.10. *Exercise.* Show that a morphism $f : X \to Y$ is locally of finite type if for *every* affine open subsets $\operatorname{Spec} A \subset X$, $\operatorname{Spec} B \subset Y$, with $f(\operatorname{Spec} A) \subset \operatorname{Spec} B$, A is a finitely generated B-algebra. (Hint: use the affine communication lemma on $f^{-1}(\operatorname{Spec} B)$.)

Example: the "structure morphism" $\mathbb{P}^n_A \to \operatorname{Spec} A$ is of finite type, as $\mathbb{P}^n_A$ is covered by $n+1$ open sets of the form $\operatorname{Spec} A[x_1, \dots, x_n]$. More generally, $\operatorname{Proj} S_* \to \operatorname{Spec} A$ (where $S_0 = A$) is of finite type.

More generally still: our earlier definition of schemes of "finite type over k" (or "finite type k-schemes") is now a special case of this more general notion: a scheme X is of finite type over k means that we are given a morphism $X \to \operatorname{Spec} k$ (the "structure morphism") that is of finite type.

Here are some properties enjoyed by morphisms of finite type.

1.11. Exercises. These exercises are important and not hard.

- Show that a closed immersion is a morphism of finite type.
- Show that an open immersion is locally of finite type. Show that an open immersion into a Noetherian scheme is of finite type. More generally, show that a quasicompact open immersion is of finite type.
- Show that a composition of two morphisms of finite type is of finite type.
- Suppose we have a composition of morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$, where f is quasicompact, and $g \circ f$ is finite type. Show that f is finite type.
- Suppose $f : X \to Y$ is finite type, and Y is Noetherian. Show that X is also Noetherian.

A morphism $f : X \to Y$ is **affine** if for every affine U of Y, $f^{-1}(U)$ is an affine scheme. Clearly affine morphisms are quasicompact. Also, clearly closed immersions are affine: if $X \to Y$ is a closed immersion, then the preimage of an affine open set $\operatorname{Spec} R$ of Y is (isomorphic to) some $\operatorname{Spec} R/I$, by the definition of closed subscheme.

1.12. *Proposition (the property of "affineness" is affine-local on the target).* A morphism $f : X \to Y$ is affine if there is a cover of Y by open affine sets U such that $f^{-1}(U)$ is affine.

Proof. As usual, we use the Affine Communication Theorem. We check our two criteria. First, suppose $f : X \to Y$ is affine over $\operatorname{Spec} S$, i.e. $f^{-1}(\operatorname{Spec} S) = \operatorname{Spec} R$. Then $f^{-1}(\operatorname{Spec} S_s) = \operatorname{Spec} R_{f\# s}$.

Second, suppose we are given $f : X \to \operatorname{Spec} S$ and $(f_1, \dots, f_n) = S$ with X_{f_i} affine ($\operatorname{Spec} R_i$, say). We wish to show that X is affine too. Define R as the kernel of S-modules

$$R_1 \times \cdots \times R_n \to R_{12} \times \cdots R_{(n-1)n}$$

where $X_{f_i f_j} = \operatorname{Spec} R_{ij}$. Then R is clearly an S-module, and has a ring structure. We define a morphism $\operatorname{Spec} R \to \operatorname{Spec} S$. Note that $R_{f_i} = R_i$. Then we define $\operatorname{Spec} R \to \operatorname{Spec} S$ via $\operatorname{Spec} R_i \to \operatorname{Spec} R_{f_i} \hookrightarrow \operatorname{Spec} S$. The morphisms glue. □

This has some non-obvious consequences, as shown in the next exercise.

1.13. *Exercise.* Suppose X is an affine scheme, and Y is a closed subscheme locally cut out by one equation (e.g. if Y is an effective Cartier divisor). Show that $X - Y$ is affine. (This is clear if Y is globally cut out by one equation f; then if $X = \operatorname{Spec} R$ then $Y = \operatorname{Spec} R_f$. However, Y is not always of this form.)

1.14. Example. Here is an explicit consequence. We showed earlier that on the cone over the smooth quadric surface $\operatorname{Spec} k[w, x, y, z]/(wz - xy)$, the cone over a ruling $w = x = 0$ is not cut out scheme-theoretically by a single equation, by considering Zariski-tangent spaces. We now show that it isn't even cut out set-theoretically by a single equation.

For if it were, its complement would be affine. But then the closed subscheme of the complement cut out by $y = z = 0$ would be affine. But this is the scheme $y = z = 0$ (also known as the wx-plane) minus the point $w = x = 0$, which we've seen is non-affine. (For comparison, on the cone $\operatorname{Spec} k[x, y, z]/(xy - z^2)$, the ruling $x = z = 0$ is not cut out scheme-theoretically by a single equation, but it *is* cut out set-theoretically by $x = 0$.) Verify all this!

We remark here that we have shown that if $f : X \to Y$ is an affine morphism, then $f_* \mathcal{O}_X$ is a quasicoherent sheaf of algebras (a quasicoherent sheaf with the structure of an algebra "over $\mathcal{O}_X$"). We'll soon reverse this process to obtain Spec of a quasicoherent sheaf of algebras.

A morphism $f : X \to Y$ is **finite** if for every affine $\operatorname{Spec} R$ of Y, $f^{-1}(\operatorname{Spec} R)$ is the spectrum of an R-algebra that is a finitely-generated R-*module*. Clearly finite morphisms are affine. Note that $f_* \mathcal{O}_X$ is a finite type quasicoherent sheaf of algebras (= coherent if X is locally Noetherian).

1.15. *Exercise (the property of finiteness is affine-local on the target).* Show that a morphism $f : X \to Y$ is finite if there is a cover of Y by open affine sets $\operatorname{Spec} R$ such that $f^{-1}(\operatorname{Spec} R)$ is the spectrum of a finite R-algebra.

(Hint: Use Exercise 1.12, and that $f_* \mathcal{O}_X$ is finite type.)

1.16. *Easy exercise.* Show that closed immersions are finite morphisms.

Degree of a finite morphism at a point. Suppose $f : X \to Y$ is a finite morphism. $f_* \mathcal{O}_X$ is a finite type (quasicoherent) sheaf on Y, and the rank of this sheaf at a point p is called the *degree* of the finite morphism at p. This is a upper semicontinuous function (we've shown that the rank of a finite type sheaf is uppersemicontinuous in an exercise when we discussed rank).

1.17. *Exercise.* Show that the rank at p is non-zero if and only if $f^{-1}(p)$ is non-empty.

1.18. *Exercise.* Show that finite morphisms are *closed*, i.e. the image of any closed subset is closed.

A morphism is **quasifinite** if it is of finite type, and for all $y \in Y$, the scheme $X_y = f^{-1}(y)$ is finite over y.

1.19. *Exercise.* (a) Show that if a morphism is finite then it is quasifinite. (b) Show that the converse is not true. (Hint: $\mathbb{A}^1 - \{0\} \to \mathbb{A}^1$.)

1.20. Images of morphisms. I want to go back to the point that the image of a finite morphism is closed. Something more general is true. We answer the question: what can the image of a morphism look like? We know it can be open (open immersion), and closed

(closed immersions), locally closed (locally closed immersions). But it can be weirder still: Consider $\mathbb{A}^2 \to \mathbb{A}^2$ given by $(x, y) \mapsto (x, xy)$. then the image is the plane, minus the y-axis, plus the origin. It can be stranger still, and indeed if S is *any* subset of a scheme Y, it can be the image of a morphism: let X be the disjoint union of spectra of the residue fields of all the points of S, and let $f : X \to Y$ be the natural map. This is quite pathological (e.g. likely horribly noncompact), and we will show that if we are in any reasonable situation, the image is essentially no worse than arose in the previous example.

We define a *constructible subset* of a scheme to be a subset which belongs to the smallest family of subsets such that (i) every open set is in the family, (ii) a finite intersection of family members is in the family, and (iii) the complement of a family member is also in the family. So for example the image of $(x, y) \mapsto (x, xy)$ is constructible.

Note that if $X \to Y$ is a morphism of schemes, then the preimage of a constructible set is a constructible set.

1.21. Exercise. Suppose X is a Noetherian scheme. Show that a subset of X is constructible if and only if it is the finite disjoint union of locally closed subsets.

Chevalley's Theorem. Suppose $f : X \to Y$ is a morphism of finite type of Noetherian schemes. Then the image of any constructible set is constructible.

I might give a proof in the notes eventually. See Atiyah-Macdonald, Exercise 7.25 for the key algebraic argument. Next quarter, we will see that in good situations (e.g. if the source is projective over k and the target is quasiprojective) then the image is closed.

We end with a useful fact about images of schemes that didn't naturally fit in anywhere in the previous exposition.

1.22. *Fast important exercise.* Show that the image of an irreducible scheme is irreducible.

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