FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 14

CONTENTS

Last day: discrete valuation rings (conclusion), cultural facts to know about regular local rings, the distinguished affine base of the topology, 2 definitions of quasicoherent sheaf.

Today: quasicoherence is affine-local, (locally) free sheaves and vector bundles, invertible sheaves and line bundles, torsion-free sheaves, quasicoherent sheaves of ideals and closed subschemes.

Last day, we defined the distinguished affine base of the Zariski topology of a scheme.

We showed that the information contained in a sheaf was precisely the information contained in a sheaf on the distinguished affine base.

0.1. *Theorem. —*

(a) *A sheaf on the distinguished affine base $\mathcal{F}^b$ determines a unique sheaf $\mathcal{F}$, which when restricted to the affine base is $\mathcal{F}^b$. (Hence if you start with a sheaf, and take the sheaf on the distinguished affine base, and then take the induced sheaf, you get the sheaf you started with.)*
(b) *A morphism of sheaves on an affine base determines a morphism of sheaves.*
(c) *A sheaf of $\mathcal{O}_X$-modules "on the distinguished affine base" yields an $\mathcal{O}_X$-module.*

We then gave two definitions of quasicoherent sheaves.

Definition 1. An $\mathcal{O}_X$-module $\mathcal{F}$ is a **quasicoherent sheaf** if for every affine open $\operatorname{Spec} R$ and distinguished affine open $\operatorname{Spec} R_f$ thereof, the restriction map $\phi : \Gamma(\operatorname{Spec} R, \mathcal{F}) \to$

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$\Gamma(\operatorname{Spec} R_f, \mathcal{F})$ factors as:

$$\phi : \Gamma(\operatorname{Spec} R, \mathcal{F}) \to \Gamma(\operatorname{Spec} R, \mathcal{F})_f \cong \Gamma(\operatorname{Spec} R_f, \mathcal{F}).$$

Definition 2. An $\mathcal{O}_X$-module $\mathcal{F}$ is a *quasicoherent sheaf* if for every affine open $\operatorname{Spec} R$,

$$\mathcal{F}|_{\operatorname{Spec} R} \cong \Gamma(\widetilde{\operatorname{Spec} R, \mathcal{F}}).$$

This isomorphism is as sheaves of $\mathcal{O}_X$-modules.

By part (c) of the above Theorem, an $\mathcal{O}_X$-module on the distinguished affine base yields an $\mathcal{O}_X$-module, so these two notions are equivalent. Thus to give a quasicoherent sheaf, I just need to give you a module for each affine open, and have them behave well with respect to restriction. (That's a priori a little weaker than definition 2, where we actually need an $\mathcal{O}_X$-module.)

Last time I proved:

0.2. *Proposition. — Definitions 1 and 2 are the same.*

1. ONWARDS!

1.1. *Proposition (quasicoherence is affine-local). — Let X be a scheme, and $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-modules. Then let P be the property of affine open sets that $\mathcal{F}|_{\operatorname{Spec} R} \cong \Gamma(\widetilde{\operatorname{Spec} R, \mathcal{F}})$. Then P is an affine-local property.*

Proof. By the Affine Communication Lemma , we must check two things. Clearly if $\operatorname{Spec} R$ has property P, then so does the distinguished open $\operatorname{Spec} R_f$: if M is an R-module, then $\tilde{M}|_{\operatorname{Spec} R_f} \cong \tilde{M}_f$ as sheaves of $\mathcal{O}_{\operatorname{Spec} R_f}$-modules (both sides agree on the level of distinguished opens and their restriction maps).

We next show the second hypothesis of the Affine Communication Lemma. Suppose we have modules $M_1, \ldots, M_n$, where M_i is an R_{f_i}-module, along with isomorphisms $\phi_{ij} : (M_i)_{f_j} \to (M_j)_{f_i}$ of $R_{f_i f_j}$-modules ($i \neq j$; where $\phi_{ij} = \phi_{ji}^{-1}$). We want to construct an M such that $\tilde{M}$ gives us $\tilde{M}_i$ on $D(f_i) = \operatorname{Spec} R_{f_i}$, or equivalently, isomorphisms $\Gamma(D(f_i), \tilde{M}) \cong M_i$, with restriction maps

$$\begin{array}{ccc} \Gamma(D(f_i), \tilde{M}) & & \Gamma(D(f_j), \tilde{M}) \\ \downarrow & & \downarrow \\ \Gamma(D(f_i), \tilde{M})_{f_j} & \stackrel{\cong}{\longleftrightarrow} & \Gamma(D(f_j), \tilde{M})_{f_i} \end{array}$$

that agree with ϕ_{ij}.

We already know what M should be. Consider elements of $M_1 \times \cdots \times M_n$ that "agree on overlaps"; let this set be M. Then

$$0 \to M \to M_1 \times \cdots \times M_n \to M_{12} \times M_{13} \times \cdots \times M_{(n-1)n}$$

is an exact sequence (where $M_{ij} = (M_i)_{f_j} \cong (M_j)_{f_i}$, and the latter morphism is the "difference" morphism). So M is a kernel of a morphism of R-modules, hence an R-module. We show that $M_i \cong M_{f_i}$; for convenience we assume $i = 1$. Localization is exact, so

$$(1) \quad 0 \to M_{f_1} \to M_1 \times (M_2)_{f_1} \times \cdots \times (M_n)_{f_1} \to M_{12} \times \cdots \times (M_{23})_{f_1} \times \cdots \times (M_{(n-1)n})_{f_1}$$

Then by interpreting this exact sequence, you can verify that the kernel is M_1. I gave one proof in class, and I'd like to give two proofs here. We know that $\cup_{i=2}^n D(f_1 f_i)$ is a distinguished cover of $D(f_1) = \operatorname{Spec} R_1$. So we have an exact sequence

$$0 \to M_1 \to (M_1)_{f_2} \times \cdots \times (M_1)_{f_n} \to (M_1)_{f_2 f_3} \times \cdots \times (M_1)_{f_{n-1} f_n}.$$

Put two copies on top of each other, and add vertical isomorphisms, alternating between identity and the negative of the identity:

$$\begin{array}{ccccccc}
0 & \longrightarrow & M_1 & \longrightarrow & (M_1)_{f_2} \times \cdots \times (M_1)_{f_n} & \longrightarrow & (M_1)_{f_2 f_3} \times \cdots \times (M_1)_{f_{n-1} f_n} \\
 & & \downarrow \mathrm{id} & & \downarrow -\mathrm{id} & & \\
0 & \longrightarrow & M_1 & \longrightarrow & (M_1)_{f_2} \times \cdots \times (M_1)_{f_n} & \longrightarrow & (M_1)_{f_2 f_3} \times \cdots \times (M_1)_{f_{n-1} f_n}
\end{array}$$

Then the *total complex* of this *double complex* is exact as well (**exercise**). (The total complex is obtained as follows. The terms are obtained by taking the direct sum in each southwest-to-northeast diagonal. This is a baby case of something essential so check it, if you've never seen it before!). But this is the same sequence as (1), except M_{f_1} replaces M_1, so we have our desired isomorphism.

Here is a second proof that the sequence

$$(2) \quad 0 \to M_1 \to M_1 \times (M_2)_{f_1} \times \cdots \times (M_n)_{f_1} \to M_{12} \times \cdots \times (M_{23})_{f_1} \times \cdots \times (M_{(n-1)n})_{f_1}$$

is exact. To check exactness of a complex of R-modules, it suffices to check exactness "at each prime $\mathfrak{p}$". In other words, if a complex is exact once tensored with $R_{\mathfrak{p}}$ for all $\mathfrak{p}$, then it was exact to begin with. Now note that if N is an R-module, then $(N_{f_i})_{\mathfrak{p}}$ is 0 if $f_i \in \mathfrak{p}$, and $N_{\mathfrak{p}}$ otherwise. Hence after tensoring with $R_{\mathfrak{p}}$, each term in (2) is either 0 or $N_{\mathfrak{p}}$, and the reader will quickly verify that the resulting complex is exact. (If any reader thinks I should say a few words as to why this is true, they should let me know, and I'll add a bit to these notes. I'm beginning to think that I should re-work some of my earlier arguments, including for example base gluability and base identity of the structure sheaf, in this way.) □

At this point, you probably want an example. I'll give you a boring example, and save a more interesting one for the end of the class.

Example: $\mathcal{O}_X$ is a quasicoherent sheaf. Over each affine open $\operatorname{Spec} R$, it is isomorphic the module $M = R$. This is not yet enough to specify what the sheaf is! We need also to describe the distinguished restriction maps, which are given by $R \to R_f$, where these are the "natural" ones. (This is confusing because this sheaf is too simple!) A variation on this theme is $\mathcal{O}_X^{\oplus n}$ (interpreted in the obvious way). This is called a *rank n free sheaf*. It corresponds to a rank n trivial vector bundle.

Joe mentioned an example of an $\mathcal{O}_X$-module that is not a quasicoherent sheaf last day.

1.2. Exercise. (a) Suppose $X = \operatorname{Spec} k[t]$. Let $\mathcal{F}$ be the skyscraper sheaf supported at the origin $[(t)]$, with group $k(t)$. Give this the structure of an $\mathcal{O}_X$-module. Show that this is not a quasicoherent sheaf. (More generally, if X is an integral scheme, and $p \in X$ that is not the generic point, we could take the skyscraper sheaf at p with group the function field of X. Except in a silly circumstances, this sheaf won't be quasicoherent.)
(b) Suppose $X = \operatorname{Spec} k[t]$. Let $\mathcal{F}$ be the skyscraper sheaf supported at the generic point $[(0)]$, with group $k(t)$. Give this the structure of an $\mathcal{O}_X$-module. Show that this *is* a quasicoherent sheaf. Describe the restriction maps in the distinguished topology of X. (Joe remarked that this is a constant sheaf!)

1.3. *Important Exercise for later.* Suppose X is a Noetherian scheme. Suppose $\mathcal{F}$ is a quasicoherent sheaf on X, and let $f \in \Gamma(X, \mathcal{O}_X)$ be a function on X. Let $R = \Gamma(X, \mathcal{O}_X)$ for convenience. Show that the restriction map $\operatorname{res}_{X_f \subset X} : \Gamma(X, \mathcal{F}) \to \Gamma(X_f, \mathcal{F})$ (here X_f is the open subset of X where f doesn't vanish) is precisely localization. In other words show that there is an isomorphism $\Gamma(X, \mathcal{F})_f \to \Gamma(X_f, \mathcal{F})$ making the following diagram commute.

$$\begin{array}{ccc} \Gamma(X,\mathcal{F}) & \xrightarrow{\operatorname{res}_{X_f \subset X}} & \Gamma(X_f,\mathcal{F}) \\ {\scriptstyle \otimes_R R_f} \searrow & & \nearrow {\scriptstyle \sim} \\ & \Gamma(X,\mathcal{F})_f & \end{array}$$

All that you should need in your argument is that X admits a cover by a finite number of open sets, and that their pairwise intersections are each quasicompact. We will later rephrase this as saying that X is quasicompact and quasiseparated. (Hint: cover by affine open sets. Use the sheaf property. A nice way to formalize this is the following. Apply the exact functor $\otimes_R R_f$ to the exact sequence

$$0 \to \Gamma(X, \mathcal{F}) \to \oplus_i \Gamma(U_i, \mathcal{F}) \to \oplus \Gamma(U_{ijk}, \mathcal{F})$$

where the U_i form a finite cover of X and U_{ijk} form an affine cover of $U_i \cap U_j$.)

1.4. *Less important exercise.* Give a counterexample to show that the above statement need not hold if X is not quasicompact. (Possible hint: take an infinite disjoint union of affine schemes.)

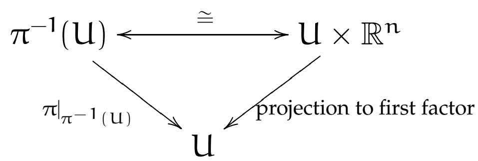
For the experts: I don't know a counterexample to this when the quasiseparated hypothesis is removed. Using the exact sequence above, I can show that there is a map $\Gamma(X_f, \mathcal{F}) \to \Gamma(X, \mathcal{F})_f$.

2. Locally free sheaves

I want to show you how that quasicoherent sheaves somehow generalize the notion of vector bundles.

(For arithmetic people: don't tune out! Fractional ideals of the ring of integers in a number field will turn out to be an example of a "line bundle on a smooth curve".)

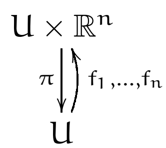
Since this is motivation, I won't make this precise, so you should feel free to think of this in the differentiable category (i.e. the category of differentiable manifolds). A rank n vector bundle on a manifold M is a fibration $\pi : V \to M$ that locally looks like the product with n-space: every point of M has a neighborhood U such that $\pi^{-1}(U) \cong U \times \mathbb{R}^n$, where the projection map is the obvious one, i.e. the following diagram commutes.



This is called a *trivialization over* U. We also want a "consistent vector space structure". Thus given trivializations over U_1 and U_2, over their intersection, the two trivializations should be related by an element of $GL(n)$ with entries consisting of functions on $U_1 \cap U_2$.

Examples of this include for example the tangent bundle on a sphere, and the moebius strip over $\mathbb{R}^1$.

Pick your favorite vector bundle, and consider its sheaf of sections $\mathcal{F}$. Then the sections over any open set form a real vector space. Moreover, given a U and a trivialization, the sections are naturally n-tuples of functions of U. [If I can figure out how to do curly arrows in xymatrix, I'll fix this.]



The open sets over which V is trivial forms a nice base of the topology.

Motivated by this, we define a *locally free sheaf of rank* n on a scheme X as follows. It is a quasicoherent sheaf that is locally, well, free of rank n. It corresponds to a vector bundle. It is determined by the following data: a cover U_i of X, and for each i, j transition functions T_{ij} lying in $GL(n, \Gamma(U_i \cap U_j, \mathcal{O}_X))$ satisfying

$$T_{ii} = Id_n, T_{ij}T_{jk} = T_{ik}$$

(which implies $T_{ij} = T_{ji}^{-1}$). Given this data, we can find the sections over any open set U as follows. Informally, they are sections of the free sheaves over each $U \cap U_i$ that agree on overlaps. More formally, for each i, they are $\vec{s}^i = \begin{pmatrix} s_1^i \\ \vdots \\ s_n^i \end{pmatrix} \in \Gamma(U \cap U_i, \mathcal{O}_X)^n$, satisfying $T_{ij}\vec{s}^i = \vec{s}^j$ on $U \cap U_i \cap U_j$.

In the differentiable category, locally free sheaves correspond precisely to vector bundles (for example, you can describe them with the same transition functions). So you should really think of these "as" vector bundles, but just keep in mind that they are not the "same", just equivalent notions.

A rank 1 vector bundle is called a *line bundle*. Similarly, a rank 1 locally free sheaf is called an *invertible sheaf*. I'll later explain why it is called invertible; but it is still a somewhat heinous term for something so fundamental.

Caution: Not every quasicoherent sheaf is locally free.

In a few sections, we will define some operations on quasicoherent sheaves that generate natural operations on vector bundles (such as dual, Hom, tensor product, etc.). The constructions will behave particularly well for locally free sheaves. We will see that the invertible sheaves on X will form a group under tensor product, called the *Picard group* of X.

We first make precise our discussion of transition functions. Given a rank n locally free sheaf $\mathcal{F}$ on a scheme X, we get transition functions as follows. Choose an open cover U_i of X so that $\mathcal{F}$ is a free rank n sheaf on each U_i. Choose a basis $e_{i,1}, \dots, e_{i,n}$ of $\mathcal{F}$ over U_i. Then over $U_i \cap U_j$, for each k, $e_{i,k}$ can be written as a $\Gamma(U_i \cap U_j, \mathcal{O}_X)$-linear combination of the $e_{j,l}$ ($1 \leq l \leq n$), so we get an $n \times n$ "transition matrix" T_{ji} with entries in $\Gamma(U_i \cap U_j, \mathcal{O}_X)$. Similarly, we get T_{ij}, and $T_{ij}T_{ji} = T_{ji}T_{ij} = I_n$, so T_{ij} and T_{ji} are invertible. Also, on $U_i \cap U_j \cap U_k$, we readily have $T_{ik} = T_{ij}T_{jk}$: both give the matrix that expresses the basis vectors of $e_{i,q}$ in terms of $e_{k,q}$. [Make sure this is right!]

2.1. Exercise. Conversely, given transition functions $T_{ij} \in GL(n, \Gamma(U_i \cap U_j, \mathcal{O}_X))$ satisfying the cocycle condition $T_{ij}T_{jk} = T_{ik}$ "on $U_i \cap U_j \cap U_k$", describe the corresponding rank n locally free sheaf.

We end this section with a few stray comments.

Caution: there are new morphisms between locally free sheaves, compared with what people usually say for vector bundles. Give example on $\mathbb{A}^1$:

$$0 \to tk[t] \to k[t] \to k[t]/(t) \to 0.$$

For vector bundle people: the thing on the left isn't a morphism of vector bundles (at least according to some definitions). (If you think it is a morphism of vector bundles, then you should still be disturbed, because its cokernel is not a vector bundle!)

2.2. Remark. Based on your intuition for line bundles on manifolds, you might hope that every point has a "small" open neighborhood on which all invertible sheaves (or locally free sheaves) are trivial. Sadly, this is not the case. We will eventually see that for the curve $y^2 - x^3 - x = 0$ in $\mathbb{A}^2_{\mathbb{C}}$, every nonempty open set has nontrivial invertible sheaves. (This will use the fact that it is an open subset of an *elliptic curve*.)

2.3. Exercise (for arithmetically-minded people only — I won't define my terms). Prove that a fractional ideal on a ring of integers in a number field yields an invertible sheaf. Show that any two that differ by a principal ideal yield the same invertible sheaf.

Thus we have described a map from the class group of the number field to the Picard group of its ring of integers. It turns out that this is an isomorphism. So strangely the number theorists in this class are the first to have an example of a nontrivial line bundle.

2.4. Exercise (for those familiar with Hartogs' Theorem for Noetherian normal schemes). Show that locally free sheaves on Noetherian normal schemes satisfy "Hartogs' theorem": sections defined away from a set of codimension at least 2 extend over that set.

3. Quasicoherent sheaves form an abelian category

The category of R-modules is an abelian category. (Indeed, this is our motivating example of our notion of abelian category.) Similarly, quasicoherent sheaves form an abelian category. I'll explain how.

When you show that something is an abelian category, you have to check many things, because the definition has many parts. However, if the objects you are considering lie in some ambient abelian category, then it is much easier. As a metaphor, there are several things you have to do to check that something is a group. But if you have a subset of group elements, it is much easier to check that it is a subgroup.

You can look at back at the definition of an abelian category, and you'll see that in order to check that a subcategory is an abelian subcategory, you need to check only the following things:

(i) 0 is in your subcategory
(ii) your subcategory is closed under finite sums
(iii) your subcategory is closed under kernels and cokernels

In our case of {quasicoherent sheaves} $\subset$ {$\mathcal{O}_X$-modules}, the first two are cheap: 0 is certainly quasicoherent, and the subcategory is closed under finite sums: if $\mathcal{F}$ and $\mathcal{G}$ are sheaves on X, and over $\operatorname{Spec} R$, $\mathcal{F} \cong \tilde{M}$ and $\mathcal{G} \cong \tilde{N}$, then $\mathcal{F} \oplus \mathcal{G} = \widetilde{M \oplus N}$, so $\mathcal{F} \oplus \mathcal{G}$ is a quasicoherent sheaf.

We now check (iii). Suppose $\alpha : \mathcal{F} \to \mathcal{G}$ is a morphism of quasicoherent sheaves. Then on any affine open set U, where the morphism is given by $\beta : M \to N$, define $(\ker \alpha)(U) = \ker \beta$ and $(\operatorname{coker} \alpha)(U) = \operatorname{coker} \beta$. Then these behave well under inversion of a single element: if

$$0 \to K \to M \to N \to P \to 0$$

is exact, then so is

$$0 \to K_f \to M_f \to N_f \to P_f \to 0,$$

from which $(\ker \beta)_f \cong \ker(\beta_f)$ and $(\operatorname{coker} \beta)_f \cong \operatorname{coker}(\beta_f)$. Thus both of these define quasicoherent sheaves. Moreover, by checking stalks, they are indeed the kernel and cokernel of α. Thus the quasicoherent sheaves indeed form an abelian category.

As a side benefit, we see that we may check injectivity, surjectivity, or exactness of a morphism of quasicoherent sheaves by checking on an affine cover.

Warning: If $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is an exact sequence of quasicoherent sheaves, then for any open set

$$0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U)$$

is exact, and we have exactness on the right is guaranteed to hold only if U is affine. (To set you up for cohomology: whenever you see left-exactness, you expect to eventually interpret this as a start of a long exact sequence. So we are expecting H^1's on the right, and now we expect that $H^1(\operatorname{Spec} R, \mathcal{F}) = 0$. This will indeed be the case.)

3.1. *Exercise.* Show that you can check exactness of a sequence of quasicoherent sheaves on an affine cover. (In particular, taking sections over an affine open $\operatorname{Spec} R$ is an exact functor from the category of quasicoherent sheaves on X to the category of R-modules. Recall that taking sections is only left-exact in general.) Similarly, you can check surjectivity on an affine cover (unlike sheaves in general).

4. Module-like constructions on quasicoherent sheaves

In a similar way, basically any nice construction involving modules extends to quasicoherent sheaves.

As an important example, we consider tensor products. **Exercise.** If $\mathcal{F}$ and $\mathcal{G}$ are quasicoherent sheaves, show that $\mathcal{F} \otimes \mathcal{G}$ is given by the following information: If $\operatorname{Spec} R$ is an affine open, and $\Gamma(\operatorname{Spec} R, \mathcal{F}) = M$ and $\Gamma(\operatorname{Spec} R, \mathcal{G}) = N$, then $\Gamma(\operatorname{Spec} R, \mathcal{F} \otimes \mathcal{G}) = M \otimes N$, and the restriction map $\Gamma(\operatorname{Spec} R, \mathcal{F} \otimes \mathcal{G}) \to \Gamma(\operatorname{Spec} R_f, \mathcal{F} \otimes \mathcal{G})$ is precisely the localization map $M \otimes_R N \to (M \otimes_R N)_f \cong M_f \otimes_{R_f} N_f$. (We are using the algebraic fact that that $(M \otimes_R N)_f \cong M_f \otimes_{R_f} N_f$. You can prove this by universal property if you want, or by using the explicit construction.)

Note that thanks to the machinery behind the distinguished affine base, sheafification is taken care of.

For category-lovers: this makes the category of quasicoherent sheaves into a monoid.

4.1. Exercise. If $\mathcal{F}$ and $\mathcal{G}$ are locally free sheaves, show that $\mathcal{F} \otimes \mathcal{G}$ is locally free. (Possible hint for this, and later exercises: check on sufficiently small affine open sets.)

4.2. *Exercise.* (a) Tensoring by a quasicoherent sheaf is right-exact. More precisely, if $\mathcal{F}$ is a quasicoherent sheaf, and $\mathcal{G}' \to \mathcal{G} \to \mathcal{G}'' \to 0$ is an exact sequence of quasicoherent sheaves, then so is $\mathcal{G}' \otimes \mathcal{F} \to \mathcal{G} \otimes \mathcal{F} \to \mathcal{G}'' \otimes \mathcal{F} \to 0$ is exact.
(b) Tensoring by a locally free sheaf is exact. More precisely, if $\mathcal{F}$ is a locally free sheaf, and $\mathcal{G}' \to \mathcal{G} \to \mathcal{G}''$ is an exact sequence of quasicoherent sheaves, then then so is $\mathcal{G}' \otimes \mathcal{F} \to \mathcal{G} \otimes \mathcal{F} \to \mathcal{G}'' \otimes \mathcal{F}$.
(c) The stalk of the tensor product of quasicoherent sheaves at a point is the tensor product of the stalks.

Note: if you have a section s of $\mathcal{F}$ and a section t of $\mathcal{G}$, you get a section $s \otimes t$ of $\mathcal{F} \otimes \mathcal{G}$. If either $\mathcal{F}$ or $\mathcal{G}$ is an invertible sheaf, this section is denoted st.

We now describe other constructions.

4.3. *Exercise.* Sheaf Hom, $\underline{\mathrm{Hom}}$, is quasicoherent, and is what you think it might be. (Describe it on affine opens, and show that it behaves well with respect to localization with respect to f. To show that $\mathrm{Hom}_A(M, N)_f \cong \mathrm{Hom}_{A_f}(M_f, N_f)$, take a "partial resolution" $A^q \to A^p \to M \to 0$, and apply $\mathrm{Hom}(\cdot, N)$ and localize.) ($\underline{\mathrm{Hom}}$ was defined earlier, and was the subject of a homework problem.) Show that $\underline{\mathrm{Hom}}$ is a left-exact functor in both variables.

Definition. $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{O}_X)$ is called the *dual* of $\mathcal{F}$, and is denoted $\mathcal{F}^\vee$.

4.4. *Exercise.* The direct sum of quasicoherent sheaves is what you think it is.

5. Some notions especially relevant for locally free sheaves

Exercise. Show that if $\mathcal{F}$ is locally free then $\mathcal{F}^\vee$ is locally free, and that there is a canonical isomorphism $(\mathcal{F}^\vee)^\vee \cong \mathcal{F}$. (Caution: your argument showing that if there is a canonical isomorphism $(\mathcal{F}^\vee)^\vee \cong \mathcal{F}$ better not also show that there is a canonical isomorphism $\mathcal{F}^\vee \cong \mathcal{F}$! We'll see an example soon of a locally free $\mathcal{F}$ that is not isomorphic to its dual. The example will be the line bundle $\mathcal{O}(1)$ on $\mathbb{P}^1$.)

Remark. This is not true for quasicoherent sheaves in general, although your argument will imply that there is always a natural morphism $\mathcal{F} \to (\mathcal{F}^\vee)^\vee$. Quasicoherent sheaves for which this is true are called *reflexive sheaves*. We will not be using this notion. Your argument may also lead to a canonical map $\mathcal{F} \otimes \mathcal{F}^\vee \to \mathcal{O}_X$. This could be called the *trace* map — can you see why?

5.1. *Exercise.* Given transition functions for the locally free sheaf $\mathcal{F}$, describe the transition functions for the locally free sheaf $\mathcal{F}^\vee$. Note that if $\mathcal{F}$ is rank 1 (i.e. locally free), the transition functions of the dual are the inverse of the transition functions of the original; in this case, $\mathcal{F} \otimes \mathcal{F}^\vee \cong \mathcal{O}_X$.

5.2. *Exercise.* If $\mathcal{F}$ and $\mathcal{G}$ are locally free sheaves, show that $\mathcal{F} \otimes \mathcal{G}$ and $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})$ are both locally free.

5.3. *Exercise.* Show that the invertible sheaves on X, up to isomorphism, form an abelian group under tensor product. This is called the *Picard group* of X, and is denoted $\mathrm{Pic}\, X$. (For arithmetic people: this group, for the Spec of the ring of integers R in a number field, is the class group of R.)

For the next exercises, recall the following. If M is an A-module, then the *tensor algebra* $T^*(M)$ is a non-commutative algebra, graded by $\mathbb{Z}^{\geq 0}$, defined as follows. $T^0(M) = A$, $T^n(M) = M \otimes_A \cdots \otimes_A M$ (where n terms appear in the product), and multiplication is what you expect. The *symmetric algebra* $\operatorname{Sym}^* M$ is a symmetric algebra, graded by $\mathbb{Z}^{\geq 0}$, defined as the quotient of $T^*(M)$ by the (two-sided) ideal generated by all elements of the form $x \otimes y - y \otimes x$ for all $x, y \in M$. Thus $\operatorname{Sym}^n M$ is the quotient of $M \otimes \cdots \otimes M$ by the relations of the form $m_1 \otimes \cdots \otimes m_n - m'_1 \otimes \cdots \otimes m'_n$ where $(m'_1, \dots, m'_n)$ is a rearrangement of $(m_1, \dots, m_n)$. The *exterior algebra* $\wedge^* M$ is defined to be the quotient of $T^* M$ by the (two-sided) ideal generated by all elements of the form $x \otimes y + y \otimes x$ for all $x, y \in M$. Thus $\wedge^n M$ is the quotient of $M \otimes \cdots \otimes M$ by the relations of the form $m_1 \otimes \cdots \otimes m_n - (-1)^{\operatorname{sgn}} m'_1 \otimes \cdots \otimes m'_n$ where $(m'_1, \dots, m'_n)$ is a rearrangement of $(m_1, \dots, m_n)$, and the sgn is even if the rearrangement is an even permutation, and odd if the rearrangement is an odd permutation. (It is a "skew-commutative" A-algebra.) It is most correct to write $T^*_A(M)$, $\operatorname{Sym}^*_A(M)$, and $\wedge^*_A(M)$, but the "base ring" is usually omitted for convenience.

5.4. *Exercise.* If $\mathcal{F}$ is a quasicoherent sheaf, then define the quasicoherent sheaves $T^n \mathcal{F}$, $\operatorname{Sym}^n \mathcal{F}$, and $\wedge^n \mathcal{F}$. If $\mathcal{F}$ is locally free of rank m, show that $T^n \mathcal{F}$, $\operatorname{Sym}^n \mathcal{F}$, and $\wedge^n \mathcal{F}$ are locally free, and find their ranks.

You can also define the sheaf of non-commutative algebras $T^* \mathcal{F}$, the sheaf of algebras $\operatorname{Sym}^* \mathcal{F}$, and the sheaf of skew-commutative algebras $\wedge^* \mathcal{F}$.

5.5. Important exercise. If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of locally free sheaves, then for any r, there is a filtration of $\operatorname{Sym}^r \mathcal{F}$:

$$\operatorname{Sym}^r \mathcal{F} = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^r \supset F^{r+1} = 0$$

with quotients

$$F^p / F^{p+1} \cong (\operatorname{Sym}^p \mathcal{F}') \otimes (\operatorname{Sym}^{r-p} \mathcal{F}'')$$

for each p.

5.6. Exercise. Suppose $\mathcal{F}$ is locally free of rank n. Then $\wedge^n \mathcal{F}$ is called the *determinant (line) bundle*. Show that $\wedge^r \mathcal{F} \times \wedge^{n-r} \mathcal{F} \to \wedge^n \mathcal{F}$ is a perfect pairing for all r.

5.7. Exercise. If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of locally free sheaves, then for any r, there is a filtration of $\wedge^r \mathcal{F}$:

$$\wedge^r \mathcal{F} = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^r \supset F^{r+1} = 0$$

with quotients

$$F^p / F^{p+1} \cong (\wedge^p \mathcal{F}') \otimes (\wedge^{r-p} \mathcal{F}'')$$

for each p. In particular, $\det \mathcal{F} = (\det \mathcal{F}') \otimes (\det \mathcal{F}'')$.

5.8. Exercise (torsion-free sheaves). An R-module M is torsion-free if $rm = 0$ implies $r = 0$ or $m = 0$. Show that this satisfies the hypotheses of the affine communication lemma. Hence we make a definition: a quasicoherent sheaf is *torsion-free* if for one (or by the affine communication lemma, for any) affine cover, the sections over each affine open are torsion-free. By definition, "torsion-freeness is affine-local". Show that a quasicoherent sheaf is torsion-free if all its stalks are torsion-free. Hence "torsion-freeness" is "stalk-local." [This exercise is wrong! "Torsion-freeness" is should be defined as "torsion-free stalks" — it is (defined as) a "stalk-local" condition. Here is a better exercise. Show that if M is torsion-free, then so is any localization of M. In particular, M_f is torsion-free, so this notion satisfies half the hypotheses of the affine communication lemma. Also, $M_{\mathfrak{p}}$ is torsion-free, so this implies that $\tilde{M}$ is torsion-free. Find an example on a two-point space showing that R might not be torsion-free even though $\mathcal{O}_{\text{Spec } R} = \tilde{R}$ is torsion-free.]

6. Quasicoherent sheaves of ideals, and closed subschemes

I then defined quasicoherent sheaves of ideals, and closed subschemes. But I'm happier with the definition I gave in class 15, so I'll leave the discussion until then.

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