FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 13

RAVI VAKIL

CONTENTS

Last day: Jacobian criterion, Euler test, characterizations of discrete valuation rings = Noetherian regular local rings

Today: discrete valuation rings (conclusion), cultural facts to know about regular local rings, the distinguished affine base of the topology, 2 definitions of quasicoherent sheaf.

Problem set 5 is out today.

I'd like to start with some words on height versus codimension. Suppose R is an integral domain, and $\mathfrak{p}$ is a prime ideal. Thus geometrically we are thinking of an irreducible topological space $\operatorname{Spec} R$, and an irreducible closed subset $\overline{[\mathfrak{p}]}$. Then we have:

$$\dim R/\mathfrak{p} + \operatorname{height} \mathfrak{p} := \dim R/\mathfrak{p} + \dim R_{\mathfrak{p}} \leq \dim R.$$

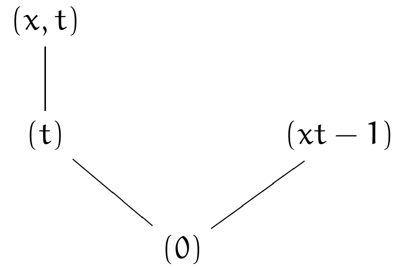
The reason is as follows: $\dim R$ is one less than the length of the longest chain of prime ideals of R. $\dim R/\mathfrak{p}$ is one less than the length of the longest chain of prime ideals containing $\mathfrak{p}$. $\dim R_{\mathfrak{p}}$ is the length of the longest chain of prime ideals contained in $\mathfrak{p}$. In the homework, you've shown that if R is a finitely generated domain over k, then we have equality, because we can compute dimension using transcendence degree. Hence through any $\mathfrak{p} \subset R$, we can string a "longest chain". Thus we even know that we have equality if R is a *localization* of a finitely generated domain over k.

However, this is false in general. In the class 9 notes, I've added an elementary example to show that you can have the following strange situation: $R = k[x]_{(x)}[t]$ has dimension 2, it is easy to find a chain of prime ideals of length 3:

$$(0) \subset (t) \subset (x, t).$$

Date: Monday, November 7, 2005. Minor updates January 30, 2007.

However, the ideal $(xt-1)$ is prime, and height 1 (there is no prime between it and (0)), and maximal.



The details are easy. Thus we have a dimension 0 subset of a dimension 2 set, but it is height 1. Thus it is dangerous to define codimension as height, because you might say something incorrect accidentally.

This example comes from a geometric picture, and if you're curious as to what it is, ask me after class.

There is one more idea I wanted to mention to you, to advertise a nice consequence of the idea of Zariski tangent space.

Problem. Consider the ring $R = k[x,y,z]/(xy-z^2)$. Show that (x,z) is not a principal ideal.

As $\dim R = 2$, and $R/(x,z) \cong k[y]$ has dimension 1, we see that this ideal is height 1 (as height behaves as codimension should for finitely generated k-domains!). Our geometric picture is that $\operatorname{Spec} R$ is a cone (we can diagonalize the quadric as $xy - z^2 = ((x+y)/2)^2 - ((x-y)/2)^2 - z^2$, at least if $\operatorname{char} k \neq 2$), and that (x,z) is a ruling of the cone. This suggests that we look at the cone point.

Solution. Let $\mathfrak{m} = (x,y,z)$ be the maximal ideal corresponding to the origin. Then $\operatorname{Spec} R$ has Zariski tangent space of dimension 3 at the origin, and $\operatorname{Spec} R/(x,z)$ has Zariski tangent space of dimension 1 at the origin. But $\operatorname{Spec} R/(f)$ must have Zariski tangent space of dimension at least 2 at the origin.

Exercise. Show that $(x,z) \subset k[w,x,y,z]/(wz-xy)$ is a height 1 ideal that is not principal. (What is the picture?)

1. Dimension 1 Noetherian regular local rings = discrete valuation rings

Last day we mostly proved the following.

Theorem. *Suppose* $(R,\mathfrak{m})$ *is a Noetherian dimension* 1 *local ring. The following are equivalent.*

(a) R *is regular.*
(b) $\mathfrak{m}$ *is principal.*
(c) *All ideals are of the form* $\mathfrak{m}^n$ *or* 0.
(d) R *is a principal ideal domain.*
(e) R *is a discrete valuation ring.*

(f) $(R, \mathfrak{m})$ *is a unique factorization domain,*
(g) R *is integrally closed in its fraction field* $K = \mathrm{Frac}(R)$.

I didn't state (d) in class, but I included it as an exercise, as it is easy to connect to the others. Other than that, I connected (a) to (e), and showed that they implies (f), which in turn implies (g). All of the arguments were quite short. I didn't show that (g) implies (a)-(e), but I included it in the notes for Friday's class. I find this the trickiest part of the argument, but it is still quite short, less than half a page.

I'd like to repeat what I said on Friday about the consequences of this characterization of discrete valuation rings (DVR's).

Whenever you see a Noetherian regular local ring of dimension 1, we have a valuation on the fraction field. If the valuation of an element is $n > 0$, we say that the element has a *zero of order* n. If the valuation is $-n < 0$, we say that the element has a *pole of order* n.

Definition. More generally: suppose X is a locally Noetherian scheme. Then for any regular height(=codimension) 1 points (i.e. any point p where $\mathcal{O}_{X,p}$ is a regular local ring), we have a valuation ν. If f is any non-zero element of the fraction field of $\mathcal{O}_{X,p}$ (e.g. if X is integral, and f is a non-zero element of the function field of X), then if $\nu(f) > 0$, we say that the element has a *zero of order* $\nu(f)$, and if $\nu(f) < 0$, we say that the element has a *pole of order* $-\nu(f)$.

We aren't yet allowed to discuss order of vanishing at a point that is not regular codimension 1. One can make a definition, but it doesn't behave as well as it does when have you have a discrete valuation.

Exercise. Suppose X is an integral Noetherian scheme, and $f \in \mathrm{Frac}(\Gamma(X, \mathcal{O}_X))^*$ is a non-zero element of its function field. Show that f has a finite number of zeros and poles. (Hint: reduce to $X = \mathrm{Spec}\, R$. If $f = f_1/f_2$, where $f_i \in R$, prove the result for f_i.)

Now I'd like to discuss the geometry of normal Noetherian schemes. Suppose R is an Noetherian integrally closed domain. Then it is *regular in codimension* 1 (translation: all its codimension at most 1 points are regular). If R is dimension 1, then obviously R is nonsingular=regular=smooth.

Example: $\mathrm{Spec}\, \mathbb{Z}[i]$ is smooth. Reason: it is dimension 1, and $\mathbb{Z}[i]$ is a unique factorization domain, hence its Spec is normal.

Remark: A (Noetherian) scheme can be singular in codimension 2 and still normal. Example: you have shown that the cone $x^2+y^2 = z^2$ in $\mathbb{A}^3$ is normal (PS4, problem B4), but it is clearly singular at the origin (the Zariski tangent space is visibly three-dimensional).

So integral (locally Noetherian) schemes can be singular in codimension 2. But their singularities turn out to be not so bad. I mentioned earlier, before we even knew what normal schemes were, that they satisfied "Hartogs Theorem", that you could extend functions over codimension 2 sets.

Remark: We know that for Noetherian rings we have inclusions:

$$\{\text{regular in codimension 1}\} \supset \{\text{integrally closed}\} \supset \{\text{unique factorization domain}\}.$$

Here are two examples to show you that these inclusions are strict.

Exercise. Let R be the subring $k[x^3, x^2, xy, y] \subset k[x, y]$. (The idea behind this example: I'm allowing all monomials in $k[x, y]$ except for x.) Show that it is not integrally closed (easy — consider the "missing x"). Show that it is regular in codimension 1 (hint: show it is dimension 2, and when you throw out the origin you get something nonsingular, by inverting x^2 and y respectively, and considering R_{x^2} and R_y).

Exercise. You have checked that if $k = \mathbb{C}$, then $k[w, x, y, z]/(wx-yz)$ is integrally closed (PS4, problem B5). Show that it is not a unique factorization domain. (The most obvious possibility is to do this "directly", but this might be hard. Another possibility, faster but less intuitive, is to prove the intermediate result that *in a unique factorization domain, any height* 1 *prime is principal*, and considering an exercise from earlier today that $w = z = 0$ is not principal.)

2. Good facts to know about regular local rings

There are some other important facts to know about regular local rings. In this class, I'm trying to avoid pulling any algebraic facts out of nowhere. As a rule of thumb, anything that you wouldn't see in Math 210, I consider "pulled out of nowhere". Even the harder facts from 210, I'm happy to give you a proof of, if you ask — none of those facts require more than a page of proof. To my knowledge, the only facts I've pulled out of nowhere to date are Krull's Principal Ideal Theorem, and its transcendence degree form. I might even type up a short proof of Krull's Theorem, and put it in the notes, so even that won't come out of nowhere.

Now, smoothness is an easy intuitive concept, but it is algebraically hard — harder than dimension. A sign of this is that I'm going to have to pull three facts out of nowhere. I think it's good for you to see these facts, but I'm going to try to avoid using these facts in the future. So consider them as mainly for culture.

Suppose $(R, \mathfrak{m})$ is a Noetherian regular local ring.

Fact 1. Any localization of R at a prime is also a regular local ring (Eisenbud's *Commutative Algebra*, Cor. 19.14, p. 479).

Hence to check if $\operatorname{Spec} R$ is nonsingular, then it suffices to check at closed points (at maximal ideals). For example, to check if $\mathbb{A}^3$ is nonsingular, you can check at all closed points, because all other points are obtained by localizing further. (You should think about this — it is confusing because of the order reversal between primes and closed subsets.)

Exercise. Show that on a Noetherian scheme, you can check nonsingularity by checking at closed points. (Caution: a scheme in general needn't have any closed points!) You are allowed to use the unproved fact from the notes, that any localization of a regular local ring is regular.

2.1. Less important exercise. Show that there is a nonsingular hypersurface of degree d. Show that there is a Zariski-open subset of the space of hypersurfaces of degree d. The two previous sentences combine to show that the nonsingular hypersurfaces form a Zariski-open set. Translation: almost all hypersurfaces are smooth.

Fact 2. ("leading terms", proved in an important case) The natural map $\mathrm{Sym}^n(\mathfrak{m}/\mathfrak{m}^2) \to \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is an isomorphism. Even better, the following diagram commutes:

$$\begin{array}{ccc} (\mathfrak{m}^i/\mathfrak{m}^{i+1}) \times (\mathfrak{m}^j/\mathfrak{m}^{j+1}) & \xrightarrow{\times} & \mathfrak{m}^{i+j}/\mathfrak{m}^{i+j+1} \\ \downarrow \sim & & \downarrow \sim \\ \mathrm{Sym}^i(\mathfrak{m}/\mathfrak{m}^2) \times \mathrm{Sym}^j(\mathfrak{m}/\mathfrak{m}^2) & \xrightarrow{\times} & \mathrm{Sym}^{i+j}(\mathfrak{m}/\mathfrak{m}^2) \end{array}$$

Easy Exercise. Suppose $(R, \mathfrak{m}, k)$ is a regular Noetherian local ring of dimension n. Show that $\dim_k(\mathfrak{m}^i/\mathfrak{m}^{i+1}) = \binom{n+i-1}{i}$.

Exercise. Show that Fact 2 also implies that $(R, \mathfrak{m})$ is a domain. (Hint: show that if $f, g \neq 0$, then $fg \neq 0$, by considering the leading terms.)

I don't like facts pulled out of nowhere, so I want to prove it in an important case. Suppose $(R, \mathfrak{m})$ is a Noetherian local ring containing its residue field k: $k \hookrightarrow R \twoheadrightarrow R/\mathfrak{m} = k$. (For example, if k is algebraically closed, this is true for all local rings of finite type k-schemes at maximal ideals, by the Nullstellensatz. But it is not true if $(R, \mathfrak{m}) = (\mathbb{Z}_p, p\mathbb{Z}_p)$, as the residue field $\mathbb{F}_p$ is not a subring of $\mathbb{Z}_p$.)

Suppose R is a regular of dimension n, with $x_1, \ldots, x_n \in R$ generating $\mathfrak{m}/\mathfrak{m}^2$ as a vector space (and hence $\mathfrak{m}$ as an ideal, by Nakayama's lemma). Then we get a natural map $k[t_1, \ldots, t_n] \to R$, taking t_i to x_i.

2.2. *Theorem. — Suppose* $(R, \mathfrak{m})$ *is a Noetherian regular local ring containing its residue field* k*:* $k \hookrightarrow R \twoheadrightarrow R/\mathfrak{m} = k$. *Then* $k[t_1, \ldots, t_n]/(t_1, \ldots, t_n)^m \to R/\mathfrak{m}^m$ *is an isomorphism for all* m.

Proof: See Section 3.

To interpret this better, and to use it: define the inverse limit $\hat{R} := \lim_{\leftarrow} R/\mathfrak{m}^n$. This is the *completion* of R at $\mathfrak{m}$. (We can complete any ring at any ideal of course.) For example, if $S = k[x_1, \ldots, x_n]$, and $\mathfrak{n} = (x_1, \ldots, x_n)$, then $\hat{S} = k[[x_1, \ldots, x_n]]$, power series in n variables. We have a good intuition for for power series, so we will be very happy with the next result.

2.3. *Theorem. — Suppose* R *contains its residue field* k*:* $k \hookrightarrow R \twoheadrightarrow R/\mathfrak{m} = k$ *. Then the natural map* $k[[t_1, \dots, t_n]] \to \hat{R}$ *taking* t_i *to* x_i *is an isomorphism.*

This follows immediately from the previous theorem, as both sides are inverse limits of the same things. I'll now give some consequences.

Note that $R \hookrightarrow \hat{R}$. Here's why. (Recall the interpretation of inverse limit: you can interpret $\hat{R}$ as a subring of $R/\mathfrak{m} \times R/\mathfrak{m}^2 \times R/\mathfrak{m}^3 \times \cdots$ such that if $j > i$, the jth element maps to the ith factor under the natural quotient map.) What can go to 0 in $\hat{R}$? It is something that lies in $\mathfrak{m}^n$ for all n. But $\cap_i \mathfrak{m}^i = 0$ (a fact I stated in class when discussing Nakayama — *I owe you a proof of this*), so the map is injective. (Important note: We aren't assuming regularity of R in this argument!!)

Thus we can think of the map $R \to \hat{R}$ as a power series expansion.

This implies the "leading term" fact in this case (where the local ring contains the residue field). (**Exercise:** Prove this. This isn't hard; it's a matter of making sure you see what the definitions are.) Hence in this case we have proved that R is a domain.

We go back to stating important **facts** that we will try not to use.

Fact 3. Not only is $(R, \mathfrak{m})$ a domain, it is a unique factorization domain, which we have shown implies integrally closed in its fraction field. Reference: Eisenbud Theorem 19.19, p. 483. This implies that regular schemes are normal. Reason: integrally closed iff all local rings are integrally closed domains. I'll explain why later.

3. Promised proof of Theorem 2.2

Let's now set up the proof of Theorem 2.2, with a series of exercises.

3.1. Exercise. If S is a Noetherian ring, show that $S[[t]]$ is Noetherian. (Hint: Suppose $I \subset S[[t]]$ is an ideal. Let $I_n \subset S$ be the coefficients of t^n that appear in the elements of I form an ideal. Show that $I_n \subset I_{n+1}$, and that I is determined by $(I_0, I_1, I_2, \dots)$.)

3.2. Corollary. $k[[t_1, \dots, t_n]]$ is a Noetherian local ring.

3.3. Exercise. Show that $\dim k[[t_1, \dots, t_n]]$ is dimension n. (Hint: find a chain of $n+1$ prime ideals to show that the dimension is at least n. For the other inequality, use Krull.)

3.4. Exercise. If R is a Noetherian local ring, show that $\hat{R} := \lim_{\leftarrow} R/\mathfrak{m}^n$ is a Noetherian local ring. (Hint: Suppose $\mathfrak{m}/\mathfrak{m}^2$ is finite-dimensional over k, say generated by $x_1, \dots, x_n$. Describe a surjective map $k[[t_1, \dots, t_n]] \to \hat{R}$.)

We now outline the proof of the Theorem, as an extended exercise. (This is hastily and informally written.)

Suppose $\mathfrak{p} \subset R$ is a prime ideal. Define $\hat{\mathfrak{p}} \subset \hat{R}$ by $\mathfrak{p}/\mathfrak{m}^m \subset R/\mathfrak{m}^m$. Show that $\hat{\mathfrak{p}}$ is a prime ideal of $\hat{R}$. (Hint: if $f, g \notin \mathfrak{p}$, then let m_f, m_g be the first "level" where they are not in $\mathfrak{p}$ (i.e. the smallest m such that $f \notin \mathfrak{p}/\mathfrak{m}^{m+1}$). Show that $fg \notin \mathfrak{p}$ by showing that $fg \notin \mathfrak{p}/\mathfrak{m}^{m_f+m_g+1}$.)

Show that if $\mathfrak{p} \subset \mathfrak{q}$, then $\hat{\mathfrak{p}} \subset \hat{\mathfrak{q}}$. Hence show that $\dim \hat{R} \geq \dim R$. But also $\dim \hat{R} \leq \dim \mathfrak{m}/\mathfrak{m}^2 = \dim R$. Thus $\dim \hat{R} = \dim R$.

We're now ready to prove the Theorem. We wish to show that $k[[t_1, \ldots, t_n]] \to \hat{R}$ is injective; we already know it is surjective. Suppose $f \in k[[t_1, \ldots, t_n]] \mapsto 0$, so we get a map $k[[t_1, \ldots, t_n]]/f$ surjects onto $\hat{R}$. Now f is not a zero-divisor, so by Krull, the left side has dimension $n-1$. But then any quotient of it has dimension at most $n-1$, contradiction. □

4. Toward quasicoherent sheaves: the distinguished affine base

Schemes generalize and geometrize the notion of "ring". It is now time to define the corresponding analogue of "module", which is a quasicoherent sheaf.

One version of this notion is that of a sheaf of $\mathcal{O}_X$-modules. They form an abelian category, with tensor products. (That might be called a tensor category — I should check.)

We want a better one — a subcategory of $\mathcal{O}_X$-modules. Because these are the analogues of modules, we're going to define them in terms of affine open sets of the scheme. So let's think a bit harder about the structure of affine open sets on a general scheme X. I'm going to define what I'll call the *distinguished affine base* of the Zariski topology. This won't be a base in the sense that you're used to. (For experts: it is a first example of a *Grothendieck topology*.) It is more akin to a base.

The open sets are the affine open subsets of X. We've already observed that this forms a base. But forget about that.

We like distinguished opens $\operatorname{Spec} R_f \hookrightarrow \operatorname{Spec} R$, and we don't really understand open immersions of one random affine in another. So we just remember the "nice" inclusions.

Definition. The *distinguished affine base* of a scheme X is the data of the affine open sets and the distinguished inclusions.

(Remark we won't need, but is rather fundamental: what we are using here is that we have a collection of open subsets, and *some* subsets, such that if we have any $x \in U, V$ where U and V are in our collection of open sets, there is some W containing x, and contained in U and V such that that $W \hookrightarrow U$ and $W \hookrightarrow V$ are both in our collection of inclusions. In the case we are considering here, this is the key Proposition in Class 9 that given any two affine opens $\operatorname{Spec} A$, $\operatorname{Spec} B$ in X, $\operatorname{Spec} A \cap \operatorname{Spec} B$ could be covered by affine opens that were simultaneously distinguished in $\operatorname{Spec} A$ and $\operatorname{Spec} B$. This is a *cofinal* condition.)

We can define a sheaf on the distinguished affine base in the obvious way: we have a set (or abelian group, or ring) for each affine open set, and we know how to restrict to distinguished open sets.

Given a sheaf $\mathcal{F}$ on X, we get a sheaf on the distinguished affine base. You can guess where we're going: we'll show that all the information of the sheaf is contained in the information of the sheaf on the distinguished affine base.

As a warm-up: We can recover stalks. Here's why. $\mathcal{F}_x$ is the direct limit $\lim_{\rightarrow}(f \in \mathcal{F}(U))$ where the limit is over all open sets contained in U. We compare this to $\lim_{\rightarrow}(f \in \mathcal{F}(U))$ where the limit is over all affine open sets, and all distinguished inclusions. You can check that the elements of one correspond to elements of the other. (Think carefully about this! It corresponds to the fact that the basic elements are cofinal in this directed system.)

4.1. Exercise. Show that a section of a sheaf on the distinguished affine base is determined by the section's germs.

4.2. *Theorem. —*

(a) *A sheaf on the distinguished affine base $\mathcal{F}^b$ determines a unique sheaf $\mathcal{F}$, which when restricted to the affine base is $\mathcal{F}^b$. (Hence if you start with a sheaf, and take the sheaf on the distinguished affine base, and then take the induced sheaf, you get the sheaf you started with.)*
(b) *A morphism of sheaves on an affine base determines a morphism of sheaves.*
(c) *A sheaf of $\mathcal{O}_X$-modules "on the distinguished affine base" yields an $\mathcal{O}_X$-module.*

Proof of (a). (Two comments: this is very reminiscent of our sheafification argument. It also trumps our earlier theorem on sheaves on a nice base.)

Suppose $\mathcal{F}^b$ is a sheaf on the distinguished affine base. Then we can define stalks.

For *any* open set U of X, define

$$\mathcal{F}(U) := \{(f_x \in \mathcal{F}^b_x)_{x \in U} : \forall x \in U, \exists U_X \text{ with } x \subset U_x \subset U, F^x \in \mathcal{F}^b(U_x) : F^x_y = f_y \forall y \in U_x\}$$

where each U_x is in our base, and F^x_y means "the germ of F^x at y". (As usual, those who want to worry about the empty set are welcome to.)

This is a sheaf: convince yourself that we have restriction maps, identity, and gluability, really quite easily.

I next claim that if U is in our base, that $\mathcal{F}(U) = \mathcal{F}^b(U)$. We clearly have a map $\mathcal{F}^b(U) \rightarrow \mathcal{F}(U)$. For the map $\mathcal{F}(U) \rightarrow \mathcal{F}^b(U)$: **gluability exercise** (a bit subtle).

These are isomorphisms, because elements of $\mathcal{F}(U)$ are determined by stalks, as are elements of $\mathcal{F}^b(U)$.

(b) Follows as before.

(c) **Exercise.** □

5. Quasicoherent sheaves

We now define a **quasicoherent sheaf**. In the same way that a scheme is defined by "gluing together rings", a quasicoherent sheaf over that scheme is obtained by "gluing together modules over those rings". We will give two equivalent definitions; each definition is useful in different circumstances. The first just involves the distinguished topology.

Definition 1. An $\mathcal{O}_X$-module $\mathcal{F}$ is a **quasicoherent sheaf** if for every affine open $\operatorname{Spec} R$ and distinguished affine open $\operatorname{Spec} R_f$ thereof, the restriction map $\phi : \Gamma(\operatorname{Spec} R, \mathcal{F}) \to \Gamma(\operatorname{Spec} R_f, \mathcal{F})$ factors as:

$$\phi : \Gamma(\operatorname{Spec} R, \mathcal{F}) \to \Gamma(\operatorname{Spec} R, \mathcal{F})_f \cong \Gamma(\operatorname{Spec} R_f, \mathcal{F}).$$

The second definition is more directly "sheafy". Given a ring R and a module M, we defined a sheaf $\tilde{M}$ on $\operatorname{Spec} R$ long ago — the sections over $D(f)$ were M_f.

Definition 2. An $\mathcal{O}_X$-module $\mathcal{F}$ is a *quasicoherent sheaf* if for every affine open $\operatorname{Spec} R$,

$$\mathcal{F}|_{\operatorname{Spec} R} \cong \Gamma(\widetilde{\operatorname{Spec} R}, \mathcal{F}).$$

(The "wide tilde" is suposed to cover the entire right side $\Gamma(\operatorname{Spec} R, \mathcal{F})$.) This isomorphism is as sheaves of $\mathcal{O}_X$-modules.

Hence by this definition, the sheaves on $\operatorname{Spec} R$ correspond to R-modules. Given an R-module M, we get a sheaf $\tilde{M}$. Given a sheaf $\mathcal{F}$ on $\operatorname{Spec} R$, we get an R-module $\Gamma(X, \mathcal{F})$. These operations are inverse to each other. So in the same way as schemes are obtained by gluing together rings, quasicoherent sheaves are obtained by gluing together modules over those rings.

By Theorem 4.2, we have:

Definition 2'. An $\mathcal{O}_X$-module on the distinguished affine base yields an $\mathcal{O}_X$-module.

5.1. *Proposition. — Definitions 1 and 2 are the same.*

Proof. Clearly Definition 2 implies Definition 1. (Recall that the definition of $\tilde{M}$ was in terms of the distinguished topology on $\operatorname{Spec} R$.) We now show that Definition 1 implies Definition 2. We use Theorem 4.2. By Definition 1, the sections over any distinguished open $\operatorname{Spec} R_f$ of $\mathcal{M}$ on $\operatorname{Spec} R$ is precisely $\Gamma(\operatorname{Spec} R, \mathcal{M})_f$, i.e. the sections of $\Gamma(\widetilde{\operatorname{Spec} R}, \mathcal{M})$ over $\operatorname{Spec} R_f$, and the restriction maps agree. Thus the two sheaves agree. □

We like Definition 1 because it says that to define a quasicoherent sheaf of $\mathcal{O}_X$-modules is that we just need to know what it is on all affine open sets, and that it behaves well under inverting single elements.

One reason we like Definition 2 is that it glues well.

5.2. *Proposition (quasicoherence is affine-local).* — *Let* X *be a scheme, and* $\mathcal{M}$ *a sheaf of* $\mathcal{O}_X$*-modules. Then let* P *be the property of affine open sets that* $\mathcal{M}|_{\operatorname{Spec} R} \cong \Gamma(\widetilde{\operatorname{Spec} R, \mathcal{M}})$*. Then* P *is an affine-local property.*

We will prove this next day.

E-mail address: vakil@math.stanford.edu