FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 10

CONTENTS

Last day: Krull's Principal Ideal Theorem, height, affine communication lemma, properties of schemes: locally Noetherian, Noetherian, finite type S-scheme, locally of finite type S-scheme, normal

Today: finite type A-scheme, locally of finite type A-scheme, projective schemes over A or k.

Problem set 4 is out today (on the web), and problem set 3 is due today. As always, feedback is most welcome. How are the problem sets pitched? I don't want to make them too grueling, but I'd like to give you enough so that you can get a grip on the concepts. I've noticed that some of you are going after the hardest questions, and others are trying easier questions, and that's fine with me.

There is a notion that I have been using implicitly, and I should have made it explicit by now. It's the notion of what I mean by when two schemes are the *isomorphic*. An *isomorphism* of two schemes $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ is the following data: (i) it is a homeomorphism between X and Y (the identification of the *sets* and *topologies*). Then we can think of $\mathcal{O}_X$ and $\mathcal{O}_Y$ are sheaves (of rings) on the same space, via this homeomorphism. (ii) It is the data of an isomorphism of sheaves $\mathcal{O}_X \leftrightarrow \mathcal{O}_Y$.

Last day, I introduced the affine communication lemma. this lemma will come up repeatedly in the future.

0.1. *Affine communication theorem. — Let P be some property enjoyed by some affine open sets of a scheme X, such that*

(i) *if* $\operatorname{Spec} R \hookrightarrow X$ *has* P *then for any* $f \in R$, $\operatorname{Spec} R_f \hookrightarrow X$ *does too.*
(ii) *if* $(f_1, \dots, f_n) = R$, *and* $\operatorname{Spec} R_{f_i} \hookrightarrow X$ *has* P *for all* i, *then so does* $\operatorname{Spec} R \hookrightarrow X$.

Suppose that $X = \cup_{i \in I} \operatorname{Spec} R_i$ *where* $\operatorname{Spec} R_i$ *is an affine, and* R_i *has property* P. *Then every other open affine subscheme of* X *has property* P *too.*

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By choosing P appropriately, we define some important properties of schemes. I gave several examples. Here is one last example.

0.2. *Proposition.* — *Suppose* R *is a ring, and* $(f_1, \dots, f_n) = R$. *Suppose* A *is a ring, and* R *is an* A*-algebra. (i) If* R *is a finitely generated* A*-algebra, then so is* R_{f_i}. *(ii) If each* R_{f_i} *is a finitely-generated* A*-algebra, then so is* R.

This of course leads to a corresponding definition.

0.3. Important Definition. Suppose X is a scheme, and A is a ring (e.g. A is a field k), and $\Gamma(X, \mathcal{O}_X)$ is an A-algebra. Note that $\Gamma(U, \mathcal{O}_X)$ is an A-algebra for all non-empty U. Then we say that X is an A*-scheme*, or a *scheme over* A. Suppose X is an A-scheme. If X can be covered by affine opens $\operatorname{Spec} R$ where R is a *finitely generated* A-algebra, we say that X is *locally of finite type over* A, or that it is a *locally of finite type* A*-scheme*. (My apologies for this cumbersome terminology; it will make more sense later.) If furthermore X is quasicompact, X is *finite type over* A, or a *finite type* A*-scheme*.

Proof of Proposition 0.2. (i) is clear: if R is generated over A by $r_1, \dots, r_n$, then R_f is generated over A by $r_1, \dots, r_n, 1/f$.

(ii) Here is the idea; I'll leave this as an **exercise** for you to make this work. We have generators of R_{f_i}: r_{ij}/f_i^j, where $r_{ij} \in R$. I claim that $\{r_{ij}\}_{ij} \cup \{f_i\}_i$ generate R as a A-algebra. Here's why. Suppose you have any $r \in R$. Then in R_{f_i}, we can write r as some polynomial in the r_{ij}'s and f_i, divided by some huge power of f_i. So "in each R_{f_i}, we have described r in the desired way", except for this annoying denominator. Now use a partition of unity type argument to combine all of these into a single expression, killing the denominator. Show that the resulting expression you build still agrees with r in each of the R_{f_i}. Thus it is indeed r. □

1. Projective k-schemes and A-schemes: a concrete example

I now want to tell you about an important class of schemes.

Our building blocks of schemes are affine schemes. For example, affine finite type k-schemes correspond to finitely generated k-algebras. Once you pick generators of the algebra, say $x_1, \dots, x_n$, then you can think of the scheme as sitting in n-space. More precisely, suppose R is a finitely-generated k-algebra, say

$$R = k[x_1, \dots, x_n]/I.$$

Then at least as a topological space, it is a closed subset of $\mathbb{A}^n$, with set $V(I)$. (We will later be able to say that it is a *closed subscheme*, but we haven't yet defined this phrase.)

Different choices of generators give us different ways of seeing $\operatorname{Spec} R$ as sitting in some affine space. These affine schemes already are very interesting. But when you glue them together, you can get even more interesting things. I'll now tell you about projective schemes.

As a warm-up, let me discuss $\mathbb{P}^n_k$ again.

Intuitive idea: We think of closed points of $\mathbb{P}^n$ as $[x_0; x_1; \cdots; x_n]$, not all zero, with an equivalence relation $[x_0; \cdots; x_n] = [\lambda x_0; \cdots; \lambda x_n]$. $x_0^2 + x_2^2$ isn't a function on $\mathbb{P}^n$. But $x_0^2 + x_2^2 = 0$ makes sense. And $(x_0^2 + x_2^2)/(x_1^2 + x_2 x_3)$ *is* a function on $\mathbb{P}^2 - \{x_1^2 + x_2 x_3 = 0\}$. We have $n+1$ patches, corresponding to $x_i = 0$ ($0 \leq i \leq n$). Where $x_0 \neq 0$, we have a patch $[x_0; x_1; x_2] = [1; u_1; u_2]$, and similarly for $x_1 \neq 0$ and $x_2 \neq 0$.

More precisely: We defined $\mathbb{P}^n$ by gluing together $n+1$ copies of $\mathbb{A}^n$. Let me show you this in the case of $\mathbb{P}^2_k =$ "$\{[x_0; x_1; x_2]\}$". Let's pick co-ordinates wisely. The first patch is $U_0 = \{x_0 \neq 0\}$. We imagine $[x_0; x_1; x_2] = [1; x_{1/0}; x_{2/0}]$. The patch will have coordinates $x_{1/0}$ and $x_{2/0}$, i.e. it is $\operatorname{Spec} k[x_{1/0}, x_{2/0}]$.

Similarly, the second patch is $U_1 = \{x_1 \neq 0\} = \operatorname{Spec} k[x_{0/1}, x_{2/1}]$. We imagine $[x_0; x_1; x_2] = [x_{0/1}; 1; x_{2/1}]$.

Finally, the third patch is $U_2 = \{x_2 \neq 0\} = \operatorname{Spec} k[x_{0/2}, x_{1/2}]$, with "$[x_0; x_1; x_2] = [x_{0/2}; x_{1/2}; 1]$".

We glue U_0 along $x_{1/0} \neq 0$ to U_1 along $x_{0/1} \neq 0$. Our identification (from $[1; x_{1/0}; x_{2/0}] = [x_{0/1}; 1; x_{2/1}]$) is given by $x_{1/0} = 1/x_{0/1}$ and $x_{2/0} = x_{2/1} x_{1/0}$. $U_{01} := U_0 \cap U_1 = \operatorname{Spec} k[x_{1/0}, x_{2/0}, 1/x_{1/0}] \cong \operatorname{Spec} k[x_{0/1}, x_{2/1}, 1/x_{0/1}]$, where the isomorphism was as just described.

We similarly glue together U_0 and U_2, and U_1 and U_2. You could show that all this is compatible, and you could imagine that this is annoying to show. I'm not going to show you the details, because I'll give you a slick way around this naive approach fairly soon.

Suppose you had a homogeneous polynomial, such as $x_0^2 + x_1^2 = x_2^2$. (Intuition: I want a homogeneous polynomial, because in my intuitive notion of projective space as $[x_0; \cdots; x_n]$, I can make sense of where a homogeneous polynomial vanishes, but I can't make as good sense of where an inhomogeneous polynomial vanishes.)

Then I claim that this defines a scheme "in" projective space (in the same way that $\operatorname{Spec} k[x_1, \ldots, x_n]/I$ was a scheme "in" $\mathbb{A}^n$). Here's how. In the patch U_0, I interpret this as $1 + x_{1/0}^2 = x_{2/0}^2$. In patch U_1, I interpret it as $x_{0/1}^2 + 1 = x_{2/1}^2$. On the overlap U_{01}, these two equations are the same: the first equation in $\operatorname{Spec} k[x_{1/0}, x_{2/0}, 1/x_{1/0}]$ is the second equation in $\operatorname{Spec} k[x_{0/1}, x_{2/1}, 1/x_{0/1}]$ [do algebra], unsurprisingly. So piggybacking on that annoying calculation that $\mathbb{P}^2$ consists of 3 pieces glued together nicely is the fact that this scheme consists of three schemes glued together nicely. Similarly, any homogeneous polynomials $x_0, \ldots, x_n$ describes some nice scheme "in" $\mathbb{P}^n$.

1.1. Exercise. Show that an irreducible homogeneous polynomial in $n+1$ variables (over a field k) describes an integral scheme of dimension $n-1$. We think of this as a "hypersurface in $\mathbb{P}^n_k$". Definition: The degree of the hypersurface is the degree of the polynomial. (Other definitions: degree 1 = hyperplane, degree 2, 3, … = quadric hypersurface, cubic, quartic, quintic, sextic, septic, octic, …; a quadric curve is usually called a conic curve, or a conic for short.) Remark: $x_0^2 = 0$ is degree 2.

I could similarly do this with a bunch of homogeneous polynomials. For example:

1.2. Exercise. Show that $wx = yz, x^2 = wy, y^2 = xz$ describes an irreducible curve in $\mathbb{P}^3_k$ (the twisted cubic!).

1.3. Tentative definitions. Any scheme described in this way ("in $\mathbb{P}^n_k$") is called a *projective* k*-scheme*. We're not using anything about k being a ring, so similarly if A is a ring, we can define a projective A-scheme. (I did the case $A = k$ first because that's the more classical case.) If I is the ideal in $A[x_0, \dots, x_n]$ generated by these homogeneous polynomials, we say that the scheme we have constructed is $\operatorname{Proj} A[x_0, \dots, x_n]/I$.

1.4. Examples of projective k-schemes "in" $\mathbb{P}^2_k$: $x = 0$ ("line"), $x^2 + y^2 = z^2$ ("conic"). $wx = yz$ ("smooth quadric surface"). $y^2 z = x^3 - xz$ ("smooth cubic curve"). ($\mathbb{P}^2_k$)

You imagine that we will have a map $\operatorname{Proj} A[x_0, \dots, x_n]/I$ to $\operatorname{Spec} A$. And indeed we will once we have a definition of morphisms of schemes.

The *affine cone* of $\operatorname{Proj} R$ is $\operatorname{Spec} R$. The picture to have in mind is an actual cone. (I described it in the cases above, §1.4.) Intuitively, you could imagine that if you discarded the origin, you would get something that would project onto $\operatorname{Proj} R$. That will be right, but right now we don't know what maps of schemes are.

The *projective cone* of $\operatorname{Proj} R$ is $\operatorname{Proj} R[T]$, where T is one more variable. For example, the cone corresponding to the conic $\operatorname{Proj} k[x, y, z]/(x^2 + y^2 = z^2)$ is $\operatorname{Proj} k[x, y, z, T]/(x^2 + y^2 + z^2)$. I then discussed this in the cases above, in §1.4.

1.5. Exercise. Show that the projective cone of $\operatorname{Proj} R$ has an open subscheme that is the affine cone, and that its complement $V(T)$ can be associated with $\operatorname{Proj} R$ (as a topological space). (More precisely, setting $T = 0$ cuts out a scheme isomorphic to $\operatorname{Proj} R$.)

2. A MORE GENERAL NOTION OF Proj

Let's abstract these notions. In the examples we've been doing, we have a graded ring $S = k[x_0, \dots, x_n]/I$ where I is a *homogeneous ideal* (i.e. I is generated by homogeneous elements of $k[x_0, \dots, x_n]$). Here we are taking the usual grading on $k[x_0, \dots, x_n]$, where each x_i has weight 1. Then S is also a graded ring, and we'll call its graded pieces S_0, S_1, etc. (In a graded ring: $S_m \times S_n \to S_{m+n}$. Note that S_0 is a subring, and S is a S_0-algebra.)

Notice in our example that $S_0 = k$, and S is generated over S_0 by S_1.

2.1. Definition. Assume for the rest of the day that S_* is a graded ring (with grading $\mathbb{Z}^{\geq 0}$). It is automatically a module over S_0. Suppose S_0 is a module over some ring A. (Imagine that $A = S_0 = k$.) Now $S_+ := \oplus_{i>0} S_i$ is an ideal, which we will call the *irrelevant ideal*; suppose that it is a finitely generated ideal.

2.2. Exercise. Show that S_* is a finitely-generated S_0-algebra.

Here is an example to keep in mind: $S_* = k[x_0, x_1, x_2]$ (with the usual grading). In this case we will build $\mathbb{P}^2_k$.

I will now define the scheme, that I will denote $\operatorname{Proj} S_*$. I will define it as a *set*, with a *topology*, and a *structure sheaf*. It will be enlightening to picture this in terms of the *affine cone* $\operatorname{Spec} S_*$. We will think of $\operatorname{Proj} S_*$ as the affine cone, minus the origin, modded out by multiplication by scalars.

The points of $\operatorname{Proj} S_*$ are defined to be the homogeneous prime ideals, except for any ideal containing the irrelevant ideal. (I waved my hands in the air linking this to $\operatorname{Spec} S_*$.)

We'll define the topology by defining the closed subsets. The closed subsets are of the form $V(I)$, where I is a homogeneous ideal. Particularly important open sets will the *distinguished open sets* $D(f) = \operatorname{Proj} S_* - V(f)$, where $f \in S_+$ is homogeneous. They form a base for the same reason as the analogous distinguished open sets did in the affine case.

Note: If $D(f) \subset D(g)$, then $f^n \in (g)$ for some n, and vice versa. We've done this before in the affine case .

Clearly $D(f) \cap D(g) = D(fg)$, by the same immediate argument as in the affine case.

We define $\mathcal{O}_{\operatorname{Proj} S_*}(D(f)) = (S_f)_0$, where $(S_f)_0$ means the 0-graded piece of the graded ring (S_f). As before, we check that this is well-defined (i.e. if $D(f) = D(f')$, then we are defining the same ring). In our example of $S_* = k[x_0, x_1, x_2]$, if we take $f = x_0$, we get $(k[x_0, x_1, x_2]_{x_0})_0 := k[x_{1/0}, x_{2/0}]$.

We now check that this is a sheaf. I could show that this is a sheaf on the base, and the argument would be the same. But instead, here is a trickier argument: I claim that

$$(D(f), \mathcal{O}_{\operatorname{Proj} S_*}) \cong \operatorname{Spec}(S_f)_0.$$

You can do this by showing that the distinguished base elements of $\operatorname{Proj} R$ contained in $D(f)$ are precisely the distinguished base elements of $\operatorname{Spec}(S_f)_0$, and the two sheaves have identifiable sections, and the restriction maps are the same.

2.3. Important Exercise. Do this. (Caution: don't assume $\deg f = 1$.)

2.4. *Example:* $\mathbb{P}^n_A$. $\mathbb{P}^n_A = \operatorname{Proj} A[x_0, \dots, x_n]$. This is great, because we didn't have to do any messy gluing.

2.5. Exercise. Check that this agrees with our earlier version of projective space.

2.6. *Exercise.* Show that $Y = \mathbb{P}^2 - (x^2 + y^2 + z^2 = 0)$ is affine, and find its corresponding ring (= find its ring of global sections).

We like this definition for a more abstract reason. Let V be an $n+1$-dimensional vector space over k. (Here k can be replaced by A as well.) Let $\operatorname{Sym}^{\cdot} V^* = k \oplus V^* \oplus \operatorname{Sym}^2 V^* \oplus \cdots$. The dual here may be confusing; it's here for reasons that will become apparent far later.)

If for example V is the dual of the vector space with basis associated to $x_0, \dots, x_n$, we would have $\operatorname{Sym}^{\cdot} V^* = k[x_0, \dots, x_n]$. Then we can define $\operatorname{Proj} \operatorname{Sym}^{\cdot} V^*$. (This is often called $\mathbb{P}V$.) I like this definition because it doesn't involved choosing a basis of V. [Picture of vector bundle, and its projectivization.]

If S_* is generated by S_1 (as a S_0-algebra), then $\operatorname{Proj} S_*$ "sits in $\mathbb{P}^n_A$". (Terminology: *generated in degree* 1.) $k[\operatorname{Sym}^* S_1] = k[x, y, z] \to\to S_*$ implies $S = k[x, y, z]/I$, where I is a homogeneous ideal. Example: $S_* = k[x, y, z]/(x^2 + y^2 - z^2)$. It sits naturally in $\mathbb{P}^2$.

Next day: I'll describe some nice properties of projective S_0-schemes.

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