FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 1

CONTENTS

Today: About this course. Why algebraic geometry? Motivation and program. Crash course in category theory: universal properties, Yoneda's lemma.

1. WELCOME

Welcome! This is Math 216A, Foundations of Algebraic Geometry, the first of a three-quarter sequence on the topic. I'd like to tell you a little about what I intend with this course.

Algebraic geometry is a subject that somehow connects and unifies several parts of mathematics, including obviously algebra and geometry, but also number theory, and depending on your point of view many other things, including topology, string theory, etc. As a result, it can be a handy thing to know if you are in a variety of subjects, notably number theory, symplectic geometry, and certain kinds of topology. The power of the field arises from a point of view that was developed in the 1960's in Paris, by the group led by Alexandre Grothendieck. The power comes from rather heavy formal and technical machinery, in which it is easy to lose sight of the intuitive nature of the objects under consideration. This is one reason why it used to strike fear into the hearts of the uninitiated.

The rough edges have been softened over the ensuing decades, but there is an inescapable need to understand the subject on its own terms.

This class is intended to be an experiment. I hope to try several things, which are mutually incompatible. Over the year, I want to cover the foundations of the subject fairly completely: the idea of varieties and schemes, the morphisms between them, their properties, cohomology theories, and more. I would like to do this rigorously, while

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trying hard to keep track of the geometric intuition behind it. This is the second time I will have taught such a class, and the first time I'm going to try to do this without working from a text. So in particular, I may find that I talk myself into a corner, and may tell you about something, and then realize I'll have to go backwards and say a little more about an earlier something.

Some of you have asked what background will be required, and how fast this class will move. In terms of background, I'm going to try to assume as little as possible, ideally just commutative ring theory, and some comfort with things like prime ideals and localization. *(All my rings will be commutative, and have unit!)* The more you know, the better, of course. But if I say things that you don't understand, please slow me down in class, and also talk to me after class. Given the amount of material that there is in the foundations of the subject, I'm afraid I'm going to move faster than I would like, which means that for you it will be like drinking from a firehose, as one of you put it. If it helps, I'm very happy to do my part to make it easier for you, and I'm happy to talk about things outside of class. I also intend to post notes for as many classes as I can. They will usually appear before the next class, but not always.

In particular, this will not be the type of class where you can sit back and hope to pick up things casually. The only way to avoid losing yourself in a sea of definitions is to become comfortable with the ideas by playing with examples.

To this end, I intend to give problem sets, to be handed in. They aren't intended to be onerous, and if they become so, please tell me. But they *are* intended to force you to become familiar with the ideas we'll be using.

Okay, I think I've said enough to scare most of you away from coming back, so I want to emphasize that I'd like to do everything in my power to make it better, short of covering less material. The best way to get comfortable with the material is to talk to me on a regular basis about it.

One other technical detail: you'll undoubtedly have noticed that this class is schedule for Mondays, Wednesdays, and Fridays, 9–10:30, $4\frac{1}{2}$ hours per week, not the usual 3. That's not because I'm psychotic; it was presumably a mistake. So I'm going to take advantage of it, and most weeks just meet two days a week, and I'll propose usually meeting on Mondays and Wednesday. I'll be away for some days, and so I'll make up for it by meeting on Fridays as well some weeks. I'll warn you well in advance.

Office hours: I haven't decided if it will be useful to have formal office hours rather than being available to talk after class, and also on many days by appointment. One possibility would be to have office hours on the 3rd day of the week during the time scheduled for class. Another is to have it some afternoon. I'm open to suggestions.

Okay, let's get down to business. I'd like to say a few words about what algebraic geometry is about, and then to start discussing the machinery.

Texts: Here are some books to have handy. Hartshorne's *Algebraic Geometry* has most of the material that I'll be discussing. It isn't a book that you should sit down and read, but you might find it handy to flip through for certain results. It should be at the bookstore, and is on 2-day reserve at the library. Mumford's *Red Book of Varieties and Schemes* has a good deal of the material I'll be discussing, and with a lot of motivation too. That is also on 2-day reserve in the library. The second edition is strictly worse than the 1st, because someone at Springer retyped it without understanding the math, introducing an irritating number of errors. If you would like something gentler, I would suggest Shafarevich's books on algebraic geometry. Another excellent foundational reference is Eisenbud and Harris' book *The geometry of schemes*, and Harris' earlier book *Algebraic geometry* is a beautiful tour of the subject.

For background, it will be handy to have your favorite commutative algebra book around. Good examples are Eisenbud's *Commutative Algebra with a View to Algebraic Geometry*, or Atiyah and Macdonald's *Commutative Algebra*. If you'd like something with homological algebra, category theory, and abstract nonsense, I'd suggest Weibel's book *Introduction to Homological Algebra*.

2. WHY ALGEBRAIC GEOMETRY?

It is hard to define algebraic geometry in its vast generality in a couple of sentences. So I'll talk around it a bit.

As a motivation, consider the study of manifolds. Real manifolds are things that locally look like bits of real n-space, and they are glued together to make interesting shapes. There is already some subtlety here — when you glue things together, you have to specify what kind of gluing is allowed. For example, if the transition functions are required to be differentiable, then you get the notion of a differentiable manifold.

A great example of a manifold is a submanifold of $\mathbb{R}^n$ (consider a picture of a torus). In fact, any compact manifold can be described in such a way. You could even make this your definition, and not worry about gluing. This is a good way to think about manifolds, but not the best way. There is something arbitrary and inessential about defining manifolds in this way. Much cleaner is the notion of an *abstract manifold*, which is the current definition used by the mathematical community.

There is an even more sophisticated way of thinking about manifolds. A differentiable manifold is obviously a topological space, but it is a little bit more. There is a very clever way of summarizing what additional information is there, basically by declaring what functions on this topological space are differentiable. The right notion is that of a sheaf, which is a simple idea, that I'll soon define for you. It is true, but non-obvious, that this ring of functions that we are declaring to be differentiable determines the differentiable manifold structure.

Very roughly, algebraic geometry, at least in its geometric guise, is the kind of geometry you can describe with polynomials. So you are allowed to talk about things like $y^2 = x^3 + x$, but not $y = \sin x$. So some of the fundamental geometric objects under consideration are things in n-space cut out by polynomials. Depending on how you define them, they are called *affine varieties* or *affine schemes*. They are the analogues of the patches on a manifold. Then you can glue these things together, using things that you can describe with polynomials, to obtain more general varieties and schemes. So then we'll have these algebraic objects, that we call varieties or schemes, and we can talk about maps between them, and things like that.

In comparison with manifold theory, we've really restricted ourselves by only letting ourselves use polynomials. But on the other hand, we have gained a huge amount too. First of all, we can now talk about things that aren't smooth (that are *singular*), and we can work with these things. (One thing we'll have to do is to define what we mean by smooth and singular!) Also, we needn't work over the real or complex numbers, so we can talk about arithmetic questions, such as: what are the rational points on $y^2 = x^3 + x^2$? (Here, we work over the field $\mathbb{Q}$.) More generally, the recipe by which we make geometric objects out of things to do with polynomials can generalize drastically, and we can make a geometric object out of rings. This ends up being surprisingly useful — all sorts of old facts in algebra can be interpreted geometrically, and indeed progress in the field of commutative algebra these days usually requires a strong geometric background.

Let me give you some examples that will show you some surprising links between geometry and number theory. To the ring of integers $\mathbb{Z}$, we will associate a smooth curve $\operatorname{Spec} \mathbb{Z}$. In fact, to the ring of integers in a number field, there is always a smooth curve, and to its orders (subrings), we have singular = non-smooth curves.

An old flavor of Diophantine question is something like this. Given an equation in two variables, $y^2 = x^3 + x^2$, how many rational solutions are there? So we're looking to solve this equation over the field $\mathbb{Q}$. Instead, let's look at the equation over the field $\mathbb{C}$. It turns out that we get a complex surface, perhaps singular, and certainly non-compact. So let me separate all the singular points, and compactify, by adding in points. The resulting thing turns out to be a compact oriented surface, so (assuming it is connected) it has a genus g, which is the number of holes it has. For example, $y^2 = x^3 + x^2$ turns out to have genus 0. Then Mordell conjectured that if the genus is at least 2, then there are at most a finite number of rational solutions. The set of complex solutions somehow tells you about the number of rational solutions! Mordell's conjecture was proved by Faltings, and earned him a Fields Medal in 1986. As an application, consider Fermat's Last Theorem. We're looking for integer solutions to $x^n + y^n = z^n$. If you think about it, we are basically looking for rational solutions to $X^n + Y^n = 1$. Well, it turns out that this has genus $\binom{n-1}{2}$ — we'll verify something close to this at some point in the future. Thus if n is at least 4, there are only a finite number of solutions. Thus Falting's Theorem implies that for each $n \geq 4$, there are only a finite number of counterexamples to Fermat's last theorem. Of course, we now know that Fermat is true — but Falting's theorem applies much more widely — for example, in more variables. The equations $x^3 + y^2 + z^14 + xy + 17 = 0$ and $3x^{14} + x^{34}y + \cdots = 0$, assuming their complex solutions form a surface of genus at least 2, which they probably do, have only a finite number of solutions.

So here is where we are going. Algebraic geometry involves a new kind of "space", which will allow both singularities, and arithmetic interpretations. We are going to define these spaces, and define maps between them, and other geometric constructions such as vector bundles and sheaves, and pretty soon, cohomology groups.

In order to think about these notions clearly and cleanly, it really helps to use the language of categories. There is not much to know about categories to get started; it is just a very useful language.

Here is an informal definition. I won't give you the precise definition unless you really want me to. A category has some *objects*, and some maps, or *morphisms*, between them. (For the pedants, I won't worry about sets and classes. And I'm going to accept the axiom of choice.) *The prototypical example to keep in mind is the category of sets.* The objects are sets, and the morphisms are maps of sets. Another good example is that of vector spaces over your favorite filed k. The objects are k-vector spaces, and the morphisms are linear transformations.

For each object, there is always an *identity morphism* from from the object to itself. There is a way of composing morphisms: if you have a morphism $f : A \to B$ and another $g : B \to C$, then there is a composed morphism $g \circ f : A \to C$. I could be pedantic and say that we have a map of sets $\mathrm{Mor}(A, B) \times \mathrm{Mor}(B, C) \to \mathrm{Mor}(A, C)$. Composition is associative: $(h \circ g) \circ f = h \circ (g \circ f)$. When you compose with the identity, you get the same thing.

Exercise. A category in which each morphism is an isomorphism is called a *groupoid*. (a) A perverse definition of a group is: a groupoid with one element. Make sense of this. (b) Describe a groupoid that is not a group. (This isn't an important notion for this course. The point of this exercise is to give you some practice with categories, by relating them to an object you know well.)

Here are a couple of other important categories. If R is a ring, then R-modules form a category. In the special case where R is a field, we get the category of vector spaces. There is a category of rings, where the objects are rings, and the morphisms are morphisms of rings (which I'll assume send 1 to 1).

If we have a category, then we have a notion of isomorphism between two objects (if we have two morphisms $f : A \to B$ and $g : B \to A$, *both* of whose compositions are the identity on the appropriate object), and a notion of automorphism.

3.1. **Functors.** A *covariant functor* is a map from one category to another, sending objects to objects, and morphisms to morphisms, such that everything behaves the way you want it to; if $F : \mathcal{A} \to \mathcal{B}$, and $a_1, a_2 \in \mathcal{A}$, and $m : a_1 \to a_2$ is a morphism in $\mathcal{A}$, then $F(m)$ is a morphism from $F(a_1) \to F(a_2)$ in B. Everything composes the way it should.

Example: If $\mathcal{A}$ is the category of complex vector spaces, and $\mathcal{B}$ is the category of sets, then there is a forgetful functor where to a complex vector space, we associate the set of its elements. Then linear transformations certainly can be interpreted as set maps.

A *contravariant functor* is just the same, except the arrows switch directions: in the above language, $F(m)$ is now an arrow from $F(a_2)$ to $F(a_1)$.

Example: If $\mathcal{A}$ is the category complex vector spaces, then taking duals gives a contravariant functor $\mathcal{A} \to \mathcal{A}$. Indeed, to each linear transformation $V \to W$, we have a dual transformation $W^* \to V^*$.

3.2. **Universal properties.** Given some category that we come up with, we often will have ways of producing new objects from old. In good circumstances, such a definition can be made using the notion of a *universal property*. Informally, we wish that there is an object with some property. We first show that if it exists, then it is essentially unique, or more precisely, is unique up to unique isomorphism. Then we go about constructing an example of such an object.

A good example of this, that you may well have seen, is the notion of a tensor product of R-modules. The way in which it is often defined is as follows. Suppose you have two R-modules M and N. Then the tensor product $M \otimes_R N$ is often first defined for people as follows: elements are of the form $m \otimes n$ ($m \in M$, $n \in N$), subject to relations $(m_1+m_2)\otimes n = m_1 \otimes n + m_2 \otimes n$, $m \otimes (n_1+n_2) = m \otimes n_1 + m \otimes n_2$, $r(m \otimes n) = (rm) \otimes n = m \otimes n$ (where $r \in R$).

Special case: if R is a field k, we get the tensor product of vector spaces.

Exercise (if you haven't seen tensor products before). Calculate $\mathbb{Z}/10 \otimes_{\mathbb{Z}} \mathbb{Z}/12$. (The point of this exercise is to give you a very little hands-on practice with tensor products.)

This is a weird definition!! And this is a clue that it is a "wrong" definition. A better definition: notice that there is a natural R-bilinear map $M \times N \to M \otimes_R N$. *Any* R-bilinear map $M \times N \to C$ factors through the tensor product uniquely: $M \times N \to M \otimes_R N \to C$. This is kind of clear when you think of it.

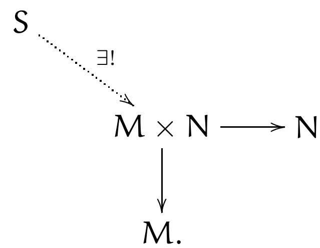
I could almost take this as the *definition* of the tensor product. Because if I could create something satisfying this property, $(M \otimes_R N)'$, and you were to create something else $(M \otimes_R N)''$, then by my universal property for $C = (M \otimes_R N)''$, there would be a unique map $(M \otimes_R N)' \to (M \otimes_R N)''$ interpolating $M \times N \to (M \otimes_R N)''$, and similarly by your universal property there would be a unique universal map $(M \otimes_R N)'' \to (M \otimes_R N)'$. The composition of these two maps in one order

$$(M \otimes_R N)' \to (M \otimes_R N)'' \to (M \otimes_R N)'$$

has to be the identity, by the universal property for $C = (M \otimes_R N)'$, and similarly for the other composition. Thus we have shown that these two maps are inverses, and our two spaces are isomorphic. In short: our two definitions may not be the *same*, but there is a canonical isomorphism between them. Then the "usual" construction works, but someone else may have another construction which works just as well.

I want to make three remarks. First, if you have never seen this sort of argument before, then you might think you get it, but you don't. So you should go back over the notes, and think about it some more, because it is rather amazing. Second, the language I would use to describe this is as follows: There is an R-bilinear map $t : M \times N \to M \otimes_R N$, unique up to unique isomorphism, defined by the following universal property: for any R-bilinear map $s : M \times N \to C$ there is a unique $f : M \otimes_R N \to C$ such that $s = f \circ t$. Third, you might notice that I didn't use much about the R-module structure, and indeed I can vary this to get a very general statement. This takes us to a powerful fact, that is very zen: it is very deep, but also very shallow. It's hard, but easy. It is black, but white. I'm going to tell you about it, and it will be mysterious, but then I'll show you some concrete examples.

Here is a motivational example: the notion of **product**. You have likely seen product defined in many cases, for example the notion of a product of manifolds. In each case, the definition agreed with your intuition of what a product should be. We can now make this precise. I'll describe product in the category of sets, in a categorical manner. Given two sets M and N, there is a unique set $M \times N$, along with maps to M and N, such that for *any other set* S *with maps to* M *and* N, this map must factor *uniquely* through $M \times N$:



You can immediately check that this agrees with the usual definition. But it has the advantage that we now have a definition in any category! The product may not exist, but if it does, then we know that it is unique up to unique isomorphism! (Explain.) This is handy even in cases that you understand. For example, one way of defining the product of two manifolds M and N is to cut them both up in to charts, then take products of charts, then glue them together. But if I cut up the manifolds in one way, and you cut them up in another, how do we know our resulting manifolds are the "same"? We could wave our hands, or make an annoying argument about refining covers, but instead, we should just show that they are indeed products, and hence the "same" (aka isomorphic).

3.3. **Yoneda's Lemma.** I want to begin with an easy fact that I'll state in a complicated way. Suppose we have a category $\mathcal{C}$. This isn't scary — just pick your favorite friendly low-brow category. Pick an object in your category $A \in \mathcal{C}$. Then for any object $C \in \mathcal{C}$, we have a set of morphisms $\mathrm{Mor}(C, A)$. If we have a morphism $f : B \to C$, we get a map of sets

(1) $$\mathrm{Mor}(C, A) \to \mathrm{Mor}(B, A),$$

just by composition: given a map from C to A, we immediately get a map from B to A by precomposing with f. In fancy language, we have a contravariant functor from the category $\mathcal{C}$ to the category of sets $\underline{\mathrm{Sets}}$. Yoneda's lemma, or at least part of it, says that this functor determines A up to unique isomorphism. Translation: If we have two objects A and A', and isomorphisms

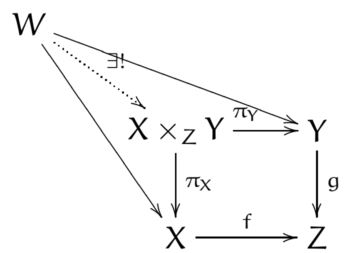
(2) $$i_C : \mathrm{Mor}(C, A) \to \mathrm{Mor}(C, A')$$

that commute with the maps (1), then the i_C must be induced from a unique morphism $A \to A'$.

Important Exercise. Prove this. This sounds hard, but it really is not. This statement is so general that there are really only a couple of things that you could possibly try. For example, if you're hoping to find an isomorphism $A \to A'$, where will you find it? Well, you're looking for an element $\mathrm{Mor}(A, A')$. So just plug in $C = A$ to (2), and see where the identity goes. (Everyone should prove Yoneda's Lemma once in their life. This is your chance.)

Remark. There is an analogous statement with the arrows reversed, where instead of maps into A, you think of maps *from* A.

Example: Fibered products. Suppose we have morphisms $X, Y \to Z$. Then the *fibered product* is an object $X \times_Z Y$ along with morphisms to X and Y, where the two compositions $X \times_Z Y \to Z$ agree, such that given any other object W with maps to X and Y (whose compositions to Z agree), these maps factor through some unique $W \to X \times_Z Y$:



The right way to interpret this is first to think about what it means in the category of sets. I'll tell you it, and let you figure out why I'm right: $X \times_Z Y = \{(x \in X, y \in Y) : f(x) = g(y)\}$.

In any category, we can make this definition, and we know thanks to Yoneda that if it exists, then it is unique up to unique isomorphism, and so we should reasonably be allowed to give it the name $X \times_Z Y$. We know what maps to it are: they are precisely maps to X and maps to Y that agree on maps to Z.

(Remark for experts: if our category has a final object, then the fibered product over the final object is just the product.)

The notion of fibered product will be important for us later.

Exercises on fibered product. (a) Interpret fibered product in the category of sets: If we are given maps from sets X and Y to the set Z, interpret $X \times_Z Y$. (This will help you build intuition about this concept.)
(b) A morphism $f : X \to Y$ is said to be a a **monomorphism** if any two morphisms $g_1, g_2 : Z \to X$ such that $f \circ g_1 = f \circ g_2$ must satisfy $g_1 = g_2$. This is the generalization of an injection of sets. Prove that a morphism is a monomorphism if and only if the natural morphism $X \to X \times_Y X$ is an isomorphism. (We may then take this as the definition of monomorphism.) (Monomorphisms aren't very central to future discussions, although they will come up again. This exercise is just good practice.)
(c) Suppose $X \to Y$ is a monomorphism, and $W, Z \to X$ are two morphisms. Show that $W \times_X Z$ and $W \times_Y Z$ are canonically isomorphic. (We will use this later when talking about fibered products.)
(d) Given $X \to Y \to Z$, show that there is a natural morphism $X \times_Y X \to X \times_Z X$. (This is trivial once you figure out what it is saying. The point of this exercise is to see why it is trivial.)

Important Exercise. Suppose $T \to R, S$ are two ring morphisms. Let I be an ideal of R. We get a morphism $R \to R \otimes_T S$ by definition. Let I^e be the extension of I to $R \otimes_T S$. (These are the elements $\sum_j i_j \otimes s_j$ where $i_j \in I$, $s_j \in S$. But it is more elegant to solve this exercise using the universal property.) Show that there is a natural isomorphism

$$R/I \otimes_T S \cong (R \otimes_T S)/I^e.$$

Hence the natural morphism $S \otimes_T R \to S \otimes_T R/I$ is a surjection. As an application, we can compute tensor products of finitely generated k algebras over k. For example,

$$k[x_1, x_2]/(x_1^2 - x_2) \otimes_k k[y_1, y_2]/(y_1^3 + y_2^3) \cong k[x_1, x_2, y_1, y_2]/(x_1^2 - x_2, y_1^3 + y_2^3).$$

Exercise. Define coproduct in a category by reversing all the arrows in the definition of product. Show that coproduct for sets is disjoint union.

I then discussed adjoint functors briefly. I will describe them again briefly next day.

Next day: more examples of universal properties, including direct and inverse limits. Groupification. Sheaves!

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