MODERN ALGEBRA (MATH 210) PROBLEM SET 7

1. Are the following ideals prime in $\mathbb{C}[x, y]$? (a) $(x, y-1)$, (b) (x, y^2) (c) $(y - x^2, y - 1)$.

2. Must any finite integral domain be a field?

3. Prove that the quotient ring $\mathbb{Z}[i]/I$ is finite for any nonzero ideal I of $\mathbb{Z}[i]$. (*Hint:* Use the fact that $I = (\alpha)$ for some nonzero α and then use the division algorithm to see that every coset of I is represented by an element of norm less than $N(\alpha)$.)

4.

(a) Prove that if an integer is the sum of two rational squares, then it is the sum of two integer squares. (For example, $13 = (1/5)^2 + (18/5)^2 = 2^2 + 3^2$.)
(b) Determine all the representations of the integer $2130797 = 17^2 \cdot 73 \cdot 101$ as the sum of two squares.
(c) Find $\gcd(47 - 13i, 53 + 56i)$ in $\mathbb{Z}[i]$.

5. Let I and J be ideals of a commutative ring R.

(a) Prove that $I + J$ is the smallest ideal of R containing both I and J.
(b) Prove that IJ is an ideal contained in $I \cap J$.
(c) Give an example where $IJ \neq I \cap J$.
(d) Prove that if $I + J = R$ then $IJ = I \cap J$.

6. *("Characterization of Noetherian rings")* Show that a ring in which all ideals are finitely generated cannot have an infinite sequence of ideals

$$I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \cdots .$$

Conversely, show that if a ring has no infinite sequence of ideals, then all ideals are finitely generated.

7. Note that $\omega = \frac{-1+\sqrt{-3}}{2}$ is a cube root of 1, and $\omega^2 + \omega + 1 = 0$.

(a) Prove that the subset $\{x + y\omega \in \mathbb{Z}[\omega] : x + y \text{ is divisible by } 3\}$ is an ideal of $\mathbb{Z}[\omega]$. Is it prime?
(b) Show that $\mathbb{Z}[\omega]$ is a Euclidean domain, hence a Unique Factorization Domain. (Hint: recall the proof for the Gaussian integers $\mathbb{Z}[i]$.)
(c) Factor 2, 3, 5, and 7 into primes in $\mathbb{Z}[\omega]$. (Which one of them has a repeated prime factor? This prime factor is key to an elementary proof of Fermat's Last Theorem for $n = 3$. If you'd like to see it, just ask me.)

This set is due Monday, Nov. 29 at noon at Jarod Alper's door, 380–J.

Date: Saturday, November 20, 2004.