MODERN ALGEBRA (MATH 210A) PRACTICE MIDTERM

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1. Suppose G is a group. Then $[G, G]$ is defined to be the subgroup generated by terms of the form $[x, y] = xyx^{-1}y^{-1}$. (This is the *commutator subgroup*.) Show that $[G, G]$ is a normal subgroup, and that $G/[G, G]$ is abelian.

2. Suppose the center of G has index n. Show that every conjugacy class has at most n elements.

3. Let $(\mathbb{Z}/24)^*$ be those integers (modulo 24) relatively prime to 24. Show that this set forms an abelian group. According to the classification of finitely generated abelian groups, $(\mathbb{Z}/24)^*$ is congruent to a product of cyclic groups of prime power order. Explicitly describe it in such a way.

4. Let A be an abelian normal subgroup of G and let B be any subgroup of G. Prove that $A \cap B \lhd AB$.

5. Let $K_4 = \mathbb{Z}/2 \times \mathbb{Z}/2$. Show that $\mathrm{Aut}(K_4) \cong S_3$. Let $\phi : S_3 \to \mathrm{Aut}(K_4)$ be an isomorphism. Show that $K_4 \rtimes_\phi S_3 \cong S_4$. (*Hint:* Show that S_4 is a semidirect product of K_4 and S_3, and figure out the induced action of S_3 on K_4.)

6. Suppose $M, N \lhd G$, $G = MN$. Show that $G/M \cap N \cong G/M \times G/N$.

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Date: Friday, October 29, 2004.