SOLUTION SKETCHES TO 18.034 PRACTICE PROBLEMS

Problem 1. Consider the differential equation

$$M(x,y) + N(x,y)y' = 0, \tag{1}$$

where M, N, M_y, N_x are continuous on the entire plane.

(a) What does it mean for equation (1) to be exact? What does this tell you about solutions to the the differential equation?

(b) What is the condition required on M and N such that equation (1) is exact?

(c) If equation (1) is not exact, then it may be made so by multiplying by an integrating factor $\mu(x)$ (a function only of x). Show that this integrating factor will make (1) exact if

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu.$$

Hence this method will only work if $\frac{M_y - N_x}{N}$ is a function of x only.

(d) Apply this technique to show that any solution to the differential equation $(3xy + y^2) + (x^2 + xy)y' = 0$ lies on a curve of the form $x^3y + x^2y^2/2 = \text{constant}$.

Solution. (b) The condition is $M_y = N_x$, see Theorem 2.8.1 on p. 84. (c) See p. 87 of the text. (d) In this case, $\frac{M_y - N_x}{N} = 1/x$, so we want to solve $d\mu/dx = \mu/x$. $\mu = x$ works, from which we get $x^3y + \frac{1}{2}x^2y^2 = c$.

Problem 2. Consider the differential equation

$$t^2y'' - (t+2)ty' + (t+2)y = 0.$$

Notice that $y_1 = t$ is a solution. Find all solutions on the interval $t > 0$. Hint: Suppose you had another solution y_2. Find the Wronskian, using Abel's theorem, and from this construct a differential equation satisfied by y_2.

Solution. Rewrite the equation as

$$y'' + (-2/t - 1)y' + (2/t^2 + 1/t)y = 0.$$

By the existence and uniqueness theorem, there is a two-dimensional family of solutions on this range. $y_1 = t$ is one solution, so we seek another solution linearly independent from this one. If y_2 is another solution, the then Wronskian $W(y_1, y_2)$ is some multiple of t^2e^t. We will find a y_2 where $W(y_1, y_2) = t^2e^t$, i.e.

$$ty_2' - y_2 = t^2e^t.$$

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Use an integrating factor of $1/t$, and integrate to get $y_2/t = e^t + C$. Thus $y_2 = te^t$ is another solution. (Check that it works!) Thus all solutions are of the form $At + Bte^t$.

Problem 3. Consider the differential equation

$$xy'' + 2y' + xy = 0.$$

(a) Which values of x are ordinary points? Regular singular points? (Hint: there is one; call it x_0.) Irregular singular points (i.e. neither ordinary nor regular singular)?

(b) Solve the *indicial equation* at the regular singular point x_0 to get the *exponents of the singularity* r_1, r_2.

(c) Find two linearly independent solutions of the form

$$(x - x_0)^{r_i} \sum_{n=0}^{\infty} a_n x^n.$$

(d)* Show that if $y(-\pi) \neq 0$, then as t approaches 0 from the left (i.e. from the negative side), $y(t)$ becomes (either positively or negatively) infinite. (Hint: Express the solutions from (c) in terms of elementary functions.)

Solution. (a) x is regular unless $x = 0$, in which case it is regular singular.

(b) The indicial equation is $r(r-1) + 2r = 0$, so $r = -1$ or 0.

(c) One solution corresponding to $r = -1$ (there is some choice involved) is

$$y = 1/x - x/2! + x^3/4! - x^5/6! \ldots$$

and the solution corresponding to $r = 0$ is

$$y = 1 - x^2/3! + x^4/5! \ldots.$$

(d) The solutions are $\sin(x)/x$ and $\cos(x)/x$. A solution for $x < 0$ is of the form $(A\sin(x) + B\cos(x))/x$. Note that $\sin(x)/x$ doesn't blow up near $x = 0$. The condition given shows that $B \neq 0$, so the solution blows up.

Problem 4. Consider the differential equation coming from a door with friction

$$y'' + \gamma y' + y = 0.$$

Here γ corresponds to the friction; assume it is small but positive. Show that as γ increases, the period of oscillation increases.

Solution. The frequency of the oscillation is the imaginary part of the roots of $r^2 + \gamma r + 1 = 0$, which turns out to be $(1 - (\gamma/2)^2)^{1/2}$. As γ increases, the frequency decreases, so the period increases. (See Feb. 28 class.)

Problem 5. Show that t^3 and t^4 can't both be solutions to a differential equation of the form $y'' + qy' + ry = 0$ where q and r are continuous functions defined on the real numbers. Can t^3 and t^4 be solutions to a differential equation of the form $py'' + qy' + ry = 0$ where p, q and r are continuous functions defined on the real numbers?

Solution. Their Wronskian vanishes at 0, so they can't be a solution to a differential equation of the form $y''+qy'+ry = 0$ where q and r are continuous functions defined on the real numbers. However, they *are* solutions to the differential equation $t^2y'' - 6ty' + 12y = 0$.

Problem 6. Solve the system of equations

$$x' = 4x + y,$$

$$y' = -x + 2y.$$

Sketch the phase portrait.

Solution. The eigenvalues are 3 and 3; the system is incomplete, and has only one eigenvector $(1, -1)$.

The solution is

$$e^{3t}\left(c_1(t, 1-t) + c_2(1, -1)\right).$$

Problem 7. Consider the system of differential equations

$$x' = -x - y,$$

$$y' = x - y.$$

(a)What are the eigenvectors and eigenvalues?

(b) Sketch the phase portrait.

(c) Find a fundamental matrix $\Psi(t)$ for the system. If

$$A = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$$

(d) Show that $e^{2\pi A} = \begin{pmatrix} e^{-2\pi} & 0 \\ 0 & e^{-2\pi} \end{pmatrix}$, where e^{At} is defined to be $\Psi(t)\Psi(0)^{-1}$.

Solution. (a) $\lambda_1 = -1 + i$, $\lambda_2 = -1 - i$, $v_1 = (i, 1)$, $v_2 = (-i, 1)$.

(b) Spiraling in counterclockwise.

(c) One possibility is

$$\begin{pmatrix} ie^{(-1+i)t} & -ie^{(-1-i)t} \\ e^{(-1+i)t} & e^{(-1-i)t} \end{pmatrix}.$$

Problem 8. (a) Find the coefficient of x^n in the power series expansion of $(1+x)^r$, where r is a real number.

(b) Show that $e^{ax}e^{bx} = e^{(a+b)x}$ by explicitly multiplying out the power series expansions of both sides.

(c) Show that $2\sin(x)\cos(x) = \sin(2x)$ by explicitly multiplying out the power series expansions of both sides.

Solution. (a) The coefficient of x^n is $1/n!$ times the nth derivative of $(1+x)^r$ — which is $r(r-1)\cdots(r-n+1)(1+x)^{r-n}$ — evaluated at 0. Hence the answer is $r(r-1)\dots(r-n+1)/n!$. (b)

$$e^{ax} = \sum_{i=0}^{\infty} a^i x^i/i!$$

$$e^{bx} = \sum_{j=0}^{\infty} b^j x^j/j!$$

so

$$e^{ax}e^{bx} = \sum_{k=0}^{\infty} \sum_{i+j=k} a^i b^j x^k/(i!j!).$$

On the other hand,

$$e^{(a+b)x} = \sum_{k=0}^{\infty} (a+b)^k x^k/k!.$$

So these two are the same if their coefficient of x^k is the same, or equivalently if

$$(a+b)^k = \sum_{i+j=k} \frac{k!}{i!j!} a^i b^j.$$

But this is just the binomial theorem.

(c) Similar to (b).

Problem 9. consider the differential equation

$$(x-1)^2 y'' - (x-1)y' + 2y = 0. \tag{2}$$

(a) What is a regular singular point? Show that 1 is a regular singular point of equation (2).

(b) Find all solutions away from $x = 1$.

(c) Sketch a non-zero solution of equation (2) near $x = 1$.

(d) Why do solutions of $(x-1)^2y'' - (x-1)y' + (2 + (x-1)^4)y = 0$ have similar behavior?

Solution.

(a) See Section 5.4.

(b) This is an Euler equation. We solve $r(r-1) - r + 2 = 0$, i.e. $r^2 - 2r + 2 = 0$, which has solutions $1 + i$ and $1 - i$. Hence solutions are of the form $y = |x|(c_1 \cos(\ln|x|) + c_2 \sin(\ln|x|))$.

(c) The solution oscillates more greater and greater frequency and and greater and greater amplitude.

(d) See Sections 5.6–5.7.

Problem 10. (a) By the method of variation of parameters show that the solution of the initial value problem

$$y'' + 2y' + 2y = f(t),$$

$y(0) = 0$, $y'(0) = 0$ is

$$y = \int_0^t e^{-(t-\tau)} f(\tau) \sin(t - \tau) d\tau.$$

(b) Show that if $f(t) = \delta(t - \tau)$, then the solution of part (a) reduces to

$$y = u_\pi(t) e^{-(t-\pi)} \sin(t - \pi).$$

Sketch this solution.

(c) Use the Laplace transform to solve the given initial value problem with $f(t) = \delta(t - \pi)$ and confirm that the solution agrees with the result of part (b).

Solution. See 6.5 Problem 21.