# 18.034 MIDTERM 2: SKETCHES OF SOLUTIONS

Explain your answers clearly; show all steps. Calculators may not be used. All problems have equal value. Please put your name on every sheet. Good luck!

**1.** (a) $y_1$, $y_2$, and $y_3$ are 3 solutions of the differential equation $(1-t)y''' + y'' + t^2y' + t^3y = 0$ on the interval $1 < t < \infty$. Calculate the function $W(y_1, y_2, y_3)(t)$ given that $W(y_1, y_2, y_3)(2) = 3$.

*Solution.* Rewrite the differential equation as $y''' + \frac{1}{1-t}y'' + \frac{t^2}{1-t}y' + \frac{t^3}{1-t}y = 0$. Then by Abel's theorem, $W = c\exp(-\int \frac{1}{1-t}dt) = c(1-t)$ for some constant $t$. From the condition $W(2) = 3$, we get $W(t) = 3(t-1)$.

(b) The equation $y' + a(x)y = 0$ has for a solution

$$\phi(x) = e^{-\int_{x_0}^{x} a(t)dt}.$$

(Here let $a$ be continuous on an interval $I$ containing $x_0$.) This suggests trying to find a solution of

$$L(y) = y'' + a_1(x)y' + a_2(x)y = 0$$

of the form

$$\phi(x) = e^{\int_{x_0}^{x} p(t)dt}$$

where $p$ is a function to be determined. Show that $\phi$ is a solution of $L(y) = 0$ if and only if $p$ satisfies the first-order non-linear equation $y' = -y^2 - a_1(x)y - a_2(x)$. (Remark: This last equation is called a *Riccati equation.*)

*Solution.* $\phi(x) = e^{\int p(t)dt}$, so $\phi'(x) = p(x)e^{\int p(t)dt}$ and $\phi''(x) = p'(x)e^{\int p(t)dt} + p(x)^2e^{\int p(t)dt}$. If $\phi$ satisfies the differential equation, then

$$e^{\int p}\left(p' + p^2 + a_1p + a_2\right) = 0$$

from which the result follows.

**2.** (a) Consider the equation $y'' - \frac{2}{x^2}y = 0$ (for $0 < x < \infty$). Find all solutions. (*Hint:* Try functions of the form $y = x^r$. How do you know you've found *all* the solutions?)

(b) Find all solutions to the equation $y'' - \frac{2}{x^2}y = x$. *Hint:* Use "variation of parameters". Suppose $\phi_1$ and $\phi_2$ are linearly independent solutions to the homogeneous version of the equation (see (a)). Look for a solution of the form $\phi(x) = u_1(x)\phi_1(x) + u_2(x)\phi_2(x)$.

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*Date*: April 14, 2000, 1:05-1:55 pm.

*Solution.* (a) $x^2$ and $1/x$ both work, so $C_1 x^2 + C_2/x$ work. These are all the solutions by the existence and uniqueness theorem (see for example Theorem 3.2.4).

(b) The general answer is $x^3/4 + Ax^3 + B/x$. This can be found using the "Variation of Parameters" formula, see Theorem 3.7.1.

Alternatively, here is the argument, explicitly. Let $\phi_1(x) = x^2$, $\phi_2(x) = 1/x$ be a basis for the space of solutions. We seek a single solution to the differential equation, as we already know the solutions to the homogeneous version.

We look for a solution $\phi = u_1\phi_1 + u_2\phi_2$, such that

$$u_1'\phi_1 + u_2'\phi_2 = 0 \tag{1}$$

Then $\phi' = u_1\phi_1' + u_2\phi_2'$, and

$$\phi'' = (u_1\phi_1'' + u_2\phi_2'') + (u_1'\phi_1' + u_2'\phi_2').$$

As $\phi" - (2/x^2)\phi = x$, we have

$$u_1'\phi_1' + u_2'\phi_2' = x. \tag{2}$$

Rewriting (1) and (2):

$$\begin{aligned} u_1'x^2 + u_2'/x &= 0 \\ u_1'(2x) + u_2'(-1/x^2) &= x \end{aligned}$$

and solving this systems gives $u_1' = \frac{1}{3}$, $u_2' = -\frac{1}{3}x^2$.

Take $u_1 = \frac{1}{3}x$, $u_2 = -\frac{1}{12}x^4$. Then

$$\phi = \frac{1}{3}x(x^2) - \frac{1}{12}x^4(\frac{1}{x}) = \frac{1}{4}x^3.$$

To be safe, we check that $\phi(x) = x^3/4$ really does satisfy the differential equation.

**3.** Iterate $x \to \sqrt{1+x}$. Start with $x = 0$. What happens?

*Solution.* The Contraction Mapping Theorem applies to the interval $0 \le x < \infty$, as if $f(x) = \sqrt{1+x}$ then $f$ maps the interval to itself, and $f'(x) = 1/(2\sqrt{1+x})$, so $|f'(x)| \le 1/2$. Hence we approach a fixed point $x_0$, satisfying $x_0 = \sqrt{1+x_0}$. Squaring and solving, we get $x_0 = (1 \pm \sqrt{5})/2$. As $x_0$ must lie in the interval, $x_0 = (1+\sqrt{5})/2$, the golden mean.

**4.** (a) State the Existence and Uniqueness Theorem for differential equations of the form $y' = f(x, y)$.

(b) Consider the differential equation $y' = t^2(y+1)$ on the interval $\mathbb{R}$, with initial condition $y(0) = 0$. Find a solution $y = \phi(t)$ defined for all $t \in \mathbb{R}$. If the first few Picard iterates (used in the proof of the Existence and Uniqueness Theorem described in (a)) are $\phi_0(t) = 0$, $\phi_1(t)$, $\phi_2(t)$, find $\phi_1(t)$ and $\phi_2(t)$.

(c) Explain why the $\phi_1(t)$ and $\phi_2(t)$ you found are approximations to $\phi(t)$.

*Solution.* (a) See practice midterm.

(b) From $y'/(y+1) = t$ we have $\ln|y+1| = t^2/3$, from which $y = e^{t^3/3} - 1$.

$\phi_{k+1}(t) = \int_0^t s^2(\phi_k(s)+1)ds$, from which inductively $\phi_1(t) = t^3/3$, $\phi_2(t) = t^3/3 + t^6/18$.

(c) The power series expansion (or Taylor series expansion) for $e^{t^3/3}$ begins

$$1 + \frac{(t^3/3)}{1!} + \frac{(t^3/3)^2}{2!} + \frac{(t^3/3)^3}{3!} + \cdots,$$

so the power series expansion for $e^{t^3/3} - 1$ begins

$$\frac{(t^3/3)}{1!} + \frac{(t^3/3)^2}{2!} + \frac{(t^3/3)^3}{3!} + \cdots.$$

In this case, the first few Picard iterates (and indeed all iterates) are partial sums of the power series.

**5.** Consider the equation $y'' + \cos(x)y' + \sin(x)y = 0$.

(a) Let $\phi(x)$ be a nontrivial solution, and let $\psi(x) = \phi(x+2\pi)$. Prove that $\psi(x)$ is also a solution.

(b) Show that $\phi(x)$ is a periodic solution of period $2\pi$ if, and only if, $\phi(0) = \phi(2\pi)$ and $\phi'(0) = \phi'(2\pi)$.

(c) Let $\phi_1(x)$, $\phi_2(x)$ be two solutions satisfying $\phi_1(0) = 1$, $\phi_1'(0) = 0$, $\phi_2(0) = 0$, $\phi_2'(0) = 1$. Show that there are constants $a$ and $b$ such that

$$\phi_1(x+2\pi) = a\phi_1(x) + b\phi_2(x).$$

(*Hint:* See (a).)

*Solution.* (a) $\psi(x)$ satisfies the differential equation $y'' + \cos(x-2\pi)y' + \sin(x-2\pi)y = 0$, which is the original differential equation.

(b) If $\phi(x)$ is a periodic solution of period $2\pi$, then by periodicity, $\phi(0) = \phi(2\pi)$ and $\phi'(0) = \phi'(2\pi)$. Conversely, if $\phi(0) = \phi(2\pi)$ and $\phi'(0) = \phi'(2\pi)$, then $\phi(0) = \psi(0)$ and $\phi'(0) = \psi'(0)$. As $\psi(x)$ and $\phi(x)$ have the same initial conditions and satisfy the same differential equation, by the Existence and Uniqueness Theorem (for second-order linear equations with continuous coefficients), $\phi(x) = \psi(x) = \phi(x+2\pi)$, i.e. $\phi$ is periodic.

(c) By the Wronskian test, $\phi_1(x)$ and $\phi_2(x)$ are linearly independent solutions of the differential equation, and hence form a basis for the solution space. As $\phi_1(x+2\pi)$ is also a solution, it is a linear combination of $\phi_1$ and $\phi_2$.

**6.** Let $A = \begin{pmatrix} 3 & 1 \\ -5 & -3 \end{pmatrix}$. Find the eigenvalues of $A$. Find eigenvectors of $A$ corresponding to each of the eigenvalues. Calculate $A^{2000}$.

*Solution.* The eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = -2$, and the corresponding eigenvectors are $\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 1 \\ -5 \end{pmatrix}$ respectively. (Any nonzero multiples of these are correct.) $A^{2000}\vec{v}_1 = 2^{2000}\vec{v}_1$ and $A^{2000}\vec{v}_2 = 2^{2000}\vec{v}_2$. As any vector $\vec{v}$ is a linear combination of $\vec{v}_1$ and $\vec{v}_2$, $A^{2000}\vec{v} = 2^{2000}\vec{v}$. Thus

$$A^{2000} = 2^{2000} I = \begin{pmatrix} 2^{2000} & 0 \\ 0 & 2^{2000} \end{pmatrix}.$$