18.034 MIDTERM 1: SKETCHES OF SOLUTIONS

Explain your answers clearly; show all steps. Calculators may not be used. All problems have equal value.

1. (a) Find the modulus (absolute value) and argument of $1+\sqrt{3}i$. Calculate $(1+\sqrt{3}i)^6$.
   (b) On the complex plane, sketch the location of the five solutions of $z^5=1$, i.e. the fifth roots of 1.
   Solution. (a) The modulus is 2, and the argument is $\pi/3$ (possibly plus some multiple of 2π). Hence $(1+\sqrt{3}i)^6 = 2^6 e^{2\pi i} = 64$. (Several people multiplied it out.) (b) They are the vertices of a regular pentagon, on the unit circle, with one vertex at (1,0).
2. Consider the autonomous differential equation $y' = (y^2-1)e^y$. What are the equilibrium values of y? Which are stable and which are unstable? For which values of $y(0)$ will the solution stay bounded? What will the limiting value of y be in these cases? (Your answer will depend on $y(0)$.) Sketch the integral curves (in the xy-plane).
   Solution. $y=1$ is unstable, $y=-1$ is stable. (0 is *not* an equilibrium value!) The solution will stay bounded for $y(0)\leq 1$. In these cases, the limit is -1 unless $y(0)=1$, in which case the limit is 1.
3. (a) Find the general solution of the differential equation
   $$y' = \frac{x+y}{x}.$$
   Hint: this equation is homogeneous, so try an appropriate substitution.
   Solution. Let $v=y/x$, so $xv'+v=1+v$, so $v'=1/x$, from which $v=\ln|x|+C$, so $y=x(\ln|x|+C)$. (Don't forget the absolute value signs in $\ln|x|$!)
   (b) For which constants k is
   $$ke^x\cos y + (e^x\sin y - \sin y)y' = 0$$
   an exact differential equation? In those cases, find the general solution ot the differential equation.
   Solution. Using the closed condition, $k=-1$. The solution is $(-e^x+1)\cos y = C$. (The solution is *not* $y=(-e^x+1)\cos y + C$!)
4. Consider the family of curves $y^2-x^2=c$, as c runs through the real numers. Find another family of curves that is everywhere orthogonal to this one. (In other words, at almost every point in the plane, the tangent to the corresponding curve in the first family is perpendicular to the tangent to the corresponding curve in the second family.)

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Solution. In this family, $y' = x/y$ (away from (0,0)), so in the orthogonal family, $y' = -y/x$. This is a separable equation: $-y'/y = 1/x$, so $-\ln|y| = \ln|x| + C$, from which $xy = C_2$. However, you must also check what happens when $x = 0$ and $y = 0$; otherwise you will only have the cases $C_2 \neq 0$. If you are careful with cases, you will see that something weird happens at $(x, y) = (0, 0)$. (Why shouldn't this be surprising?)

5. A not-uncommon calculus mistake is to believe that the product rule for derivatives says that $(fg)' = f'g'$. If $f(x) = x^2$, determine, with proof, whether there exists a nonzero function $g(x)$ defined on the set $(0, 2) = \{x|0 < x < 2\}$ such that the wrong product rule is true for x in $(0, 2)$.

   Solution. We are looking for g satisfying $(x^2 g)' = 2xg'$, i.e. $2xg + x^2 g' = 2xg'$, i.e. (where $x \neq 0, 2$) $g' - (2/(x-2))g = 0$. Multiply by integrating factor $e^{\int -2/(x-2)dx} = (x-2)^{-2}$. Hence one solution on $(0, 2)$ is $1/(x-2)^2$.

   Many of you got as far as the differential equation, and then pointed out that by the existence and uniqueness theorem, there is a solution, which is all you were asked to show.

6. Solve the differential equation $y'' + 4y = \cos(2.02t)$ with the initial conditions $y(0) = y'(0) = 0$. Sketch the solution. (Hint: you should see beats. How long are they, i.e. what is the period?) You may use the fact that $2.02^2 = 4.0804$, and $1/.0804 \cong 12.4$.

   Solution. The solution is $y = 1/.0804(\cos(2t) - \cos(2.02t)) = 1/.0804 \times 2\sin(2.01t)\sin(.01t)$. So the amplitude is about 24.8, the big beats have period 200π, and the small oscillations have period about π.