18.024 PRACTICE QUIZ IV

Quiz IV will be on Wednesday from 10 to 11:30 am. On Tuesday, Benoit will have office hours (2–251) from 10 am to 1 pm, and I will have office hours (2–271) from 4 to 7 pm.

1. Find the mass of the hemisphere $x^2 + y^2 + z^2 = a^2$, $z \geq 0$, if the density, in mass per unit area, is $\delta = z^2$. (This is a surface integral, not a volume integral!)

2. Let C be a simple closed curve in the xy-plane, traversed in the *clockwise* direction, and let I_z denote the moment of inertia about the z-axis of the region enclosed by C. Show that

$$I_z = q \oint_C x^3\,dy - y^3\,dx$$

for some q (and find q).

3. The set of points satisfying the equation $4x^2 + 4xy + 2y^2 - 2y = 3$ is an ellipse. Find the area of the region bounded by the ellipse. (A suitable linear transformation will carry this ellipse to a circle.)

4. Suppose S is the portion of the surface $z = 1 - x^2 - y^2$ above the xy-plane, and $\vec{F} = (e^{x+y+z}, -e^{x+y+z}, x^2 + y^2)$. Calculate $\iint_S \vec{F} \cdot \vec{n}\,dA$, where $\vec{n}$ is in the upwards direction. Hint: Use Stokes' theorem (note that $\vec{F} = \vec{\nabla} \times \vec{G}$, where $\vec{G}(x,y,z) = (-y^3/3, x^3/3, e^{x+y+z})$) or the Divergence theorem to show that it is equal to $\iint_{S'} \vec{F} \cdot \vec{n}\,dA$, where S' is the unit disc in the xy-plane given by $x^2+y^2 \leq 1$, $z = 0$, and $\vec{n} = \vec{k}$ is the upward normal, and then calculate this integral.

In the version handed out, the formula for $\vec{G}$ had a typo.

5. You put a perfectly spherical egg is through an egg slicer, resulting in n slices of identical height. But you forgot to peel it first! Show that the amount of eggshell in each slice is the same.

6. Let S_1 be the unit disc $z = 0$, $x^2 + y^2 \leq a^2$ in the xy-plane; let S_2 be the upper hemisphere of radius a; let $\vec{n}_i$ be the unit upward normal to S_i. Evaluate (for $i = 1, 2$) $\iint_{S_i} (\vec{F} \cdot \vec{n}_i)\,dA$, if $\vec{F}$ is the vector field

$$\vec{F} = (y^2 + z^2)\vec{i} + (x^2 + 2z^2)\vec{j} + (3z + 2)\vec{k}.$$

Date: Spring 2001.

Extra practice question. Suppose X is a convex region in the plane that is "sufficiently big", bounded by a piecewise-differentiable curve C (oriented counter-clockwise). A chord of length 1 slides around X, and its midpoint sweeps out a smaller curve C'. *Theorem.* The area between C and C' is $\pi/4$

(a) Prove the theorem in the case when C is a circle of large radius R. Prove the theorem in the case when C is a large rectangle. Hint: see the figures on the handout (not in the pdf file).

(b) Prove the theorem in general as follows. Suppose the chord moves around C as $0 \leq t \leq 1$, counterclockwise. Let $t \mapsto (\alpha(t), \beta(t))$ $(0 \leq t \leq 1)$ be the "counterclockwise (leading) endpoint" of the chord, and let $t \mapsto (\gamma(t), \beta(t))$ be the "clockwise endpoint". Explain why, as $0 \leq t \leq 1$, $(\alpha(t), \beta(t)) - (\gamma(t), \delta(t))$ describes a circle of radius 1, counterclockwise. Show that the area inside C is $\int \alpha \, d\beta$ (i.e. $\int_0^1 \alpha(t)\beta'(t)\, dt$), and also $\int \gamma \, d\delta$. Show that $\int (\alpha - \gamma)\, d(\beta - \gamma) = \pi$. Show that the area inside C' is

$$\int \left(\frac{\alpha + \gamma}{2} \right) d \left(\frac{\beta + \delta}{2} \right).$$

(Assume everything in sight is a Green's region.) Prove the theorem.