Ralph L. Cohen

Bundles, Homotopy, and Manifolds

An introduction to graduate level algebraic and differential topology

Contents

Introduction

Differential Topology is the study of the topology of differentiable manifolds and differentiable mappings between them. This subject is of central importance throughout much of mathematics, especially those areas with a geometric perspective, such as differential geometry, geometric analysis, and algebraic geometry.

In these notes we will assume the reader is familiar with the basics of algebraic topology, such as the fundamental group, homology, and cohomology. The text by Hatcher [41] is an excellent reference for these topics. Perhaps the most basic theorem concerning the algebraic topology of manifolds is the Poincaré Duality theorem. Because not every student having completed a first course in algebraic topology will have seen the Poincaré Duality theorem, we begin these notes with a brief discussion of this topic in chapter 1, where we basically summarize the approach to this important theorem contained in Section 3.3 of [41].

The main content of these notes will be a variety of topics in Differential and Algebraic topology. The main philosophy of the presentation here is to show that there is no clear dividing line between these important areas of topology. A modern study of Differential Topology relies on the techniques of Algebraic Topology, and many important questions in Algebraic Topology come from the study of differentiable manifolds.

The main goal of these notes is to introduce graduate students to topics and methods of Differential and Algebraic topology, going from very basic discussions to more specific topics of modern research. Our objective is that the reader will obtain a literacy in these topics, so that the interested reader can then pursue these topics in more depth. This is not a traditional textbook that gives complete proofs of every theorem presented. Rather our goal is to give the reader an introduction to a variety of important topics so that (s)he obtains the familiarity necessary for deeper study and/or for applications to related areas of mathematics. So we will often sketch a proof of a theorem, when good references exist. We will therefore assume a certain mathematical maturity of the reader, so that they can look up the details, or better yet, supply the details themselves, of some of the beautiful results in Algebraic and Differential Topology discussed here.

The topics covered in these notes include the following:

- The basics of differentiable manifolds (tangent spaces, vector fields, tensor fields, differential forms)

- Fiber bundles in general, Lie groups, principal bundles, vector bundles, and their classification via universal bundles. Automorphisms of principal bundles (gauge transformations) and their classifying spaces
- Characteristic classes of vector bundles and their calculation
- Embeddings, immersions, tubular neighborhoods, and normal bundles
- Basic homotopy theory including homotopy groups, Serre fibrations, obstruction theory, Eilenberg-MacLane spaces, and spectral sequences
- Transversality, and Intersection theory using Poincaré duality
- Stable homotopy theory
- The Pontrjagin-Thom construction and cobordism theory (including the topology of cobordism categories and their relation to diffeomorphisms of manifolds)
- Morse theory, including flow categories and their classifying spaces.

These notes emanated from a variety of graduate courses I've given over the years at Stanford University. The author is grateful to the students in these courses for their inspiration and for their feedback.

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Topological Manifolds and Poincaré Duality

The subject of much of this book is the topology of manifolds. Manifolds of dimension n are topological spaces that have a well defined local topology (they are locally homeomorphic to $\mathbb{R}^n$), but globally, two n-dimensional manifolds may have very different topologies.

Nonetheless we will find that the homological structure of manifolds is quite striking. In particular they satisfy an important, unifying property, called "Poincaré Duality". The discussion and proof of this property is the subject of this chapter. As the reader will see, this property will be used throughout the book, and is used in a basic way in many areas of topology and geometry.

Definition 1.1. *An n-dimensional (topological) manifold is a Hausdorff space M^n that is locally homeomorphic to $\mathbb{R}^n$. That is, each point $x \in M^n$ has a neighborhood U_x which is homeomorphic to $\mathbb{R}^n$, or equivalently, to the open ball $B^n = \{v \in \mathbb{R}^n : |v| < 1\}$. A specific homeomorphism $\phi : U_x \to \mathbb{R}^n$ is called a chart* around x. An open cover of M^n consisting of charts is called an *atlas.*

1.0.1 Orientations

We observe that the local-Euclidean property of manifolds has a manifestation homologically. Namely, suppose M^n is a connected, n-dimensional manifold, and let $x \in M^n$. Then the relative homology:

$$\begin{aligned} H_q(M^n, M - x) &\cong H_q(U, U - x) \quad \text{by excision} \\ &\cong H_q(\mathbb{R}^n, \mathbb{R}^n - \{0\}) \quad \text{by the local-Euclidean property} \\ &\cong H_{q-1}(\mathbb{R}^n - \{0\}) \cong \begin{cases} \mathbb{Z} & \text{if } q = 0, n \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

In particular, observe that the dimension n, is determined homologically.

Definition 1.2. *Let M^n be an n-dimensional manifold. A local orientation* of M^n at x is a choice of generator of $H_n(M^n, M^n - \{x\}) \cong \mathbb{Z}$.

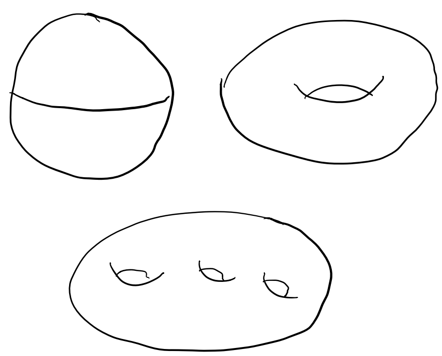

FIGURE 1.1
These surfaces are all 2-dimensional manifolds, as they are all locally homeomorphic to $\mathbb{R}^2$. However their global topologies are quite different.

Notice that there are two choices of local orientations at any point $x \in M^n$, and a choice of orientation is equivalent to choosing an isomorphism $\Phi_x : H_n(M^n, M^n - \{x\}) \xrightarrow{\cong} \mathbb{Z}$.

Definition 1.3. *A manifold M^n is orientable*, if there is a continuous choice of local orientations at each point $x \in M^n$. A specific choice of such a continuous choice of local orientations is called a (global) orientation of M^n.

Of course this definition is not yet complete, because we have not yet defined what is meant by a "continuous choice of local orientations". To make this precise, we use the theory of covering spaces.

For $x \in M^n$, let $Or_x(M^n)$ be the set of local orientations of M^n at x. That is, it is the set of generators of $H_n(M^n, M^n - x)$. As observed above, this is a set with two elements, as there are two possible choices of generators for the infinite cyclic group. Said another, but equivalent way, $Or_x(M^n)$ is the set of isomorphisms, $\sigma : H_n(M^n, M^n - x) \xrightarrow{\cong} \mathbb{Z}$.

Let $Or(M^n)$ be the space of all local orientations on M^n. That is, as a set,

$$Or(M^n) = \bigcup_{x \in M^n} Or_x(M^n). \tag{1.1}$$

Proposition 1.1. *There is a natural topology on $Or(M^n)$ with respect to which the map $p : Or(M^n) \to M^n$ defined by $p(v) = x$ if and only if $v \in Or_x(M^n)$, is a two-fold covering space.*

Before we prove this proposition, we note that we can, as a result, define what we mean by a "continuous choice of local orientations". That is, such a continuous choice would simply be a continuous cross section $\sigma : M^n \to Or(M^n)$ of this covering space. This means that σ is a continuous map with the property that $p(\sigma(x)) = x$ for all $x \in M^n$. Notice that such a continuous section $x \to \sigma(x) \in Or_x(M^n)$ is precisely a continuous choice of local orientation as x varies over all points of $x \in M^n$. The continuity is reflected by the topology of $Or(M^n)$ stated in Proposition 1.1.

We now prove Proposition 1.1.

Proof. Let $\mathcal{U} = \{(U_\alpha, \phi_{U_\alpha}) : \alpha \in \Lambda\}$ be an open cover of M^n by charts. That is, $M = \bigcup_{\alpha \in \Lambda} U_\alpha$, and each $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ is a homeomorphism. Notice that for each pair $\alpha, \beta \in \Lambda$, there is a continuous map

$$\psi_{\alpha,\beta} : U_\alpha \cap U_\beta \to Homeo(\phi_{U_\alpha}(U_\alpha \cap U_\beta); \phi_{U_\beta}(U_\alpha \cap U_\beta)$$

where the target is the group of homeomorphisms between these two open subspaces of $\mathbb{R}^n$. This group of homeomorphisms is endowed with the compact-open topology. Each such homeomorphism determines an isomorphism

$$H_n(\phi_\alpha(U_\alpha \cap U_\beta); \phi_\alpha(U_\alpha \cap U_\beta) - \{\phi_\alpha(x)\}) \xrightarrow{\cong} H_n(\phi_\beta(U_\alpha \cap U_\beta); \phi_\beta(U_\alpha \cap U_\beta) - \{\phi_\beta(x)\}).$$

By excision, this in turn determines a self-isomorphism

$$H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}) \xrightarrow{\cong} H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}).$$

Notice that since $H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}) \cong \mathbb{Z}$, The group of such self isomorphisms consists of the identity and minus the identity. That is, this isomorphism group is $\mathbb{Z}/2$.

Thus $\psi_{\alpha,\beta}$ determines a continuous locally constant (i.e constant on each path component) map

$$\Psi_{\alpha,\beta} : U_\alpha \cap U_\beta \to \mathbb{Z}/2 = \{\pm 1\}.$$

We then define

$$Or(M^n) = \coprod_{\alpha \in \Lambda} U_\alpha \times Gen(H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}))/\sim \tag{1.2}$$

where $Gen(H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\})$ is the two-point set of generators of this homology group, and the equivalence relation $\sim$ is defined by the following: If $x \in U_\alpha \cap U_\beta$ and $\gamma \in Gen(H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}))$, then

$$(x, \gamma) \sim (x, \Psi_{\alpha,\beta}(x)(\gamma))$$

where $(x, \gamma) \in U_\alpha \times Gen(H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}))$ and $(x, \Psi_{\alpha,\beta}(x)(\gamma)) \in U_\beta \times Gen(H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}))$.

$Or(M^n)$, as defined by (1.2) then is given the quotient topology.

□

Exercise. Finish the proof of Proposition 1.1. Specifically show that as sets, the two definitions of $Or(M^n)$ given in (1.1) and (1.2) are the same, and that the map

$$p : Or(M^n) \to M^n$$
$$(x, \gamma) \to x$$

is a two-fold covering map.

Notice that if M^n is orientable, which is to say, the orientation double cover admits a section, $\sigma : M^n \to Or(M^n)$, then it has another orientation, called the *opposite orientaion,* and written $-\sigma$, whose value on a point $x \in M^n$ is the unique point in $Or_x(M^n)$ that is *not* equal to $\sigma(x)$.

Corollary 1.2. *A manifold M^n admits an orientation if and only if the orientation double covering $p : Or(M^n) \to M^n$ is trivial. That is, it admits an isomorphism of covering spaces, to the trivial double covering space, $\pi : M \times \mathbb{Z}/2 \to M$ defined by projecting onto the first coordinate.*

Proof. Suppose M^n is orientable. Then the orientation double cover $p : Or(M^n) \to M^n$ admits a continuous section $\sigma : M^n \to Or(M^n)$. We can then define a trivialization Θ of the covering space

$$\begin{array}{ccc} M^n \times \mathbb{Z}/2 & \xrightarrow{\Theta} & Or(M^n) \\ \pi \downarrow & & \downarrow p \\ M^n & \xrightarrow[=]{} & M^n \end{array}$$

by $\Theta(x, 1) = \sigma(x)$, and $\Theta(x, -1) = -\sigma(x)$.

Conversely, assume that $Or(M^n)$ is trivial. That is, $Or(M^n)$ is isomorphic to $M \times \mathbb{Z}/2$ as covering spaces. Since $\pi : M^n \times \mathbb{Z}/2 \to M^n$ clearly admits two distinct sections, then so does $p : Or(M^n) \to M^n$. □

It will be quite helpful to have the following homological implications of orientability.

Theorem 1.3. *Let M^n be an n-manifold and $A \subset M^n$ a compact subspace. Then*

1. *If $\alpha : M^n \to Or(M^n)$ is a section of the orientation double cover (i.e an orientation of M^n), then there exists a unique homology class $\alpha_A \in H_n(M, M - A)$ whose image in $H_n(M, M - x)$ is $\alpha(x)$ for every $x \in A$.*
2. *$H_i(M, M - A) = 0$ for $i > n$.*

Observation. A compact manifold is often called "closed". Notice that if M^n is a closed oriented manifold, we can let $A = M^n$ and then the above theorem implies that exists a unique *"orientation class"* or *"fundamental class"* $[M^n] = \alpha_M \in H_n(M) \cong \mathbb{Z}$ with the property that the restriction of $[M^n]$ to $H_n(M^n, M^n - x)$ is the value of the orientation $\alpha(x)$.

Proof. We sketch the proof here. We refer the reader to Hatcher [41] Lemma 3.27.

The idea of the proof follows a theme that is often followed in studying homological properties of manifolds. Namely, one proves the theorem first for $\mathbb{R}^n$, which will imply a local version of the theorem for every manifold, and then use "patching arguments" such as the Mayer-Vietoris sequence, to prove the theorem for general manifolds.

We break down the proof of this theorem into four steps.

Step 1. We first observe that if the theorem is true for A and B (both compact), as well as $A \cap B$, then the theorem is true for $A \cup B$.

Consider the following Mayer-Vietoris sequence:

$$0 \to H_n(M, M-(A\cup B)) \xrightarrow{\Phi} H_n(M, M-A)\oplus H_n(M, M-B)$$
$$\xrightarrow{\Psi} H_n(M, M-(A\cap B)) \to \cdots$$

Here we are using the facts that $(M-A)\cup(M-B) = M-(A\cap B)$ and $(M-A)\cap(M-B) = M-(A\cup B)$.

Notice that the zero on the left side is the assumption that $H_{n+1}(M, M-(A\cap B)) = 0$.

Notice that $\Psi(\alpha_A \oplus \alpha_B) = 0$, since by assumption, α_A and α_B restrict to the same class in $H_n(M, M-(A\cap B))$. Using the fact that Φ is a monomorphism, one can conclude that there is a unique class $\alpha_{A\cup B} \in H_n(M^n, M^n-(A\cup B))$ that restricts to α_A in $H_n(M^n, M^n-A)$ and to α_B in $H_n(M^n, M^n-B)$. This completes Step 1.

Step 2. Assume the theorem is true for $M^n = \mathbb{R}^n$. We then prove the theorem for general n-manifolds M^n.

Notice that a compact set $A \subset M^n$ can be written as a finite union $A = A_1\cup\cdots\cup A_k$, where each A_i is a subspace of a chart $A_i \subset U_i$. We apply the result of Step 1 to $(A_1\cup\cdots\cup A_{k-1})$ and A_k. Notice that the intersection of these two spaces is $(A_1\cap A_k)\cup\cdots\cup(A_{k-1}\cap A_k)$. This is a union of $k-1$ compact subspaces, each of which is contained in a chart. By induction, we could conclude the validity of the result in this step, if we knew it to be true for $k=1$, i.e compact subsets A that are contained in a chart, $A \subset U$. But in this case,

$$H_n(M^n, M^n-A) \cong H_n(U, U-A)$$

by excision, which is isomorphic to $H_n(\mathbb{R}^n, \mathbb{R}^n - C)$, where C is a compact subspace of $\mathbb{R}^n$. But by the assumptions of this step, we know the theorem to be true in this case.

We are therefore reduced to proving the theorem for $M^n = \mathbb{R}^n$.

Step 3. Assume $M^n = \mathbb{R}^n$, and prove the theorem for the case $A = A_1\cup\cdots\cup A_k$ where each A_i is convex. The same argument as was used to prove Step 2 reduces this to the case when A is itself convex. In this case

$$H_(\mathbb{R}^n, \mathbb{R}^n - A) \cong H_*(\mathbb{R}^n, \mathbb{R}^n - x)$$

since A is contractible with a canonical contraction to any $x \in A$. In particular $\mathbb{R}^n - A \simeq \mathbb{R}^n - x$.

We leave the general case of an arbitrary compact subspace $A \subset \mathbb{R}^n$ to the reader. This argument is carried out in detail in Hatcher's book [41].

□

We observe that if R is any commutative ring with unit, we could have done the entire discussion above using homology with R-coefficients. That is, we may define a covering space

$$p : Or(M^n; R) \to M^n$$

with the property that

$$p^{-1}(x) = Or_x(M^n; R) = Gen(H_n(M^n, M^n - x; R)).$$

By $Gen(H_n(M^n, M^n - x; R)$ we mean the following. By choosing a chart U around x, one has an isomorphism $H_n(M^n, M^n - x; R) \cong H_n(U, U - x; R) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - x; R) \cong R$. A generator of R is an element $u \in R$ such that $R{\cdot}u = R$. $Gen(H_n(M^n, M^n - x; R)$ is the preimage of the group of generators of R under this isomorphism. We observe that this group of "generators" is well defined. That is, it is independent of the choice of chart, even though the chart is what defines the isormorphism of

$$Gen(H_n(M^n, M^n - x; R)$$

with $Gen(R)$.

Definition 1.4. *If R is a commutative ring with unit, then an R-orientation of an n-dimensional manifold M^n is a section of the "R-orientation covering space" $p : Or(M^n; R) \to M^n$.*

Observations.

1. By sending $1 \in \mathbb{Z}$ to $1 \in R$, there is always a canonical ring homomorphism $\mathbb{Z} \to \mathbb{R}$. This induces a map of covering spaces $Or(M^n) \to Or(M^n; R)$. Thus if M^n is ($\mathbb{Z}$) orientable, it is orientable with respect to any commutative ring with unit R. In fact a choice of ($\mathbb{Z}$) orientation of M^n induces an R-orientation.

2. Let $R = \mathbb{Z}/2$. Then since $Gen(\mathbb{Z}/2) = \{1\}$ is the trivial, one-element group, then the covering space $p : Or(M^n; \mathbb{Z}/2) \to M^n$ is a homeomorphism. Thus it has a unique section. So every manifold is $\mathbb{Z}/2$-orientable, and has a unique $\mathbb{Z}/2$-orientation.

3. Finally observe that Theorem 1.3 can be generalized to a statement about R-orientations for any commutative ring R. In particular when $R = \mathbb{Z}/2$ one has the following consequence.

Corollary 1.4. *Let M^n be a connected, closed n-dimensional manifold. Then*

$$H_n(M^n; \mathbb{Z}/2) \cong \mathbb{Z}/2.$$

1.0.2 Poincaré Duality

Poincaré duality states that for a closed, orientable n-dimensional manifold M^n, the k^{th}-cohomology group and the $(n-k)^{th}$ homology group are isomorphic. The isomorphism is given by the "cap product" with the fundamental, or orientation class $[M^n] \in H_n(M)$. Before we state the Poincaré Duality theorem more carefully, and in more generality, we recall the cap product operation. We refer the reader to any introductory text in algebraic topology for details.

Let X be any topological space, and R aa commutative ring with unit. The cap product operation is an operation of the form

$$\cap : C_k(X;R) \times C^\ell(X;R) \longrightarrow C_{k-\ell}(X;R) \quad \text{for } k \geq \ell.$$

Let $[v_0, \cdots v_k]$ represent the k- simplex spanned by vectors $v_0, \cdots v_k \in \mathbb{R}^N$, where N is large. Let $\sigma \in C_k(X;R)$, and $\phi \in C^\ell(X;R)$. Then one defines

$$\sigma \cap \phi = \phi(\sigma_{|_{[v_0,\cdots,v_\ell]}}) \cdot \sigma_{|_{[v_\ell,\cdots,v_k]}} \tag{1.3}$$

One will then find that the boundary of this cap product chain is given by

$$\partial(\sigma \cap \phi) = (-1)^\ell(\partial\sigma \cap \phi - \sigma \cap \delta\phi) \tag{1.4}$$

where $\partial : C_r(X;R) \to C_{r-1}(X;R)$ is the boundary operator and $\delta : C^p(X;R) \to C^{p+1}(X;R)$ is the coboundary operator. Notice that this formula quickly implies that the cap product of a cycle with a cocycle is a cycle, and hence induces an operation

$$\cap : H_k(X;R) \times H^\ell(X;R) \longrightarrow H_{k-\ell}(X;R). \tag{1.5}$$

And indeed it gives operations on relative (co)homology:

$$\begin{aligned} \cap : & H_k(X,A;R) \times H^\ell(X;R) \longrightarrow H_{k-\ell}(X,A;R) \\ & H_k(XA;R) \times H^\ell(X,A;R) \longrightarrow H_{k-\ell}(X;R) \end{aligned} \tag{1.6}$$

The reader can check that the cap product satisfies the following rather odd naturality property:

$$f_*(\alpha) \cap \phi = f_*(\alpha \cap f^*(\phi)). \tag{1.7}$$

This property becomes more reasonable (and easier to remember) when one realizes that it simply says that if $f : X \to Y$, then the following diagram commutes:

$$\begin{array}{ccc} H_k(X) \times H^\ell(X) & \xrightarrow{\ \cap\ } & H_{k-\ell}(X) \\ \downarrow f_* \qquad \uparrow f^* & & \downarrow f_* \\ H_k(Y) \times H^\ell(Y) & \xrightarrow[\ \cap\ ]{} & H_{k-\ell}(Y) \end{array}$$

Exercise. Show that the cap product is adjoint to the cup product in cohomology. That is, prove that for $\phi \in H^\ell(X;R)$, $\sigma \in H_k(X;R)$, and $\psi \in H^{k-\ell}(X;R)$, then

$$\langle \psi \cup \phi; \sigma \rangle = \pm \langle \psi, \sigma \cap \phi \rangle. \tag{1.8}$$

Here $<,>$ represents the evaluation pairing of cohomology on homology.

The following is the basic statement of Poincaré Duality:

Theorem 1.5. *(Poincaré Duality) If M^n is a closed, R-oriented n-dimensional manifold with fundamental class $[M^n] \in H_n(M^n;R)$, then the map*

$$D = [M^n] \cap \_ : H^k(M^n : R) \to H_{n-k}(M^n;R)$$

is an isomorphism for all k.

Exercise. Show that the Poincaré Duality theorem implies that if F is a field and M^n is a closed F-oriented manifold with fundamental class $[M^n] \in H_n(M^n;F)$, then the pairing

$$\begin{aligned} H^k(M^n;F) \times H^{n-k}(M^n;F) &\longrightarrow F \\ \phi \times \psi &\to \langle \phi \cup \psi, [M^n] \rangle \end{aligned} \tag{1.9}$$

is nonsingular for every $k = 0, \cdots, n$.

In order to prove the Poincaré Duality theorem for compact manifolds, it actually is useful to generalize the theorem to the setting of noncompact manifolds. In this setting, however, one must use the notion of "cohomology with compact supports".

Roughly, a cochain with compact supports is one which is zero on chains living outside some compact set. More carefully,

$$C^i_c(X;G) = \bigcup_{K \text{ compact}} C^i(X, X-K;G).$$

(Strictly speaking, by the union sign we mean the colimit.) The ordinary coboundary map defines a cochain complex

$$\cdots \to C^i_c(X;G) \xrightarrow{\delta} C^{i+1}_c(X;G) \xrightarrow{\delta} \cdots \tag{1.10}$$

The resulting cohomology is written as $H^*_c(X;G)$.

Exercise. Show that

$$H^*_c(\mathbb{R}^n;G) \cong \tilde{H}^*(S^n;G)$$

and more generally that

$$H^*_c(X;G) \cong \tilde{H}^*(X \cup \infty;G)$$

where $X \cup \infty$ is the one-point compactification of X. Here we must assume that the point at infinity in the one-point compactification has a contractible open neighborhood.

Notice that by Theorem 1.3, that if M^n is an R-orientable n-manifold with orientation α, then for every compact subspace $K \subset M^n$, there is a well-defined orientation class $\alpha_K \in H_n(M^n; M^n - K; R)$ that restricts to the R-orientation $\alpha(x) \in H_n(M^n, M^n - \{x\}; R)$. Consider the cap product

$$H^k(M^n, M^n - K; R) \times H_n(M^n, M^n - K; R) \to H_{n-k}(M^n; R).$$

Capping with α_K defines an operation

$$\cap\alpha_K : H^k(M^n, M^n - K; R) \to H_{n-k}(M^n; R).$$

Taking the colimit over K defines a duality operation from the cohomology with compact supports:

$$D_{M^n} : H^k_c(M^n; R) \to H_{n-k}(M^n; R).$$

The following is the generalized form of Poincaré that we will prove:

Theorem 1.6. *Let M^n be an R-oriented manifold. Then the duality map*

$$D_{M^n} : H^k_c(M^n; R) \to H_{n-k}(M^n; R).$$

is an isomorphism for all k.

The proof of Theorem 1.6 (and thereby Theorem 1.5) involves a "patching" argument, for which we will need a lemma involving the Mayer Vietoris sequence.

Notice that if K and L are compact subspaces of M, we have the set theoretic properties,

$$(M - K) \cup (M - L) = M - (K \cap L) \quad \text{and}$$
$$(M - K) \cap (M - L) = M - (K \cup L).$$

So in cohomology there is a Mayer-Vietoris sequence

$$\cdots \to H^k(M; M - (K \cap L)) \to H^k(M, M - K) \oplus H^k(M, M - L) \qquad (1.11)$$
$$\to H^k(M, M - (K \cup L)) \to H^{k+1}(M, M - (K \cap L)) \to \cdots$$

Now suppose $M^n = U \cup W$, where both U and W are open subsets. By taking a limit over compact subsets, Mayer-Vietoris sequence (1.11) yields the following Mayer-Vietoris sequence of cohomologies with compact supports:

$$\cdots \to H^k_c(U \cap W) \to H^k_c(U) \oplus H^k_c(W) \to H^k_c(M^n) \to H^{k+1}_c(U \cap W) \to \cdots$$

We leave to the reader to check the following lemma.

Lemma 1.7. *Let M^n be an R- oriented n-manifold with $M = U \cup W$, where both U and W are open subsets. Then there is a commutative diagram of Mayer-Vietoris sequences:*

$$\begin{array}{ccccccccc}
H_c^k(U \cap W) & \longrightarrow & H_c^k(U) \oplus H_c^k(W) & \longrightarrow & H_c^k(M^n) & \longrightarrow & H_c^{k+1}(U \cap W) & \longrightarrow \\
\downarrow D_{U\cap W} & & \downarrow D_U \oplus D_W & & \downarrow D_{M^n} & & \downarrow D_{U\cap V} & \\
H_{n-k}(U \cap W) & \longrightarrow & H_{n-k}(U) \oplus H_{n-k}(W) & \longrightarrow & H_{n-k}(M^n) & \longrightarrow & H_{n-k-1}(U \cap W) & \longrightarrow
\end{array}$$

Here all (co)homologies are taken with R-coefficients.

We now prove Theorem 1.6.

Proof. This proof has several steps.

Step 1. If $M^n = U \cup W$, and D_U, D_W and $D_{U\cap W}$ are isomorphisms, then so is D_M,

This follows from the above Lemma 1.7 and the five lemma.

Step 2. The theorem holds for $M^n = \mathbb{R}^n$.

Proof. Think of $\mathbb{R}^n$ as the interior of the closed unit ball around the origin, B_1. Let r be a number strictly between 0 and 1. Notice that

$$H_n(B_1, B_1 - B_r) = H_n(B_r, \partial B_r) \cong H_n(B_1, \partial B_1) \cong \tilde{H}_n(S^n) \cong \mathbb{Z}.$$

Since any compact set $K \subset \mathbb{R}^n = interior(B_1)$ is a subset of B_R for some R, we see that $H_c^*(\mathbb{R}^n) \cong H^*(B_1, \partial B_1)$, and the reader can readily check that taking the cap product with the generator of $H_n(B_1, \partial B_1)$ gives the evaluation map

$$H^n(B_1, \partial B_1) \cong Hom(H_n(B_1, \partial B_1), \mathbb{Z}) \cong Hom(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}$$

where the last isomorphism s given by evaluating on a generator of $H_n(B_1, \partial B_1)$, which is to say, its fundamental class.

Step 3. The theorem holds for M^n an arbitrary open subset of $\mathbb{R}^n$.

Proof. Write M^n as a countable union of convex open sets in $\mathbb{R}^n$.

$$M^n = \bigcup_j U_j.$$

Let $V_i = \bigcup_{j<i} U_j$. Notice that both V_i and $V_i \cap U_i$ are unions of $i-1$ convex open sets. So we may make an inductive assumption that the theorem holds for manifolds that are the union of less than or equal to $i-1$ convex open sets in $\mathbb{R}^n$. So D_{V_i} and $D_{V_i \cap U_i}$ are isomorphisms. Then Step 1 implies that $D_{V_i \cup U_i}$ is an isomorphism. But $V_i \cup U_i = V_{i+1}$. This completes the inductive step.

Step 4. The theorem holds if M^n is a countable union of open sets U_i each homeomorphic to $\mathbb{R}^n$.

Proof. This follows by the same argument as in Step 3, with "open set in $\mathbb{R}^n$" replacing "convex open set in $\mathbb{R}^n$. We leave the details to the reader.

We are now done for manifolds that can be expressed as a countable union of charts. We now prove the general case.

Step 5. The general case.

Proof. Consider the collection of open sets $U \subset M^n$ for which D_U is an isomorphism. This collection is partially ordered by inclusion. Notice that the union of every totally ordered subcollection is again in this collection, by the argument in Step 3.

Zorn's Lemma implies that there is a maximal open set U for which this theorem holds. We claim that $U = M^n$. If $U \neq M^n$, let $x \in M^n - U$, and let V be a chart around x. Since V is homeomorphic to $\mathbb{R}^n$, the theorem holds for V by Step 2. It also holds for $U \cap V$ by Step 3. Therefore by Step 1, the theorem holds for $U \cup V$. This contradicts the maximality of U, so we must conclude that $U = M^n$. □

2

Fiber Bundles

In this chapter we define our basic object of study: locally trivial fibrations, or "fiber bundles". We discuss many examples, including covering spaces, vector bundles, and principal bundles. We also describe various constructions on bundles, including pull-backs, sums, and products.

Throughout all that follows, all spaces will be Hausdorff and paracompact.

2.1 Definitions and examples

Let B be connected space with a basepoint $b_0 \in B$, and $p : E \to B$ be a continuous map.

Definition 2.1. *The map $p : E \to B$ is a* **locally trivial fibration, or fiber bundle, with fiber F** if it satisfies the following properties:

1. $p^{-1}(b_0) = F$
2. $p : E \to B$ is surjective
3. For every point $x \in B$ there is an open neighborhood $U_x \subset B$ and a "fiber preserving homeomorphism" $\Psi_{U_x} : p^{-1}(U_x) \to U_x \times F$, that is a homeomorphism making the following diagram commute:

$$\begin{array}{ccc} p^{-1}(U_x) & \xrightarrow[\cong]{\Psi_{U_x}} & U_x \times F \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle proj} \\ U_x & = & U_x \end{array}$$

Some examples:

- The projection map $X \times F \longrightarrow X$ is the *trivial* fibration over X with fiber F.

- Let $S^1 \subset \mathbb{C}$ be the unit circle with basepoint $1 \in S^1$. Consider the map $f_n : S^1 \to S^1$ given by $f_n(z) = z^n$. Then $f_n : S^1 \to S^1$ is a locally trivial fibration with fiber a set of n distinct points (the n^{th} roots of unity in S^1).

- Let $exp : \mathbb{R} \to S^1$ be given by

$$exp(t) = e^{2\pi i t} \in S^1.$$

  Then exp is a locally trivial fibration with fiber the integers $\mathbb{Z}$.

- Recall that the n - dimensional real projective space $\mathbb{RP}^n$ is defined by

$$\mathbb{RP}^n = S^n / \sim$$

  where $x \sim -x$, for $x \in S^n \subset \mathbb{R}^{n+1}$.

  Let $p : S^n \to \mathbb{RP}^n$ be the projection map. This is a locally trivial fibration with fiber the two point set.

- Here is the complex analogue of the last example. Let S^{2n+1} be the unit sphere in $\mathbb{C}^{n+1}$. Recall that the complex projective space $\mathbb{CP}^n$ is defined by

$$\mathbb{CP}^n = S^{2n+1} / \sim$$

  where $x \sim ux$, where $x \in S^{2n+1} \subset \mathbb{C}^n$, and $u \in S^1 \subset \mathbb{C}$. Then the projection $p : S^{2n+1} \to \mathbb{CP}^n$ is a locally trivial fibration with fiber S^1.

- Consider the Moebeus band $M = [0,1] \times [0,1]/ \sim$ where $(t,0) \sim (1-t, 1)$. Let C be the "center circle" $C = \{(1/2, s) \in M\}$ and consider the projection

$$\begin{aligned} p : M &\to C \\ (t,s) &\to (1/2, s). \end{aligned}$$

  This map is a locally trivial fibration with fiber $[0,1]$.

Given a fiber bundle $p : E \to B$ with fiber F, the space B is called the *base space* and the space E is called the total space. We will denote this data by a triple (F, E, B).

Definition 2.2. *A map (or "morphism") of fiber bundles $\Phi : (F_1, E_1, B_1) \to (F_2, E_2, B_2)$ is a pair of basepoint preserving continuous maps $\bar{\phi} : E_1 \to E_2$ and $\phi : B_1 \to B_2$ making the following diagram commute:*

$$\begin{array}{ccc} E_1 & \xrightarrow{\bar{\phi}} & E_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ B_1 & \xrightarrow[\phi]{} & B_2 \end{array}$$

Notice that such a map of fibrations determines a continuous map of the fibers, $\phi_0 : F_1 \to F_2$.

A map of fiber bundles $\Phi : (F_1, E_1, B_1) \to (F_2, E_2, B_2)$ is an isomorphism if there is an inverse map of fibrations $\Phi^{-1} : (F_2, E_2, B_2) \to (F_1, E_1, B_1)$ so that $\Phi \circ \Phi^{-1} = \Phi^{-1} \circ \Phi = 1$.

Finally we say that a fibration (F, E, B) is *trivial* if it isomorphic to the trivial fibration $B \times F \to B$.

Exercise. Verify that all of the above examples of fiber bundles are all nontrivial except for the first one.

The notion of a locally trivial fibration is quite general and includes examples of many types. For example you may have already noticed that **covering spaces** are examples of locally trivial fibrations. In fact one may simply define a covering space to be a locally trivial fibration with discrete fiber. Two other very important classes of examples of locally trivial fiber bundles are *vector bundles* and *principal bundles.* We now describe these notions in some detail.

2.1.1 Vector Bundles

Definition 2.3. *An n- dimensional vector bundle over a field* $\mathbf{k}$ *is a locally trivial fibration $p : E \to B$ with fiber an n - dimensional* $\mathbf{k}$ *- vector space V satisfying the additional requirement that the local trivializations*

$$\psi : p^{-1}(U) \to U \times V$$

induce $\mathbf{k}$ *- linear transformations on each fiber. That is, restricted to each $x \in U$, ψ defines a* $\mathbf{k}$ *- linear transformation (and thus isomorphism)*

$$\psi : p^{-1}(x) \xrightarrow{\cong} \{x\} \times V.$$

It is common to denote the data (V, E, B) defining an n - dimensional vector bundle by a Greek letter, e.g ζ.

A "map" or "morphism" of vector bundles $\Phi : \zeta \to \xi$ is a map of fiber bundles as defined above, with the added requirement that when restricted to each fiber, $\bar{\phi}$ is a $\mathbf{k}$ - linear transformation.

Examples

- Given an n - dimensional $\mathbf{k}$ vector space V, then $B \times V \to B$ is the corresponding trivial bundle over the base space B. Notice that since all n - dimensional trivial bundles over B are isomorphic, we denote it (or more precisely, its isomorphism class) by ϵ_n.

- Consider the " Moebeus line bundle" μ defined to the the one dimensional real vector bundle ("line bundle") over the circle given as follows. Let $E = [0,1] \times \mathbb{R}/\sim$ where $(0,t) \sim (1,-t)$. Let C be the "middle" circle $C = \{(s,0) \in E\}$. Then μ is the line bundle defined by the projection

$$\begin{aligned} p : E &\to C \\ (s,t) &\to (s,0). \end{aligned}$$

- Define the real line bundle γ_1 over the projective space $\mathbb{RP}^n$ as follows. Let $x \in S^n$. Let $[x] \in \mathbb{RP}^n = S^n/\sim$ be the class represented by x. Then $[x]$ determines (and is determined by) the line through the origin in $\mathbb{R}^{n+1}$ going through x. It is well defined since both representatives of $[x]$ (x and $-x$) determine the same line. Thus $\mathbb{RP}^n$ can be thought of as the space of lines through the origin in $\mathbb{R}^{n+1}$. Let $E = \{([x], v) : [x] \in \mathbb{RP}^n, v \in [x]\}$. Then γ_1 is the line bundle defined by the projection

$$\begin{aligned} p : E &\to \mathbb{RP}^n \\ ([x], v) &\to [x]. \end{aligned}$$

  Exercise. Verify that the $\mathbb{RP}^1$ is a homeomorphic to a circle, and the line bundle γ_1 over $\mathbb{RP}^1$ is isomorphic to the Moebeus line bundle μ.

- By abuse of notation we let γ_1 also denote the *complex* line bundle over $\mathbb{CP}^n$ defined analogously to the *real* line bundle γ_1 over $\mathbb{RP}^n$ above.

- Let $Gr_k(\mathbb{R}^n)$ (respectively $Gr_k(\mathbb{C}^n)$) be the space whose points are k - dimensional subvector spaces of $\mathbb{R}^n$ (respectively $\mathbb{C}^n$). These spaces are called *"Grassmannian"* manifolds, and are topologized as follows. Let $V_k(\mathbb{R}^n)$ denote the space of *injective* linear transformations from $\mathbb{R}^k$ to $\mathbb{R}^n$. Let $V_k(\mathbb{C}^n)$ denote the analogous space of injective linear transformations $\mathbb{C}^k \hookrightarrow \mathbb{C}^n$. These spaces are called *"Stiefel manifolds"*, and can be thought of as spaces of $n \times k$ matrices of rank k. These spaces are given topologies as subspaces of the appropriate vector space of matrices. To define $Gr_k(\mathbb{R}^n)$ and $Gr_k(\mathbb{C}^n)$, we put an equivalence relation on $V_k(\mathbb{R}^n)$ and $V_k(\mathbb{C}^n)$ by saying that two transformations A and B are equivalent if they have the same image in $\mathbb{R}^n$ (or $\mathbb{C}^n$). If viewed as matrices, then $A \sim B$ if and only if there is an element $C \in GL(k,\mathbb{R})$ (or $GL(k,\mathbb{C})$) so that $A = BC$. Then the equivalence classes of these matrices are completely determined by their image in $\mathbb{R}^n$ (or $\mathbb{C}^n$), i.e the equivalence class is determined completely by a k - dimensional subspace of $\mathbb{R}^n$ (or $\mathbb{C}^n$). Thus we define

$$Gr_k(\mathbb{R}^n) = V_k(\mathbb{R}^n)/\sim \quad \text{and} \quad Gr_k(\mathbb{C}^n) = V_k(\mathbb{C}^n)/\sim$$

  with the corresponding quotient topologies.

Consider the vector bundle γ_k over $Gr_k(\mathbb{R}^n)$ whose total space E is the subspace of $Gr_k(\mathbb{R}^n) \times \mathbb{C}^n$ defined by

$$E = \{(W, \omega) : W \in Gr_k(\mathbb{R}^n) \, \text{and} \, \omega \in W \subset \mathbb{R}^n\}.$$

Then γ_k is the vector bundle given by the natural projection

$$\begin{aligned} E &\to Gr_k(\mathbb{R}^n) \\ (W, \omega) &\to W \end{aligned}$$

For reasons that will become more apparent later in these notes, the bundles γ_k are called the "universal" or "canonical" k - dimensional bundles over the Grassmannians.

- Notice that the universal bundle γ_k over the Grassmanians $Gr_k(\mathbb{R}^n)$ and $Gr_k(\mathbb{C}^n)$ come equipped with embeddings (i.e injective vector bundle maps) in the trivial bundles $Gr_k(\mathbb{R}^n) \times \mathbb{R}^n$ and $Gr_k(\mathbb{C}^n) \times \mathbb{C}^n$ respectively. We can define the orthogonal complement bundles $\gamma_k^{\perp}$ to be the $n-k$ dimensional bundles whose total spaces are given by

$$E_k^{\perp} = \{(W, \nu) \in Gr_k(\mathbb{R}^n) \times \mathbb{R}^n : \nu \perp W\}$$

  and similarly over $Gr_k(\mathbb{C}^n)$. Observe that the natural projection to the Grassmannian defines $n-k$ dimensional vector bundles (over $\mathbb{R}$ and $\mathbb{C}$ respectively).

Exercises
1. Verify that γ_k is a k -dimensional real vector bundle over $Gr_k(\mathbb{R}^n)$.
2. Define the analogous bundle (which by abuse of notation we also call γ_k) over $Gr_k(\mathbb{C}^n)$. Verify that it is a k-dimensional complex vector bundle over $Gr_k(\mathbb{C}^n)$.
3. Verify that $\mathbb{RP}^{n-1} = Gr_1(\mathbb{R}^n)$ and that the line bundle γ_1 defined above is the universal bundle. Do the analogous exercise with $\mathbb{CP}^{n-1}$ and $Gr_1(\mathbb{C}^n)$.

An important notion associated to vector bundles (and in fact all fibrations) is the notion of a (cross) section. We've already encountered this notion when the fiber bundle is a covering space in our discussion of orientations in Chapter 1.

Definition 2.4. *Given a fiber bundle*

$$p : E \to B$$

a section s is a continuous map $s : B \to E$ such that $p \circ s = identity : B \to B$.

Notice that every vector bundle has a section, namely the *zero section*

$$z : B \to E$$
$$x \to 0_x$$

where 0_x is the origin in the vector space $p^{-1}(x)$. However most geometrically interesting sections have few zero's. Indeed as we will see later, an appropriate count of the number of zero's of a section of an n - dimensional bundle over an n - dimensional manifold is an important topological invariant of that bundle (called the "Euler number"). In particular an interesting geometric question is to determine when a vector bundle has a nowhere zero section, and if it does, how many linearly independent sections it has. (Sections $\{s_1, \cdots, s_m\}$ are said to be linearly independant if the vectors $\{s_1(x), \cdots, s_m(x)\}$ are linearly independent for every $x \in B$.) These questions are classical in the case where the vector bundle is the tangent bundle, as we will see later in our discussion of differentiable manifolds. A section of the tangent bundle is called a **vector field.**The question of how many linearly independent vector fields exist on the sphere S^n was answered by J.F. Adams [3] in the early 1960's using sophisticated techniques of homotopy theory.

Exercises (from [74])

1. Let $x \in S^n$, and $[x] \in \mathbb{RP}^n$ be the corresponding element. Consider the functions $f_{i,j} : \mathbb{RP}^n \to \mathbb{R}$ defined by $f_{i,j}([x]) = x_i x_j$. Show that these functions define a diffeomorphism between $\mathbb{RP}^n$ and the submanifold of $\mathbb{R}^{(n+1)^2}$ consisting of all symmetric $(n+1) \times (n+1)$ matrices A of trace 1 satisfying $AA = A$.

2. Use exercise 1 to show that $\mathbb{RP}^n$ is compact.

3. Prove that an n -dimensional vector bundle ζ has n - linearly independent sections if and only if ζ is trivial.

2.1.2 Principal Bundles

Principal bundles are basically parameterized families of topological groups, and often Lie groups. (A Lie group is a topological group with a compatible differentiable structure. Such structures will be discussed in Chapter 3.) In order to define the notion of a principal bundle carefully we first review some basic properties of group actions.

Recall that a right action of topological group G on a space X is a map

$$\mu : X \times G \to X$$
$$(x, g) \to xg$$

satisfying the basic properties

1. $x \cdot 1 = x$ for all $x \in X$
2. $x(g_1g_2) = (xg_1)g_2$ for all $x \in X$ and $g_1,\ g_2 \in G$.

Notice that given such an action, every element g acts as a homeomorphism, since action by g^{-1} is its inverse. Thus the group action μ defines a map

$$\mu : G \to Homeo(X)$$

where $Homeo(X)$ denotes the group of homeomorphisms of X. The two conditions listed above are equivalent to the requirement that $\mu : G \to Homeo(X)$ be a group homomorphism.

Let X be a space with a right G - action. Given $x \in X$, let $xG = \{xg : g \in G\} \subset X$. This is called the *orbit* of x under the G - action. The isotropy subgroup of x, $Iso(x)$, is defined by $Iso(x) = \{g \in G : xg = x\}$ Notice that the map

$$G \to xG$$

defined by sending g to xg defines a homeomorphism from the coset space to the orbit

$$G/Iso(x) \xrightarrow{\cong} xG \subset X.$$

A group action on a space X is said to be *transitive* if the space X is the orbit of a single point, $X = xG$. Notice that if $X = x_0G$ for some $x_0 \in X$, then $X = xG$ for *any* $x \in X$. Notice furthermore that the transitivity condition is equivalent to saying that for any two points $x_1, x_2 \in X$, there is an element $g \in G$ such that $x_1 = x_2g$. Finally notice that if X has a transitive G - action, then the above discussion about isotropy subgroups implies that there exists a subgroup $H < G$ and a homeomorphism

$$G/H \xrightarrow{\cong} X.$$

Of course if X is smooth, G is a Lie group, and the action is smooth, then the above map would be a diffeomorphism.

A group action is said to be *(fixed point) free* if the isotropy groups of every point x are trivial,

$$Iso(x) = \{1\}$$

for all $x \in X$. Said another way, the action is free if and only if the only time there is an equation of the form $xg = x$ is if $g = 1 \in G$. That is, if for $g \in G$, the fixed point set $Fix(g) \subset X$ is the set

$$Fix(g) = \{x \in X : xg = x\},$$

then the action is free if and only if $Fix(g) = \emptyset$ for all $g \neq 1 \in G$.

We are now able to define principal bundles.

Definition 2.5. *Let G be a topological group. A principal G bundle* is a fiber bundle $p : E \to B$ with fiber $F = G$ satisfying the following properties.

1. The total space E has a free, fiberwise right G action. That is, it has a free group action making the following diagram commute:

$$\begin{array}{ccc} E \times G & \xrightarrow{\mu} & E \\ {\scriptstyle p\times\epsilon}\downarrow & & \downarrow{\scriptstyle p} \\ B \times \{1\} & = & B \end{array}$$

   where ϵ is the constant map.

2. The induced action on fibers

$$\mu : p^{-1}(x) \times G \to p^{-1}(x)$$

   is free and transitive.

3. There exist local trivializations

$$\psi : p^{-1}(U) \xrightarrow{\cong} U \times G$$

   that are equivariant. That is, the following diagrams commute:

$$\begin{array}{ccc} p^{-1}(U) \times G & \xrightarrow[\cong]{\psi\times 1} & U \times G \times G \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle 1\times \text{ mult.}} \\ p^{-1}(U) & \xrightarrow[\psi]{\cong} & U \times G. \end{array}$$

Notice that in a principal G - bundle, the group G acts freely on the total space E. It is natural to ask if a free group action suffices to induce a principal G - bundle. That is, suppose E is a space with a free, right G action, and define B to be the orbit space

$$B = E/G = E/\sim$$

where $y_1 \sim y_2$ if and only if there exists a $g \in G$ with $y_1 = y_2 g$ (i.e if and only if their orbits are equal: $y_1 G = y_2 G$). Define $p : E \to B$ to be the natural projection, $E \to E/G$. Then the fibers are the orbits, $p^{-1}([y]) = yG$. So for $p : E \to B$ to be a principal bundle we must check the local triviality condition.

An important example of this situation is the following (taken from the notes on principal bundles by S. Mitchell [77]): Consider the additive group of real numbers, $\mathbb{R}$, and its subgroup of rational numbers, $\mathbb{Q}$. As a subgroup of $\mathbb{R}$, $\mathbb{Q}$ acts freely on the right by translation:

$$\begin{aligned} \mathbb{R} \times \mathbb{Q} &\to \mathbb{R}. \\ (t, q) &\to t + q \end{aligned}$$

However the quotient map $\mathbb{R} \to \mathbb{R}/\mathbb{Q}$ is clearly *not* a principal $\mathbb{Q}$- bundle. For if it had locally trivial neighborhoods, then since $\mathbb{R}/\mathbb{Q}$ has the trivial topology, it would have to be globally trivial. But clearly $\mathbb{R}$ is not homeomorphic to $\mathbb{R}/\mathbb{Q} \times \mathbb{Q}$.

To avoid this type of example we simply define a subgroup H of G to be *admissible* if the quotient $G \to G/H$ is a principal bundle, i.e it has locally trivial neighborhoods. Clearly any subgroup of a discrete group is admissible. It is also known that any closed subgroup of a Lie group is admissible [82].

Proposition 2.1. *Suppose $G \to P \to B$ is a principal G-bundle. Suppose $H < G$ is an admissible subgroup. Then*

$$H \to P \to P/H$$

is a principal H-bundle.

Proof. . For any subgroup H we have that

$$P/H = P \times_G G/H$$

where the right side is the quotient of the diagonal action of G on $P \times G/H$. The fact that

$$P = P \times_G G \to P \times_G G/H = P/H$$

has local trivializations follows from that facts that $P \to P/G$ and $G \to G/H$ have local trivializations. □

Examples.

- The projection map $p : S^{2n+1} \to \mathbb{CP}^n$ is a principal S^1 - bundle.
- Let $V_k(\mathbb{R}^n)$ be the Stiefel manifold of rank k $n \times k$ matrices described above. Then the projection map

  $$p : V_k(\mathbb{R}^n) \to Gr_k(\mathbb{R}^n)$$

  is a principal $GL(k, \mathbb{R})$ - bundle. Similarly the projection map

  $$p : V_k(\mathbb{C}^n) \to Gr_k(\mathbb{C}^n)$$

  is a principal $GL(k, \mathbb{C})$ - bundle.
- Let $V_k(\mathbb{R}^n)^O \subset \mathbb{R}$ denote those $n \times k$ matrices whose k - columns are orthonormal n - dimensional vectors. This is the Stiefel manifold of orthonormal k - frames in $\mathbb{R}^n$. Then the induced projection map

  $$p : V_k(\mathbb{R}^n)^O \to Gr_k(\mathbb{R}^n)$$

is a principal $O(k)$ - bundle. Similarly, if $V_k(\mathbb{C}^n)^U$ is the space of orthonormal k - frames in $\mathbb{C}^n$ (with respect to the standard Hermitian inner product), then the projection map

$$p : V_k(\mathbb{C}^n)^U \to Gr_k(\mathbb{C}^n)$$

is a principal $U(n)$ - bundle.

- There is a homeomorphism

$$\rho : U(n)/U(n-1) \xrightarrow{\cong} S^{2n-1}$$

and the projection map $U(n) \to S^{2n-1}$ is a principal $U(n-1)$ - bundle.

To see this, notice that $U(n)$ acts transitively on the unit sphere in $\mathbb{C}^n$ (i.e S^{2n-1}). Moreover the isotropy subgroup of the point $e_1 = (1, 0, \cdots, 0) \in S^{2n-1}$ are those elements $A \in U(n)$ which have first column equal to $e_1 = (1, 0, \cdots, 0)$. Such matrices also have first row $= (1, 0, \cdots, 0)$. That is, A is of the form

$$A = \begin{pmatrix} 1 & 0 \\ 0 & A' \end{pmatrix}$$

where A' is an element of $U(n-1)$. Thus the isotropy subgroup $Iso(e_1) \cong U(n-1)$ and the result follows.

Notice that a similar argument gives a diffeomorphism $SO(n)/SO(n-1) \cong S^{n-1}$.

- There is a homeomorphism

$$\rho : U(n)/U(n-k) \xrightarrow{\cong} V_k(\mathbb{C}^n)^U.$$

The argument here is similar to the above, noticing that $U(n)$ acts transitively on $V_k(\mathbb{C}^n)^U$, and the isotopy subgroup of the $n \times k$ matrix

$$e = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

consist of matrices in $U(n)$ of them form

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & \vdots & & \ddots & & \vdots \\ 0 & 0 & 1 & \cdots & 0 & \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ & & (0) & & & & (B) & & & \end{pmatrix}$$

where B is an $(n-k) \times (n-k)$ dimensional unitary matrix.

- A similar argument shows that there are homeomorphisms

$$\rho : U(n)/\left(U(k) \times U(n-k)\right) \xrightarrow{\cong} Gr_k(\mathbb{C}^n)$$

and

$$\rho : O(n)/\left(O(k) \times O(n-k)\right) \xrightarrow{\cong} Gr_k(\mathbb{R}^n)$$

Principal bundles define other fiber bundles in the presence of group actions. Namely, suppose $p : E \to B$ be a principal G - bundle and F is a space with a cellular right group action. Then the product space $E \times F$ has the "diagonal" group action $(e, f)g = (eg, fg)$. Consider the orbit space, $E \times_G F = (E \times F)/G$. Then the induced projection map

$$p : E \times_G F \to B$$

is a locally trivial fibration with fiber F.

For example we have the following important class of fiber bundles.

Proposition 2.2. *Let G be a compact topological group and $K < H < G$ closed subgroups. Then the projection map of coset spaces*

$$p : G/K \to G/H$$

is a fiber bundle with fiber H/K.

Proof. Observe that $G/K \cong G \times_H H/K$ where H acts on H/K in the natural way. Moreover the projection map $p : G/K \to G/H$ is the projection can be viewed as the projection

$$G/K = G \times_H H/K \to G/H$$

and so is the H/K - fiber bundle induced by the H - principal bundle $G \to G/H$ via the action of H on the coset space H/K. □

Example

We know by the above examples, that $U(2)/U(1) \cong S^3$, and that $U(2)/U(1) \times U(1) \cong Gr_1(\mathbb{C}^2) = \mathbb{CP}^1 \cong S^2$. Therefore there is a principal $U(1)$ - fibration

$$p : U(2)/U(1) \to U(2)/U(1) \times U(1),$$

or equivalently, a principal $U(1) = S^1$ fibration

$$p : S^3 \to S^2.$$

This fibration is the well known "Hopf fibration", and is of central importance in both geometry and algebraic topology. In particular, as we will see later, the map from S^3 to S^2 gives an nontrivial element in the homotopy group $\pi_3(S^2)$, which from the naive point of view is quite surprising. It says, that, in a sense that can be made precise, there is a "three dimensional hole" in S^2 that cannot be filled. Many people (eg. Whitehead, see [102]) refer to this discovery as the beginning of modern homotopy theory.

The fact that the Hopf fibration is a locally trivial fibration also leads to an interesting geometric observation. First, it is not difficult to see directly (and we will prove this later) that one can take the upper and lower hemispheres of S^2 to be a cover of S^2 over which the Hopf fibration is trivial. That is, there are local trivializations,

$$\psi_+ : D^2_+ \times S^1 \to p^{-1}(D^2_+)$$

and

$$\psi_- : D^2_- \times S^1 \to p^{-1}(D^2_-)$$

where D^2_+ and D^2_- are the upper and lower hemispheres of S^2, respectively. Putting these two local trivializations together yields the following classical result:

Theorem 2.3. *The sphere S^3 is homeomorphic to the union of two solid tori $D^2 \times S^1$ whose intersection is their common torus boundary, $S^1 \times S^1$.*

As another example of fiber bundles induced by principal bundles, suppose that

$$\rho : G \to GL(n, \mathbb{R})$$

is a representation of a topological group G, and $p : E \to B$ is a principal G bundle. Then let $\mathbb{R}^n(\rho)$ denote the space $\mathbb{R}^n$ with the action of G given by the representation ρ. Then the projection

$$E \times_G \mathbb{R}^n(\rho) \to B$$

is a vector bundle.

Exercise.

Let $p : V_k(\mathbb{R}^n) \to Gr_k(\mathbb{R}^n)$ be the principal bundle described above. Let $\mathbb{R}^n$ have the standard $GL(n, \mathbb{R})$ representation. Proved that the induced vector bundle

$$p : V_k(\mathbb{R}^n) \times_{GL(n,\mathbb{R})} \mathbb{R}^n$$

is isomorphic to the universal bundle γ_k described in the last section.

In the last section we discussed sections of vector bundles and in particular vector fields. For principal bundles, the existence of a section (or lack thereof) completely determines the triviality of the bundle.

Theorem 2.4. *A principal G - bundle $p : E \to B$ is trivial if and only if it has a section.*

Proof. If $p : E \to B$ is isomorphic to the trivial bundle $B \times G \to B$, then clearly it has a section. So we therefore only need to prove the converse.

Suppose $s : B \to E$ is a section of the principal bundle $p : E \to B$. Define the map

$$\psi : B \times G \to E$$

by $\psi(b, g) = s(b)g$ where multiplication on the right by g is given by the right G - action of G on E. It is straightforward to check that ψ is an isomorphism of principal G - bundles, and hence a trivialization of E. □

2.1.3 Clutching Functions and Structure Groups

Let $p : E \to B$ be a fiber bundle with fiber F. Cover the basespace B by a collection of open sets $\{U_\alpha\}$ equipped with local trivializations $\psi_\alpha : U_\alpha \times F \xrightarrow{\cong} p^{-1}(U_\alpha)$. Let us compare the local trivializations on the intersection: $U_\alpha \cap U_\beta$:

$$U_\alpha \cap U_\beta \times F \xrightarrow[\cong]{\psi_\beta} p^{-1}(U_\alpha \cap U_\beta) \xrightarrow[\cong]{\psi_\alpha^{-1}} U_\alpha \cap U_\beta \times F.$$

For every $x \in U_\alpha \cap U_\beta$, $\psi_\alpha^{-1} \circ \psi_\beta$ determines a homeomorphism of the fiber F. That is, this composition determines a map $\phi_{\alpha,\beta} : U_\alpha \cap U_\beta \to Homeo(F)$. These maps are called the *clutching functions* of the fiber bundle. When the bundle is a real n - dimensional vector bundle then the clutching functions are of the form

$$\phi_{\alpha,\beta} : U_\alpha \cap U_\beta \to GL(n, \mathbb{R}).$$

Similarly, complex vector bundles have clutching functions that take values in $GL(n, \mathbb{C})$.

If $p : E \to B$ is a G - principal - bundle, then the clutching functions take values in G:

$$\phi_{\alpha,\beta} : U_\alpha \cap U_\beta \to G.$$

In general for a bundle $p : E \to B$ with fiber F, the group in which the clutching values take values is called the *structure group* of the bundle. If no group is specified, then the structure group is the homeomorphism group $Homeo(F)$.

The clutching functions and the associated structure group completely determine the isomorphism type of the bundle. Namely, given an open covering of a space B, and a compatible family of clutching functions $\phi_{\alpha,\beta} : U_\alpha \cap U_\beta \to G$, and a space F upon which the group acts, we can form the space

$$E = \bigcup_\alpha U_\alpha \times F / \sim$$

where if $x \in U_\alpha \cap U_\beta$, then $(x, f) \in U_\alpha \times F$ is identified with $(x, f\phi_{\alpha,\beta}(x)) \in U_\beta \times F$. E is the total space of a locally trivial fibration over B with fiber F and structure group G. If the original data of clutching functions came from locally trivializations of a bundle, then notice that the construction of E above yields a description of the total space of the bundle. Thus we have a description of the total space of a fiber bundle completely in terms of the family of clutching functions.

Suppose ζ is an n - dimensional vector bundle with projection map $p : E \to B$ and local trivializations $\psi_\alpha : U_\alpha \times \mathbb{R}^n \to p^{-1}(U_\alpha)$. Then the clutching functions take values in the general linear group

$$\phi_{\alpha,\beta} : U_\alpha \cap U_\beta \to GL(n, \mathbb{R}).$$

So the total space E has the form $E = \bigcup_\alpha U_\alpha \times \mathbb{R}^n / \sim$ as above. We can then form the corresponding principal $GL(n, \mathbb{R})$ bundle with total space

$$E_{GL} = \bigcup_\alpha U_\alpha \times GL(n, \mathbb{R})$$

with the same clutching functions. That is, for $x \in U_\alpha \cap U_\beta$, $(x, g) \in U_\alpha \times GL(n, \mathbb{R})$ is identified with $(x, g \cdot \phi_{\alpha,\beta}(x)) \in U_\beta \times GL(n, \mathbb{R})$. The principal bundle

$$p : E_{GL} \to B$$

is called the *associated principal bundle* to the vector bundle ζ, or sometimes is referred to as the *associated frame* bundle.

Observe also that this process is reversable. Namely if $p : P \to X$ is a principal $GL(n, \mathbb{R})$ - bundle with clutching functions $\theta_{\alpha,\beta} : V_\alpha \cap V_\beta \to GL(n, \mathbb{R})$, then there is an associated vector bundle $p : P_{\mathbb{R}^n} \to X$ where

$$P_{\mathbb{R}^n} = \bigcup_\alpha V_\alpha \times \mathbb{R}^n$$

where if $x \in V_\alpha \cap V_\beta$, then $(x, v) \in V_\alpha \times \mathbb{R}^n$ is identified with $(x, v \cdot \theta_{\alpha,\beta}(x)) \in V_\beta \times \mathbb{R}^n$.

This correspondence between vector bundles and principal bundles proves the following result:

Theorem 2.5. *Let $Vect_n^{\mathbb{R}}(X)$ and $Vect_n^{\mathbb{C}}(X)$ denote the set of isomorphism classes of real and complex n - dimensional vector bundles ovr X respectively. For a Lie group G let $Prin_G(X)$ denote the set of isomorphism classes of principal G - bundles. Then there are bijective correspondences*

$$Vect_n^{\mathbb{R}}(X) \xrightarrow{\cong} Prin_{GL(n,\mathbb{R})}(X)$$
$$Vect_n^{\mathbb{C}}(X) \xrightarrow{\cong} Prin_{GL(n,\mathbb{C})}(X).$$

This correspondence and theorem 1.6 allows for the following method of determining whether a vector bundle is trivial:

Corollary 2.6. *A vector bundle $\zeta : p : E \to B$ is trivial if and only if its associated principal $GL(n)$ - bundle $p : E_{GL} \to B$ admits a section.*

Clutching functions and structure groups are also useful in studying structures on principal bundles and their associated vector bundles.

Definition 2.6. *Let $p : P \to B$ be a principal G - bundle, and let $H < G$ be a subgroup. P is said to have a reduction of its structure group to H if and only if P is isomorphic to a bundle whose clutching functions take values in H:*

$$\phi_{\alpha,\beta} : U_\alpha \cap U_\beta \to H < G.$$

Let $P \to X$ be a principal G - bundle. Then P has a reduction of its structure group to $H < G$ if and only if there is a principal H - bundle $\tilde{P} \to X$ and an isomorphism of G bundles,

$$\begin{array}{ccc} \tilde{P} \times_H G & \xrightarrow{\cong} & P \\ \downarrow & & \downarrow \\ X & = & X \end{array}$$

Definition 2.7. *Let $H < GL(n, \mathbb{R})$. Then an H - structure on an n - dimensional vector bundle ζ is a reduction of the structure group of its associated $GL(n, \mathbb{R})$ - principal bundle to H.*

Examples.

- A $\{1\} < GL(n, \mathbb{R})$ - structure on a vector bundle (or its associated principal bundle) is a trivialization or *framing* of the bundle. A framed manifold is a manifold with a framing of its tangent bundle.
- Given a $2n$ - dimensional real vector bundle ζ, an *almost complex structure* on ζ is a $GL(n, \mathbb{C}) < GL(2n, \mathbb{R})$ structure on its associated principal bundle. An almost complex structure on a manifold is an almost complex structure on its tangent bundle.

We now study two examples of vector bundle structures in some detail: Euclidean structures, and orientations.

Example 1: O(n) - structures and Euclidean structures on vector bundles.

Recall that a Euclidean vector space is a real vector space V together with a positive definite quadratic function

$$\mu : V \to \mathbb{R}.$$

Specifically, the statement that μ is quadratic means that it can written in the form

$$\mu(v) = \sum_i \alpha_i(v)\beta_i(v)$$

where each α_i and $\beta_i : V \to \mathbb{R}$ is linear. The statement that μ is positive definite means that

$$\mu(v) > 0 \quad \text{for } v \neq 0.$$

Positive definite quadratic functions arise from, and give rise to inner products (i.e symmetric bilinear pairings $(v, w) \to v \cdot w$) defined by

$$v \cdot w = \frac{1}{2}(\mu(v + w) - \mu(v) - \mu(w)).$$

Notice that if we write $|v| = \sqrt{v \cdot v}$ then $|v|^2 = \mu(v)$. So in particular there is a metric on V.

This notion generalizes to vector bundles in the following way.

Definition 2.8. *A Euclidean vector bundle is a real vector bundle $\zeta : p : E \to B$ together with a map*

$$\mu : E \to \mathbb{R}$$

which when restricted to each fiber is a positive definite quadratic function. That is, μ induces a Euclidean structure on each fiber.

Exercise.

Show that an $O(n)$ - structure on a vector bundle ζ gives rise to a Euclidean structure on ζ. Conversely, a Euclidean structure on ζ gives rise to an $O(n)$ - structure.

Hint. Make the constructions directly in terms of the clutching functions.

Definition 2.9. *A smooth Euclidean structure on the tangent bundle $\mu : TM \to \mathbb{R}$ is called a Riemannian* structure on M.

Exercises.
1. *Existence theorem for Euclidean metrics.* Using a partition of unity, show that any vector bundle over a paracompact space can be given a Euclidean metric.

2. *Isometry theorem.* Let μ and μ' be two different Euclidean metrics on the same vector bundle $\zeta : p : E \to B$. Prove that there exists a homeomorphism $f : E \to E$ which carries each fiber isomorphically onto itself, so that the composition $\mu \circ f : E \to \mathbb{R}$ is equal to μ'. (*Hint.* Use the fact that every positive definite matrix A can be xpressd uniquely as the square of a positive definite matrix $\sqrt{A}$. The power series expansion

$$\sqrt{(tI + X)} = \sqrt{t}(I + \frac{1}{2t}X - \frac{1}{8t^2}X^2 + - \cdots),$$

is valid providing that the characteristic roots of $tI + X = A$ lie between 0 and $2t$. This shows that the function $A \to \sqrt{A}$ is smooth.)

Example 2: S$L(n, \mathbb{R})$ - structures and orientations.

Recall that an orientation of a real n - dimensional vector space V is an equivalence class of basis for V, where two bases $\{v_1, \cdots, v_n\}$ and $\{w_1, \cdots, w_n\}$ are equivalent (i.e determine the same orientation) if and only if the change of basis matrix $A = (a_{i,j})$, where $w_i = \sum_j a_{i,j} v_j$ has positive determinant, $det(A) > 0$. Let $Or(V)$ be the set of orientations of V. Notice that $Or(V)$ is a two point set.

For a vector bundle $\zeta : p : E \to B$, an orientation is a continuous choice of orientations of each fiber. Said more precisely, we may define the "orientation double cover" $Or(\zeta)$ to be the two - fold covering space

$$Or(\zeta) = E_{GL} \times_{GL(n,\mathbb{R})} Or(\mathbb{R}^n)$$

where E_{GL} is the associated principal bundle, and where $GL(n, \mathbb{R})$ acts on $Or(\mathbb{R}^n)$ by matrix multiplication on a basis representing the orientation.

Definition 2.10. ζ *is orientable* if the orientation double cover $Or(\zeta)$ admits a section. A choice of section is an orientation of ζ.

This definition is reasonable, in that a continuous section of $Or(\zeta)$ is a continuous choice of orientations of the fibers of ζ.

Recall that $SL(n, \mathbb{R}) < GL(n, \mathbb{R})$ and $SO(n) < O(n)$ are the subgroups consisting of matrices with positive determinants. The following is now straightforward.

Theorem 2.7. *An n - dimensional vector bundle ζ has an orientation if and only if it has a $SL(n, \mathbb{R})$ - structure. Similarly a Euclidean vector bundle is orientable if and only if it has a $SO(n)$ - structure. Choices of these structures are equivalent to choices of orientations.*

Exercise. Show that a manifiold M is orientable if and only if its its tangent bundle τM is orientable.

2.2 Pull Backs and Bundle Algebra

In this section we describe the notion of the pull back of a bundle along a continuous map. We then use it to describe constructions on bundles such as direct sums, tensor products, symmetric and exterior products, and homomorphisms.

2.2.1 Pull Backs

Let $p : E \to B$ be a fiber bundle with fiber F. Let $A \subset B$ be a subspace. The restriction of E to A, written $E_{|_A}$ is simply given by

$$E_{|_A} = p^{-1}(A).$$

The restriction of the projection $p : E_{|_A} \to A$ is clearly still a locally trivial fibration with fiber F.

This notion generalizes from inclusions of subsets $A \subset B$ to general maps $f : X \to B$ in the form of the *pull back* bundle over X, $f^*(E)$. This bundle is defined by

$$f^*(E) = \{(x, u) \in X \times E : f(x) = p(u)\}.$$

Proposition 2.8. *The map*

$$p_f : f^*(E) \to X$$
$$(x, u) \to x$$

is a locally trivial fibration with fiber F. Furthermore if $\iota : A \hookrightarrow B$ is an inclusion of a subspace, then the pull-back $\iota^*(E)$ is equal to the restriction $E_{|_A}$.

Proof. Let $\{U_\alpha\}$ be a collection of open sets in B and $\psi_\alpha : U_\alpha \times F \to p^{-1}(U_\alpha)$ local trivializations of the bundle $p : E \to B$. Then $\{f^{-1}(U_\alpha)\}$ is an open cover of X, and the maps

$$\psi_\alpha(f) : f^{-1}(U_\alpha) \times F \to p_f^{-1}(f^{-1}(U_\alpha))$$

defined by $(x, y) \to (x, \psi_\alpha(f(x), y))$ are clearly local trivializations.

This proves the first statement in the proposition. The second statement is obvious. □

We now use the pull back construction to define certain algebraic constructions on bundles.

Let $p_1 : E_1 \to B_1$ and $p_2 : E_2 \to B_2$ be fiber bundles with fibers F_1 and F_2 respectively. Then the cartesian product

$$p_1 \times p_2 : E_1 \times E_2 \to B_1 \times B_2$$

is clearly a fiber bundle with fiber $F_1 \times F_2$. In the case when $B_1 = B_2 = B$, we can consider the pull back (or restriction) of this cartesian product bundle via the diagonal map

$$\begin{aligned} \Delta : B &\hookrightarrow B \times B \\ x &\to (x, x). \end{aligned}$$

Then the pull-back $\Delta^*(E_1 \times E_2) \to B$ is a fiber bundle with fiber $F_1 \times F_2$, is defined to be the internal product, or *Whitney sum* of the fiber bundles E_1 and E_2. It is written

$$E_1 \oplus E_2 = \Delta^*(E_1 \times E_2).$$

Notice that if E_1 and E_2 are G_1 and G_2 principal bundles respectively, then $E_1 \oplus E_2$ is a principal $G_1 \times G_2$ - bundle. Similarly, if E_1 and E_2 are n and m dimensional vector bundles respectively, then $E_1 \oplus E_2$ is an $n + m$ - dimensional vector bundle. $E_1 \oplus E_2$ is called the *Whitney sum* of the vector bundles. Notice that the clutching functions of $E_1 \oplus E_2$ naturally lie in $GL(n, \mathbb{R}) \times GL(m, \mathbb{R})$ which is thought of as a subgroup of $GL(n + m, \mathbb{R})$ consisting of $(n + m) \times (n + m)$ - dimensional matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

where $A \in GL(n, \mathbb{R})$ and $B \in GL(m, \mathbb{R})$.

We now describe other algebraic constructions on vector bundles. The first is a generalization of the fact that a given a subspace of a vector space, the ambient vector space splits as a direct sum of the subspace and the quotient space.

Let $\eta : E^\eta \to B$ be a k - dimensional vector bundle and $\zeta : E^\zeta \to B$ an n - dimensional bundle. Let $\iota : \eta \hookrightarrow \zeta$ be a linear embedding of vector bundles. So on each fiber ι is a linear embedding of a k - dimensional vector space into an n - dimensional vector space. Define ζ/η to be the vector bundle whose fiber at x is E^ζ_x/E^η_x.

Exercise.

Verify that ζ/η is an $n - k$ - dimensional vector bundle over B.

Theorem 2.9. *There is a splitting of vector bundles*

$$\zeta \cong \eta \oplus \zeta/\eta.$$

Proof. Give ζ a Euclidean structure. Define $\eta^\perp \subset \zeta$ to be the subbundle whose fiber at x is the orthogonal complement

$$E_x^{\eta^\perp} = \{v \in E_x^\zeta : \, v \cdot w = 0 \,\text{for all}\; w \in E_x^\eta\}$$

Then clearly there is an isomorphism of bundles

$$\eta \oplus \eta^\perp \cong \zeta.$$

Moreover the composition

$$\eta^\perp \subset \zeta \to \zeta/\eta$$

is also an isomorphism. The theorem follows. □

Corollary 2.10. *Let ζ be a Euclidean n - dimensional vector bundle. Then ζ has a $O(k) \times O(n-k)$ - structure if and only if ζ admits a k - dimensional subbundle $\eta \subset \zeta$.*

We now describe the dual of a vector bundle. So let $\zeta : E^\zeta \to B$ be an n - dimensional bundle. Its dual, $\zeta^* : E^{\zeta^*} \to B$ is the bundle whose fiber at $x \in B$ is the dual vector space $E_x^{\zeta^*} = Hom(E^\zeta, \mathbb{R})$. If

$$\{\phi_{\alpha,\beta} : U_\alpha \cap U_\beta \to GL(n,\mathbb{R})\}$$

are clutching functions for ζ, then

$$\{\phi^*_{\alpha,\beta} : U_\alpha \cap U_\beta \to GL(n,\mathbb{R})\}$$

form the clutching functions for ζ^*, where $\phi^*_{\alpha,\beta}(x)$ is the adjoint (transpose) of $\phi_{\alpha,\beta}(x)$. The dual of a complex bundle is defined similarly.

Exercise.

Prove that ζ and ζ^* are isomorphic vector bundles. *Hint.* Give ζ a Euclidean structure.

Now let $\eta : E^\eta \to B$ be a k - dimensional, and as above, $\zeta : E^\zeta \to B$ an n - dimensional bundle. We define the tensor product bundle $\eta \otimes \zeta$ to be the bundle whose fiber at $x \in B$ is the tensor product of vector spaces, $E_x^\eta \otimes E_x^\zeta$. The clutching fucntions can be thought of as compositions of the form

$$\phi^{\eta\otimes\zeta}_{\alpha,\beta} : U_\alpha \cap U_\beta \xrightarrow{\phi^\eta_{\alpha,\beta}\times\phi^\zeta_{\alpha,\beta}} GL(k,\mathbb{R}) \times GL(n,\mathbb{R}) \xrightarrow{\otimes} GL(kn,\mathbb{R})$$

where the tensor product of two linear transformations $A : V_1 \to V_2$ and $B : W_1 \to W_2$ is the induced linear transformation $A \otimes B : V_1 \otimes W_1 \to V_2 \otimes W_2$.

With these two constructions we are now able to define the "homomorphism bundle", $Hom(\eta, \zeta)$. This will be the bundle whose fiber at $x \in B$ is the $k \cdot m$ - dimensional vector space of linear transformations

$$Hom(E_x^\eta, E_x^\zeta) \cong (E_x^\eta)^* \otimes E_x^\zeta.$$

So as bundles we can define

$$Hom(\eta, \zeta) = \eta^* \otimes \zeta.$$

Observation. A bundle homomorphsim $\theta : \eta \to \zeta$ assigns to every $x \in B$ a linear transformation of the fibers, $\theta_x : E_x^\eta \to E_x^\zeta$. Thus a bundle homomorphism can be thought of as a section of the bundle $Hom(\eta, \zeta)$. That is, there is a bijection between the space of sections, $\Gamma(Hom(\eta, \zeta))$ and the space of bundle homomorphisms, $\{\theta : \eta \to \zeta\}$.

3

General Background on Differentiable Manifolds

In geometry one most often studies manifolds that have differentiable structures. They are precisely the types of spaces on which one can do calculus and study differential equations. We begin this chapter by defining these "differentiable manifolds".

Definition 3.1. *An n-dimensional topological manifold M^n is a C^r-differentiable manifold if it admits a C^r-differentiable atlas. This is an atlas $\mathcal{A} = \{U_\alpha, \Psi_{U_\alpha})$ such that every composition of the form*

$$\Psi_{U_\beta} \circ \Psi_{U_\alpha}^{-1} : \Psi_{U_\alpha}(U_\alpha \cap U_\beta) \to U_\alpha \cap U_\beta \to \Psi_{U_\beta}(U_\alpha \cap U_\beta)$$

is a C^r-diffeomorphism of open sets in $\mathbb{R}^n$. We say that each pair of charts $(U_\alpha, \Psi_{U_\alpha})$ and $(U_\beta, \Psi_{U_\beta})$ have a "C^r-overlap".

We note that a C^r-differentiable manifold M^n with atlas $\mathcal{A}$ admits a unique maximal C^r-atlas $\tilde{\mathcal{A}}$ containing $\mathcal{A}$. Namely $\tilde{\mathcal{A}}$ consists of all charts which have C^r-overlap with every chart of $\mathcal{A}$.

Notice that with this definition, it makes sense to say that a continuous map between C^r-differentiable manifolds, $f : M^n \to N^m$ is C^r- differentiable at $x \in M^n$ if there are charts (U, Φ) around $x \in M^n$ and (V, Ψ) around $f(x) \in N$ with $f(U) \subset V$ such that the map

$$\Psi \circ f \circ \Phi^{-1} : \Phi(U) \to \Psi(V)$$

is a differentiable map between open sets $\Phi(U) \subset \mathbb{R}^n$ and $\Psi(V) \subset \mathbb{R}^m$. We say that f is C^r-differentiable if it is C^r-differentiable at every point $x \in M^n$.

For the most part, in these notes we will be studying the topology of "smooth", meaning C^∞- differentiable manifolds.

In our definition, we assume that manifolds are always **Hausdorff** topological spaces. Recall that this means that any two points x, $y \in M$ can be separated by disjoint open sets. That is, there are open sets $U_1 \subset M$ containing x and $U_2 \subset M$ containing y with $U_1 \cap U_2 = \emptyset$. Throughout these notes we will also assume our manifolds are **paracompact.** Recall that a space X is paracompact if every open cover $\mathcal{U}$ of X has a locally finite refinement. That is there is another cover $\mathcal{V}$, all of whose open sets are all contained in $\mathcal{U}$, and so that $\mathcal{V}$ is locally finite. That is, each $x \in M$ lies in only finitely many of the open sets in $\mathcal{V}$. Recall that a Hausdorff space is paracompact if and only if it admits a *partition of unity* subordinate to any open cover $\mathcal{U} = \{U_i, i \in \Lambda\}$. Such a partition of unity is a collection of maps $\rho_i : X \to [0,1]$ so that

- The support supp $(\rho_i) \subset U_i$, and
- $\sum_{i \in \Lambda} \rho_i(x) = 1 \quad$ for every $x \in X$.

3.1 History

Reference: Hirsch's book [44].

Historically, the notion of a differentiable manifold grew from geometry and function theory in the 19th century. Geometers studied curves and surfaces in $\mathbb{R}^3$, and were mainly interested in local structures, such as *curvature*, introduced by Gauss in the early part of the 19th century. Function theorists were interested in studying "level sets" of differentiable functions $F : \mathbb{R}^n \to \mathbb{R}$, i.e the spaces $F^{-1}(c) \subset \mathbb{R}^n$ for $c \in \mathbb{R}$. They observed that for "most" values of c these level sets are "smooth" and nonsingular. This was part of the analytic study of "Calculus of Variations", which let to "Morse theory" in the 20th century.

In the mid-19th century Riemann broke new ground with the study of what are now called "Riemann surfaces". These were historically the first examples of "abstract manifolds", which is to say not defined to be a subspace of some Euclidean space. Riemann surfaces represent the global nature of the analytic continuation process. Riemann also studied topological invariants of these surfaces, such as the *"connectivity"* of a surface, which is defined to be the maximal number of embedded closed curves on a surface whose union does not disconnect the surface plus one. Riemann showed in the 1860's that for compact, orientable surfaces, this number classifies the surface up to homeomorphism. In particular for a surface of genus g, Riemann's connectivity number is $2g + 1$.

In the early 20th century, Poincaré studied 3-dimensional manifolds in his famous treatise, "Analysis Situs". In that work Poincaré introduced some notions in Algebraic Topology such as the fundamental group. The famous *"Poincaré Conjecture"* which was proved by Perelman nearly a hundred years later in 2003, states that every simply connected compact 3-dimensional manifold is homeomoprhic, and indeed diffeomorphic to the sphere S^3.

Poincaré's conjecture was a statement about the *classification of manifolds.* Such a classification has been a key problem in differential topology for the past hundred years. Currently there is great interest and work on the classification of symmetries ("diffeomorphisms") of manifolds.

Herman Weyl defined abstract differentiable manifolds in 1912. But it was not until the work of H. Whitney (1936-1940) when basic geometric and topological properties of manifolds, such as existence of embeddings into Euclidean space, were proved. At that time the modern notion of differentiable manifold became firmly established as a fundamental object in mathematics.

3.2 Examples and Basic Notions

3.2.1 Examples

Consider the following standard examples of manifolds:

1. Consider the unit sphere $S^n \subset \mathbb{R}^{n+1}$. It has an atlas consisting of two charts. Let $\epsilon > 0$ be small. Then define

$$U_1 = \{(x_1, \cdots, x_{n+1}) \,:\, x_{n+1} > -\epsilon\}$$
$$U_2 = \{(x_1, \cdots, x_{n+1}) \,:\, x_{n+1} < \epsilon\}$$

   There are natural projections of U_1 and U_2 onto $B_1(0)$ with C^∞-overlaps, thus defining a smooth structure on S^n.

2. Let $\mathbb{RP}^n = S^n/\sim$ where $x \sim -x$. This is the (real) projective space. This is a C^∞-n-dimensional manifold. To see a smooth atlas we use "projective coordinates". These are obtained by viewing $\mathbb{RP}^n$ as the quotient of the nonzero elements of Euclidean space, $\mathbb{R}^{n+1}$ by the group action of the nonzero real numbers, $\mathbb{R}^\times$ given by scalar multiplication:

$$\mathbb{RP}^n = \left(\mathbb{R}^{n+1} - \{0\}\right)/\mathbb{R}^\times.$$

   We describe a point in $\mathbb{RP}^n$ as the equivalence class of a point in $\mathbb{R}^{n+1}-\{0\}$, which we denote using square brackets: $[x_0, x_1, \cdots, x_n] \in \mathbb{RP}^n$. For $0 \leq i \leq n$ define

$$U_i = \{[x_0, \cdots, x_n] \in \mathbb{RP}^n \,:\, x_i \neq 0\}.$$

   Notice that $\mathbb{RP}^n = U_0 \cup \cdots \cup U_n$ and that the map

$$\Psi_i : U_i \to \mathbb{R}^n$$
$$[x_0, \cdots, x_n] \to \left(\frac{x_0}{x_i}, \cdots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \cdots, \frac{x_n}{x_i}\right)$$

   defines a homeomorphism of U_i onto $\mathbb{R}^n$. Moreover its easily checked that these homeomorphisms have C^∞-overlaps. Thus $\{(U_i, \Psi_i), \,:\, i = 0, \cdots, n\}$ is a smooth (C^∞) atlas for $\mathbb{RP}^n$.

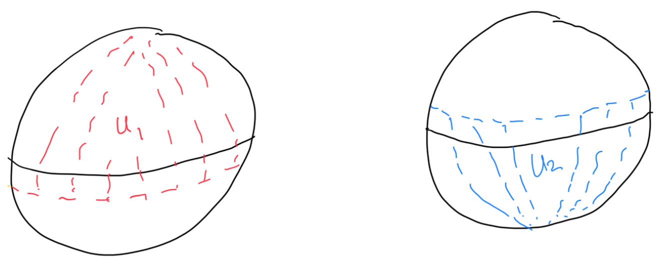

FIGURE 3.1
Charts for S^n

Exercise

Describe atlases for complex projective space $\mathbb{CP}^n$ and quaternionic projective space $\mathbb{HP}^n$, constructed similarly to the atlas described for $\mathbb{RP}^n$ described above, using the complex numbers and the quaternions respectively, instead of real numbers. Show that $\mathbb{CP}^n$ is a differentiable $2n$-dimensional manifold, and $\mathbb{HP}^n$ is a differentiable $4n$-dimensional manifold.

3.2.2 The tangent bundle

An important concept in the study of differentiable manifolds is that of a **tangent bundle.**

Definition 3.2. *Let M^n be a differentiable (C^1) n-dimensional manifold with an atlas $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$. A tangent vector* to M at $x \in M$ is an equivalence class of triples $(x, \alpha, v) \in M \times \Lambda \times \mathbb{R}^n$ under the equivalence relation

$$(x, \alpha, v) \sim (x, \beta, u)$$

if $D(\phi_\beta \phi_\alpha^{-1})(\phi_\alpha(x))(v) = u$. The *tangent space* of M at x, denoted $T_x M$ is defined to be the set of all tangent vectors at x.

Notice that the functions we are differentiating in this definition are defined on open subspaces of Euclidean space. More specifically, they are defined on open sets of the form $\phi_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{R}^n$ and take values in $\phi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{R}^n$.

We leave it to the reader to verify that $T_x M$ is an n-dimensional real vector space. One can also verify that this definition does not depend on the choice of atlas or charts. The *tangent bundle* is defined to be the union of all tangent spaces

$$TM = \bigcup_{x \in M} T_x M.$$

So far TM is defined only set-theoretically. We have yet to discuss its topology. We do so as follows:

Definition 3.3. *Let $\mathcal{U} = \{(U_\alpha, \phi_\alpha) : \alpha \in \Lambda\}$ be an atlas for a differentiable n-dimensional manifold M^n. Define the tangent bundle*

$$TM = \coprod_{\alpha \in \Lambda} U_\alpha \times \mathbb{R}^n / \sim$$

where $(x, v) \in U_\alpha \times \mathbb{R}^n$ is identified with $(x, u) \in U_\beta \times \mathbb{R}^n$ if $x \in U_\alpha \cap U_\beta$ and $D(\phi_\beta \phi_\alpha^{-1})(\phi_\alpha(x))(v) = u$. TM is given the quotient topology under this identification.

We can give the tangent bundle a more concrete definition in the setting where M^n is a subset of $\mathbb{R}^L$ for some L. (We will later prove that every manifold can be appropriately viewed as a subset of Euclidean space of sufficiently high dimension.)

Assume $M^n \subset \mathbb{R}^L$. Given $x \in M^n \subset \mathbb{R}^L$, we say that a vector $v \in \mathbb{R}^L$ is tangent to M^n at $x \in M$ if there exists an $\epsilon > 0$ and differentiable curve

$$\gamma : (-\epsilon, \epsilon) \to M^n \subset \mathbb{R}^L$$

such that $\frac{d\gamma}{dt}(0) = v$.

We define the tangent space $T_x M^n$ to be the set of all vectors tangent to X. Clearly this is an n-dimensional real vector space. Moreover we can now topologize the tangent bundle as a subspace of $\mathbb{R}^L \times \mathbb{R}^L$:

$$TM^n = \bigcup_{x \in M} T_x M^n \subset \mathbb{R}^L \times \mathbb{R}^L$$
$$v \in T_x M^n \to (x, v).$$

There is a natural continuous projection map

$$\begin{aligned} p : TM &\to M \\ v \in T_x M &\to M. \end{aligned} \tag{3.1}$$

Exercise

Prove that the two definitions of tangent bundle given above are equivalent, when the manifold M^n is a submanifold of $\mathbb{R}^L$. By "equivalent" we mean that each of the definitions define vector bundles $TM \to M$ which are isomorphic (as vector bundles).

A differentiable section of the tangent bundle $\sigma : M^n \to TM^n$ is called a *vector field.* At every point of the manifold, a section picks out a tangent vector. The question of which manifolds admit a nowhere zero vector field, and if so, how many linearly independent vector fields are possible, has long been a fundamental question in differential topology. (A collection of vector fields are linearly independent if they pick out linearly independent tangent vectors at every point.) A manifold is called *parallelizable* if its tangent bundle is trivial. Notice that a parallelizable manifold of dimension n admits n linearly independent vector fields.

Exercises

1. Show that a manifold M^n is parallelizable if and only if it admits n linearly independent vector fields.

2. Show that the unit sphere S^n admits a nowhere zero vector field if n is odd.

3. If S^n admits a nowhere zero vector field show that the identity map of S^n is homotopic to the antipodal map. For n even show that the antipodal map of S^n is homotopic to the reflection

$$r(x_1, \cdots, x_{n+1}) = (-x_1, x_2, \cdots, x_{n+1});$$

and therefore has degree -1. Combining these facts, show that S^n is not parallelizable for n even, $n \geq 2$.

3.2.3 The implicit and inverse function theorems, embeddings and immersions

We assume the reader is familiar with the following basic theorems from the analysis of differentiable maps on Euclidean space. We observe that they are local theorems, and so can be used to study differentiable manifolds and maps between them.

Theorem 3.1. *(The Implicit Function Theorem - the surjective version) Let $U \subset \mathbb{R}^m$ be an open subspace and $f : U \to \mathbb{R}^n$ a C^r-map, where $r \geq 1$. For $p \in U$, assume $f(p) = 0$. Suppose the derivative at p,*

$$Df_p : \mathbb{R}^m \to \mathbb{R}^n$$

is surjective. Then there is a local diffeomorphism ϕ of $\mathbb{R}^m$ at 0 such that $\phi(0) = p$ and

$$f \circ \phi(x_1, \cdots, x_n, \cdots, x_m) = (x_1, \cdots, x_n).$$

That is, $f \circ \phi$ is the projection onto the first n-coordinates.

There is another version of the implicit function theorem when the derivative is *injective.*

Theorem 3.2. *(The Implicit Function Theorem - the injective version) Let $U \subset \mathbb{R}^m$ be an open set and $f : U \to \mathbb{R}^n$ a C^r-map, where $r \geq 1$. Let $q \in \mathbb{R}^n$ be such that $0 \in f^{-1}(q)$. Suppose that*

$$Df_0 : \mathbb{R}^m \to \mathbb{R}^n$$

is injective. Then there is a local diffeomorphism ψ of $\mathbb{R}^n$ such that $\psi(q) = 0$ and

$$\psi \circ f(x_1, \cdots, x_m) = (x_1, \cdots, x_m, 0, 0, \cdots, 0) \in \mathbb{R}^n.$$

That is $\psi \circ f$ is the inclusion of the first m-coordinate axes.

Finally, consider the following theorem, which is equivalent to the implicit function theorems.

Theorem 3.3. *(Inverse Function Theorem) Let $U \subset \mathbb{R}^n$ be an open set and $f : U \to \mathbb{R}^n$ be a C^r-map where $r \geq 1$. If $p \in U$ is such that $Df_p : \mathbb{R}^n \to \mathbb{R}^n$ is invertible, then f is a C^r-local diffeomorphism at p. That is there is an open set $V \subset U \subset \mathbb{R}^n$ such that $f : V \to f(V)$ is a diffeomorphism.*

We end with the definition of immersion and embedding.

Definition 3.4. *Suppose $f : M^m \to N^n$ is C^r, for $r \geq 1$, where M^m and N^n are C^r manifolds of dimensions m and n, respectively. We say that f is* **immersive** at $x \in M$ if the linear map

$$Df_x : T_xM \to T_{f(x)}N$$

is injective. f is an **immersion** if f is immersive at every point $x \in M$. We use the symbol $f : M^m \looparrowright N^n$ to mean that f is an immersion.

Definition 3.5. *Suppose $f : M^m \to N^m$ is C^r, for $r \geq 1$, where M^m and N^n are C^r manifolds of dimensions m and n, respectively. We say that f is* **submersive** at $x \in M$ if the linear map

$$Df_x : T_xM \to T_{f(x)}N$$

is surjective. f is an **submersion** if f is submersive at every point $x \in M$.

Definition 3.6. *A C^r-map $f : M \to N$ is an* **embedding** if it is an immersion and f maps M homeomorphically onto its image. In this case we write $f : M \hookrightarrow N$.

Finally we have the following definition.

Definition 3.7. *Suppose N is a C^r-manifold, $r \geq 1$. A subspace $A \subset N$ is a C^r-submanifold if and only if A is the image of a C^r-embedding of some manifold into N.*

Exercises. 1. Prove that the following are C^∞- submanifolds of the space of $n \times n$ matrices, $Mat_{n,n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$. Compute their dimensions.

1. $GL_n(\mathbb{R})$
2. $SL_n(\mathbb{R})$
3. $SO(n)$.

2. (a) Let $x \in S^n$, and $[x] \in \mathbb{RP}^n$ be the corresponding element. Consider the functions $f_{i,j} : \mathbb{RP}^n \to \mathbb{R}$ defined by $f_{i,j}([x]) = x_i x_j$. Show that these functions define a diffeomorphism between $\mathbb{RP}^n$ and the submanifold of $\mathbb{R}^{(n+1)^2}$ consisting of all symmetric $(n+1) \times (n+1)$ matrices A of trace 1 satisfying $AA = A$.
(b) Use the above to show that $\mathbb{RP}^n$ is compact.

The following is an immediate corollary of Implicit Function Theorem (the injective version).

Proposition 3.4. *If $f : M \to N$ is an immersion, then it is a local embedding. That is, around every $x \in M$ there is an open neighborhood U of x so that the restriction $f : U \to N$ is an embedding.*

Exercise Let $\pi : \tilde{X} \to X$ be a covering space. Let Φ be a smooth structure on X. Prove that there is a smooth structure $\tilde{\Phi}$ on $\tilde{X}$ so that $\pi : (\tilde{X}, \tilde{\Phi}) \to (X, \Phi)$ is an immersion.

3.2.4 Manifolds with boundary

In many areas of mathematics one often confronts manifolds that have a boundary. A closed disk in $\mathbb{R}^n$ is a basic example. In this section we describe how the concepts developed above for smooth manifolds, can be generalized to "smooth manifolds with boundary".

Definition 3.8. *The "upper half space" $\mathbb{H}^n \subset \mathbb{R}^n$ is the subspace*

$$\mathbb{H}^n = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \text{ such that } x_n \geq 0\}.$$

The boundary points of $\mathbb{H}^n$ are those $(x_1, \cdots, x_n)$ with $x_n = 0$.

An n-dimensional topological manifold with boundary is then one that has charts homeomorphic to open sets in $\mathbb{H}^n$ rather than $\mathbb{R}^n$. That is, we have the following definition, which is completely analogous to Definition 1.1 above.

Definition 3.9. *An n-dimensional topological manifold manifold with boundary is a Hausdorff space M^n with the property that for every $x \in M$, there is an open neighborhood U containing x and a homeomorphism,*

$$\psi_U : U \xrightarrow{\cong} V \subset \mathbb{H}^n$$

where V is an open subspace of $\mathbb{R}^n$. The boundary of M^n, written ∂M^n consists of those points $p \in M^n$ for which there is an open neighborhood $p \in U$ and a chart $\psi_U : U \xrightarrow{\cong} V \subset \mathbb{H}^n$ where $\psi_U(p)$ is a boundary point of $\mathbb{H}^n$. Observe that the condition of $p \in M^n$ being a boundary point is independent of the particular chart used.

We leave it for the reader to check that if M^n is a topological n-manifold with boundary, then the boundary ∂M^n is a topological $(n-1)$-dimensional manifold (without boundary).

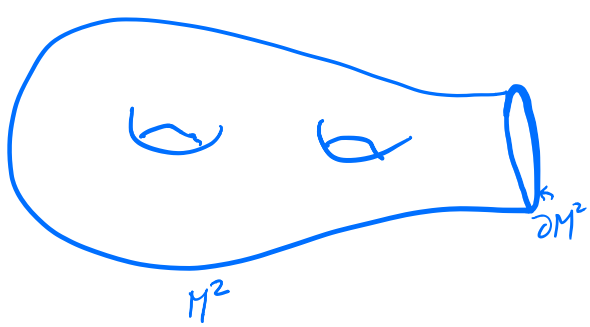

FIGURE 3.2
A 2-dimensional manifold with boundary

We need to be careful about the definition of submanifolds in the setting of manifolds with boundary. First, for $k \leq n$, consider a standard inclusion $\mathbb{H}^k \hookrightarrow \mathbb{R}^n$ mapping $(x_1, \cdots, x_k)$ to $(x_1, \cdots, x_k, 0, \cdots 0)$. A subspace $V \subset \mathbb{R}^n$ is a C^r-dimensional submanifold if each $x \in V$ belongs to the domain of a chart $\phi : U \to \mathbb{R}^n$ of $\mathbb{R}^n$ such that $V \cap U = \phi^{-1}(\mathbb{H}^k)$.

A general definition of a submanifold (with boundary) can be taken to be the following:

Definition 3.10. *Let M be a C^r-manifold, with or without boundary. A subset $N \subset M$ is a C^r-submanifold if each $x \in N$ there is an open set subset U of M containing x, a C^r embedding $g : U \hookrightarrow \mathbb{R}^n$, such that*

$$N \cap U = g^{-1}(\mathbb{H}^k),$$

A particularly important type of embedding of one manifold into another is when one restricts to the boundary of the submanifold, the image of the embedding lies in the boundary of the ambient manifold. This is called a *neat* embedding,

Definition 3.11. *An embedding $e : N \hookrightarrow M$ of C^r-manifolds is neat* if $\partial N = N \cap \partial M$ and N is covered by charts (ϕ, U) of M such that $N \cap U = \phi^{-1}(\mathbb{H}^k)$.

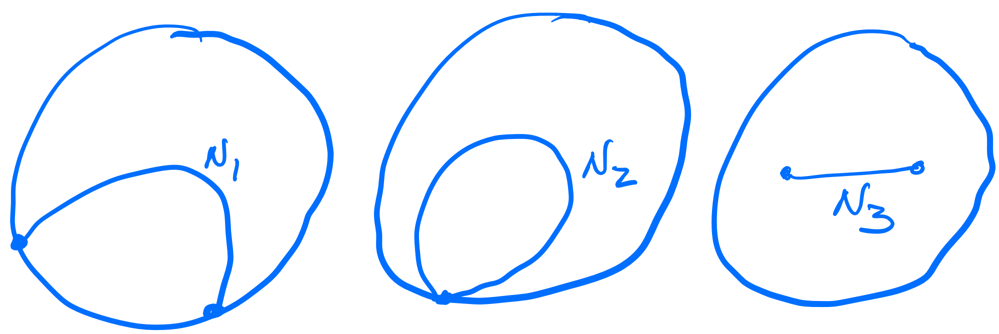

FIGURE 3.3
N_1 is neat, N_2 and N_3 are not.

3.2.5 Regular Values and transversality

We begin this section with the notion of *regular points and values* as well as *critical points and values*.

Definition 3.12. *Suppose $f : M \to N$ is a C^r map between C^r manifolds, where $r \geq 1$. A point $x \in M$ is called a regular point* if $f$is submersive at x. If $u \in M$ is not a regular point it is called a *critical point*. $f(u) \in N$ is then called a *critical value*. If $y \in N$ is not a critical value it is called a regular value. In particular every point $y \in N$ that is not in the image of f is a regular value. If $y \in N$ is a regular value, its inverse image $f^{-1}(y) \subset M$ is called a *regular level set*.

The following is one of the most fundamental theorems in differential topology:

Theorem 3.5. *(The Regular Value Theorem) Suppose $f : M^n \to N^k$ is a C^r-map between C^r manifolds of dimension n and k respectively. Here $r \geq 1$. If $y \in M$ is a regular value, then the regular level set $f^{-1}() \subset M^n$ is a C^r-submanifold of dimension $n - k$.*

Proof. Since being a manifold is a local property, it suffices to prove this theorem in the case when $M^n \subset \mathbb{R}^n$ is an open set, and $N = \mathbb{R}^m$. The theorem now follows from the surjective version of the Implicit Function Theorem. □

The Regular Value Theorem for manifolds with boundary has the following formulation.

Theorem 3.6. *Let M ba a C^r manifold with boundary, and N a C^r manifold (with or without boundary). Here we are assuming $r \geq 1$. Let $f : M \to N$ be a C^r map. If $y \in N - \partial N$ is a regular value for both f and $f_{|\partial M}$, then $f^{-1}(y)$ is a neat C^r submanifold of M.*

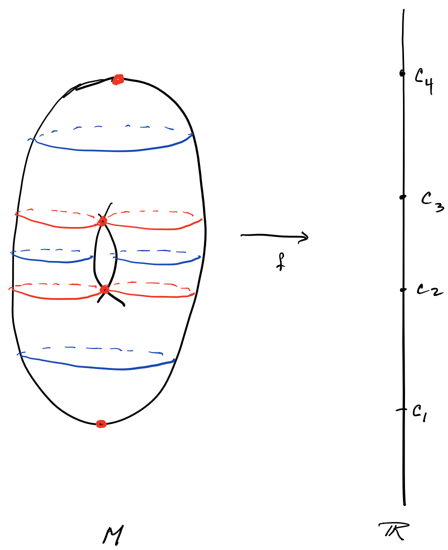

FIGURE 3.4
f is the height function from the torus to the real line. It has 4 critical values. The level sets of the critical values are shown in red, and regular sets of regular values, which are all one-dimensional submanifolds, are shown in blue.

We now want to discuss an important generalization of the concepts involved in the Regular Value Theorem. This is the concept of *transversality*. The following is probably the most conceptual setting for transversality.

Let N^n be an n-dimensional manifold, and let $A \subset N$ and $B \subset N$ be submanifolds of dimensional p and q respectively.

$$\begin{array}{ccc} B & \xrightarrow{\subset} & N \\ & & \uparrow \cup \\ & & A \end{array}$$

We say that A and B have a *transverse intersection* in N if for every $x \in A \cap B$, the tangent spaces of the submanifolds A and B at x, together span the entire tangent space of the ambient manifold N. That is,

$$T_x A + T_x B = T_x N \tag{3.2}$$

When A and B have transverse intersection we write $A \pitchfork B$. We will see that such transversal intersections are, in an appropriate sense, generic. We begin, though, with the following theorem.

Theorem 3.7. *Let A and B be submanifolds of the n-dimensional manifold N, where $Dim\, A = p$ and $Dim\, B = q$. Suppose furthermore that $A \pitchfork B$. The $A \cap B \subset N$ is a submanifold of dimension $p + q - n$.*

We will actually prove the following generalization of Theorem 3.7.

Let A^p be a p-dimensional manifold and N^n an n-dimensional manifold with a q-dimensional submanifold $B^q \subset N^n$. Let $f : A \to N$ be a smooth map. We say that f is transverse to B, and write $f \pitchfork B$ if whenever $b \in B$ is such that $f^{-1}(b)$ is nonempty, then for any $x \in f^{-1}(b)$

$$Df_x(T_x A) + T_b B = T_b N. \tag{3.3}$$

Notice that if $f : A \to N$ is an embedding, then $f \pitchfork B$ if and only if the submanifold given by the image of f has transverse intersection with B. Notice furthermore that if $B = y \in N$ is a point, viewed as a zero dimensional submanifold, then $f \pitchfork B$ if and only if $y$is a regular value of f. This is the sense in which the notion of transversality is a generalization of the notion of regular value.

The following is a strengthening of both transversality Theorem 3.7 and of the Regular Value Theorem 3.5:

Theorem 3.8. *Let $f : A^p \to N^n$ and $B^q \subset N^n$ be as above. Then if $f \pitchfork B$, then the inverse image $f^{-1}(B) \subset A$ is a submanifold of codimension $n - q$, which is the same as the codimension of B in N. That is, $f^{-1}(B)$ has dimension $p + q - n$.*

Notice that this theorem is precisely the statement of the Regular Value Theorem when B a point.

Proof. It suffices to prove this theorem locally. By the Implicit Function Theorem, we can locally replace $B^q \subset N^n$ by $U \times \{0\} \subset U \times V$, where $U \subset \mathbb{R}^q$ and $V \subset \mathbb{R}^{n-q}$ are open sets. Notice that

$$f : A^p \to U \times V$$

is transverse to $U \times \{0\}$ if and only if the composition

$$g : A^p \xrightarrow{f} U \times V \xrightarrow{project} V$$

has $0 \in V \subset \mathbb{R}^{n-q}$ as a regular value. Since $f^{-1}(U \times \{0\}) = g^{-1}(0)$, the theorem follows from the Regular Value Theorem (Theorem 3.5). $\square$

A generalization of this theorem to the setting of manifolds with boundary is the following. The above proof applies to this situation with only minor modifications.

Theorem 3.9. *Suppose $B^q \subset N^n$ is a C^r submanifold with boundary. Suppose that either B^q is neat or $B^q \subset N^n - \partial N^n$, or $B^q \subset \partial N^n$. If $f : A^p \to N^n$ is a C^r map between manifolds with boundary with both f and $f_{|\partial A^p}$ transverse to B^q, the $f^{-1}(B^q)$ is a C^r submanifold and $\partial f^{-1}(B^q) = f^{-1}(\partial B^q)$. The dimension of $f^{-1}(B^q)$ is $p + q - n$.*

3.3 Bundles and Manifolds

3.3.1 The tangent bundle of Projective Space

We now use these constructions to identify the tangent bundle of projective spaces, $T\mathbb{RP}^n$ and $T\mathbb{CP}^n$. We study the real case first.

Recall the canonical line bundle, $\gamma_1 : E^{\gamma_1} \to \mathbb{RP}^n$. If $[x] \in \mathbb{RP}^n$ is viewed as a line in $\mathbb{R}^{n+1}$, then the fiber $E^{\gamma_1}_{[x]}$ is the one dimensional space of vectors in the line $[x]$. Thus γ_1 has a natural embedding into the trivial $n+1$ - dimensional bundle $\epsilon : \mathbb{RP}^n \times \mathbb{R}^{n+1} \to \mathbb{RP}^n$ via

$$E^{\gamma_1} = \{([x], u) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} : u \in [x]\} \hookrightarrow \mathbb{RP}^n \times \mathbb{R}^{n+1}.$$

Let $\gamma_1^\perp$ be the n - dimensional orthogonal complement bundle of this embedding.

Theorem 3.10. *There is an isomorphism of the tangent bundle with the homomorphism bundle*
$$T\mathbb{RP}^n \cong Hom(\gamma_1, \gamma_1^{\perp})$$

Proof. Let $p : S^n \to \mathbb{RP}^n$ be the natural projection. For $x \in S^n$, recall that the tangent space of S^n can be described as

$$T_x S^n = \{(x, v) \in S^n \times \mathbb{R}^{n+1} : x \cdot v = 0\}.$$

Notice that $(x, v) \in T_x S^n$ and $(-x, -v) \in T_{-x} S^n$ have the same image in $T_{[x]}\mathbb{RP}^n$ under the derivative $Dp : TS^n \to T\mathbb{RP}^n$. Since p is a local diffeomorphism, $Dp(x) : T_x S^n \to T_{[x]}\mathbb{RP}^n$ is an isomorphism for every $x \in S^n$. Thus $T_{[x]}\mathbb{RP}^n$ can be identified with the space of pairs

$$T_{[x]}\mathbb{RP}^n = \{(x, v), (-x, -v) : x, v \in \mathbb{R}^{n+1},\ |x| = 1,\ x \cdot v = 0\}.$$

If $x \in S^n$, let $L_x = [x]$ denote the line through $\pm x$ in $\mathbb{R}^{n+1}$. Then a pair $(x, v), (-x, -v) \in T_{[x]}\mathbb{RP}^n$ is uniquely determined by a linear transformation

$$\ell : L_x \to L^{\perp}$$
$$\ell(tx) = tv.$$

Thus $T_{[x]}\mathbb{RP}^n$ is canonically isomorphic to $Hom(E_x^{\gamma_1}, E_x^{\gamma_1^{\perp}})$, and so

$$T\mathbb{RP}^n \cong Hom(\gamma_1, \gamma_1^{\perp}),$$

as claimed. □

The following description of the $T\mathbb{RP}^n \oplus \epsilon_1$ will be quite helpful to us in future calculations of characteristic classes.

Theorem 3.11. *The Whiney sum of the tangent bundle and a trivial line bundle, $T\mathbb{RP}^n \oplus \epsilon_1$ is isomorphic to the Whitney sum of $n+1$ copies of the canonical line bundle γ_1,*

$$T\mathbb{RP}^n \oplus \epsilon_1 \cong \oplus_{n+1}\gamma_1.$$

Proof. Consider the line bundle $Hom(\gamma_1, \gamma_1)$ over $\mathbb{RP}^n$. This line bundle is trivial since it has a canonical nowhere zero section

$$\iota(x) = 1 : E_{[x]}^{\gamma_1} \to E_{[x]}^{\gamma_1}.$$

We therefore have

$$\begin{aligned}
T\mathbb{RP}^n \oplus \epsilon_1 &\cong T\mathbb{RP}^n \oplus Hom(\gamma_1, \gamma_1) \\
&\cong Hom(\gamma_1, \gamma_1^\perp) \oplus Hom(\gamma_1, \gamma_1) \\
&\cong Hom(\gamma_1, \gamma_1^\perp \oplus \gamma_1) \\
&\cong Hom(\gamma_1, \epsilon_{n+1}) \\
&\cong \oplus_{n+1}\gamma_1^* \\
&\cong \oplus_{n+1}\gamma_1
\end{aligned}$$

as claimed. □

The following are complex analogues of the above theorems and are proved in the same way.

Theorem 3.12.

$$T\mathbb{CP}^n \cong_{\mathbb{C}} Hom_{\mathbb{C}}(\gamma_1, \gamma_1^\perp)$$

and

$$T\mathbb{CP}^n \oplus \epsilon_1 \cong \oplus_{n+1}\gamma_1^*,$$

where $\cong_{\mathbb{C}}$ *and* $Hom_{\mathbb{C}}$ *denote isomorphisms and homomorphisms of complex bundles, respectively.*

Note. γ^* is not isomorphic as *complex* vector bundles to γ_1. It is isomorphic to γ_1 with the *conjugate* complex structure. We will discuss this phenomenon more later.

3.3.2 K - theory

Let $Vect^*(X) = \oplus_{n\geq 0} Vect^n(X)$ where, as above, $Vect^n(X)$ denotes the set of isomorphism classes of n - dimensional complex bundles over X. $Vect^*_{\mathbb{R}}(X)$ denotes the analogous set of real vector bundles. In both these cases $Vect^0(X)$ denotes, by convention, the one point set, representing the unique zero dimensional vector bundle.

Now the Whitney sum operation induces pairings

$$Vect^n(X) \times Vect^m(X) \xrightarrow{\oplus} Vect^{n+m}(X)$$

which in turn give $Vect^*(X)$ the structure of an abelian monoid. Notice that it is indeed abelian because given vector bundles η and ζ we have an obvious isomorphism

$$\eta \oplus \zeta \cong \zeta \oplus \eta.$$

The "zero" in this monoid structure is the unique element of $Vect^0(X)$.

Given an abelian monoid, A, there is a construction due to Grothendieck of its *group completion* $K(A)$. Formally, $K(A)$ is the smallest abelian group equipped with a homomorphism of monoids, $\iota : A \to K(A)$. It is smallest in the sense if G is any abelian group and $\phi : A \to G$ is any homomorphism of monoids, then there is a unique extension of ϕ to a map of abelian groups $\bar{\phi} : K(A) \to G$ making the diagram commute:

$$\begin{array}{ccc} A & \xrightarrow{\iota} & K(A) \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\bar{\phi}} \\ G & = & G \end{array}$$

This formal property, called the *universal property*, characterizes $K(A)$, and can be taken to be the definition. However there is a much more explicit description. Basically the group completion $K(A)$ is obtained by formally adjoining inverses to the elements of A. That is, an element of $K(A)$ can be thought of as a formal difference $\alpha - \beta$, where $\alpha, \beta \in A$. Strictly speaking we have the following definition.

Definition 3.13. *Let $F(A)$ be the free abelian group generated by the elements of A, and let $R(A)$ denote the subgroup of $F(A)$ generated by elements of the form $a \oplus b - (a+b)$ where $a, b \in A$. Here "$\oplus$" is the group operation in the free abelian group and "$+$" is the addition in the monoid structure of A. We then define the Grothendieck group completion $K(A)$ to be the quotient group*

$$K(A) = F(A)/R(A).$$

Notice that an element of $K(A)$ is of the form

$$\theta = \sum_i n_i a_i - \sum_j m_j b_j$$

where the n_i's and m_j's are positive integers, and each a_i and $b_j \in A$. That is, by the relations in $R(A)$, we may write

$$\theta = \alpha - \beta$$

where $\alpha = \sum_i n_i a_i \in A$, and $\beta = \sum_j m_j b_j \in A$.

Notice also that the composition $\iota : A \subset F(A) \to F(A)/R(A) = K(A)$ is a homomorphism of monoids, and clearly has the universal property described above. We can now make the following definition.

Definition 3.14. *Given a space X, its complex and real (or orthogonal) K - theories are defined to be the Grothendieck group completions of the abelian monoids of isomorphism classes of vector bundles:*

$$K(X) = K(Vect^*(X))$$
$$KO(X) = K(Vect^*_{\mathbb{R}}(X)$$

An element $\alpha = \zeta - \eta \in K(X)$ is often referred to as a "virtual vector bundle" over X.

Notice that the discusion of the tangent bundles of projective spaces above (section 2.2) can be interpreted in K -theoretic language as follows:

Proposition 3.13. *As elements of* $K(\mathbb{CP}^n)$*, we have the equation*

$$[T\mathbb{CP}^n] = (n+1)[\gamma_1^*] - [1]$$

where $[m] \in K(X)$ *refers to the class represented by the trivial bundle of dimension* m*. Similarly, in the orthogonal* K *- theory* $KO(\mathbb{RP}^n)$ *we have the equation*

$$[T\mathbb{RP}^n] = (n+1)[\gamma_1] - [1].$$

Notice that for a point, $Vect^*(pt) = \mathbb{Z}^+$, the nonnegative integers, since there is precisely one vector bundle over a point (i.e vector space) of each dimension. Thus

$$K(pt) \cong KO(pt) \cong \mathbb{Z}.$$

Notice furthermore that by taking tensor products there are pairings

$$Vect^m(X) \times Vect^n(X) \xrightarrow{\otimes} Vect^{mn}(X).$$

The following is verified by a simple check of definitions.

Proposition 3.14. *The tensor product pairing of vector bundles gives* $K(X)$ *and* $KO(X)$ *the structure of commutative rings.*

Now given a bundle ζ over Y, and a map $f : X \to Y$, we saw in the previous section how to define the pull-back, $f^*(\zeta)$ over X. This defines a homomorphism of abelian monoids

$$f^* : Vect^*(Y) \to Vect^*(X).$$

After group completing we have the following:

Proposition 3.15. *A continuous map* $f : X \to Y$ *induces ring homomorphisms,*

$$f^* : K(Y) \to K(X)$$

and

$$f^* : KO(Y) \to KO(X).$$

In particular, consider the inclusion of a basepoint $x_0 \hookrightarrow X$. This induces a map of rings, called the augmentation,

$$\epsilon : K(X) \to K(x_0) \cong \mathbb{Z}.$$

This map is a split surjection of rings, because the constant map $c : X \to x_0$ induces a right inverse of ϵ, $c^* : \mathbb{Z} = K(x_0) \to K(X)$. Notice that the augmentation can be viewed as the "dimension" map in that when restricted to the monoid $Vect^*(X)$, then $\epsilon : Vect^m(X) \to \{m\} \subset \mathbb{Z}$. That is, on an element $\zeta - \eta \in K(X)$, $\epsilon(\zeta - \eta) = dim(\zeta) - dim(\eta)$. We then define the reduced K -theory as follows.

Definition 3.15. *The reduced K - theory of X, denoted $\tilde{K}(X)$ is defined to be the kernel of the augmentation map*

$$\tilde{K}(X) = ker\{\epsilon : K(X) \to \mathbb{Z}\}$$

and so consists of classes $\zeta - \eta \in K(X)$ such that $dim(\zeta) = dim(\eta)$. The reduced orthogonal K - theory, $\tilde{KO}(X)$ is defined similarly.

The following is an immediate consequence of the above observations:

Proposition 3.16. *There are natural splittings of rings*

$$K(X) \cong \tilde{K}(X) \oplus \mathbb{Z}$$
$$KO(X) \cong \tilde{KO}(X) \oplus \mathbb{Z}.$$

Clearly then the reduced K - theory is the interesting part of K - theory. Notice that a bundle $\zeta \in Vect^n(X)$ determines the element $[\zeta] - [n] \in \tilde{K}(X)$, where $[n]$ is the K - theory class of the trivial n - dimensional bundle.

The definitions of K - theory are somewhat abstract. The following discussion makes it clear precisely what K - theory measures in the case of compact spaces.

Definition 3.16. *Let ζ and η be vector bundles over a space X. ζ and η are said to be stably isomorphic* if for some m and n, there is an isomorphism

$$\zeta \oplus \epsilon_n \cong \eta \oplus \epsilon_m$$

where, as above, ϵ_k denotes the trivial bundle of dimension k. We let $\mathcal{S}Vect(X)$ denote the set of stable isomorphism classes of vector bundles over X.

Notice that $\mathcal{S}Vect(X)$ is also an abelian monoid under Whitney sum, and that since any two trivial bundles are stably isomorphic, and that adding a trivial bundle to a bundle does not change the stable isomorphic class, then any trivial bundle represents the zero element of $\mathcal{S}Vect(X)$.

Theorem 3.17. *Let X be a compact space, then $\mathcal{S}Vect(X)$ is an abelian group and is isomorphic to the reduced K -theory,*

$$\mathcal{S}Vect(X) \cong \tilde{K}(X).$$

Proof. A main component of the proof is the following result, which we will prove in the next chapter when we study the classification of vector bundles.

Theorem 3.18. *Every vector bundle over a compact space can be embedded in a trivial bundle. That is, if ζ is a bundle over a compact space X , then for sufficiently large $N > 0$, there is bundle embedding*

$$\zeta \hookrightarrow \epsilon_N.$$

We use this result in the following way in order to prove the above theorem. Let ζ be a bundle over a compact space X. Then by this result we can find an embedding $\zeta \hookrightarrow \epsilon_N$. Let $\zeta^\perp$ be the orthogonal complement bundle to this embedding. So that

$$\zeta \oplus \zeta^\perp = \epsilon_N.$$

Since ϵ_N represents the zero element in $\mathcal{S}Vect(X)$, then as an equation in $\mathcal{S}Vect(X)$ this becomes

$$[\zeta] + [\zeta^\perp] = 0.$$

Thus every element in $\mathcal{S}Vect(X)$ is invertible in the monoid structure, and hence $\mathcal{S}Vect(X)$ is an abelian group.

To prove that $\mathcal{S}Vect(X)$ is isomorphic to $\tilde{K}(X)$, notice that the natural surjection of $Vect^*(X)$ onto $\mathcal{S}Vect(X)$ is a morphism of abelian monoids, and since $\mathcal{S}Vect(X)$ is an abelian group, this surjection extends linearly to a surjective homomorphism of abelian groups,

$$\rho : K(X) \to \mathcal{S}Vect(X).$$

Since $[\epsilon_n] = [n] \in K(X)$ maps to zero in $\mathcal{S}Vect(X)$ under ρ, this map factors through a surjective homomorphism from reduced K - theory, which by abuse of notation we also call ρ,

$$\rho : \tilde{K}(X) \to \mathcal{S}Vect(X).$$

To prove that ρ is a injective (and hence an isomorphism), we will construct a left inverse to ρ. This is done by considering the composition

$$Vect^*(X) \xrightarrow{\iota} K(X) \to \tilde{K}(X)$$

which is given by mapping an n - dimensional bundle ζ to $[\zeta] - [n]$. This map clearly sends two bundles which are stably isomorphic to the same class in $\tilde{K}(X)$, and hence factors through a homomorphism

$$j : \mathcal{S}Vect(X) \to \tilde{K}(X).$$

By checking its values on bundles, it becomes clear that the composition $j \circ \rho : \tilde{K}(X) \to \mathcal{S}Vect(X) \to \tilde{K}(X)$ is the identity map. This proves the theorem. □

We end this section with the following observation. As we said above, in the next chapter we will study the classification of bundles. In the process we will show that homotopic maps induce isomorphic pull - back bundles, and therefore homotopy equivalences induce bijections, via pulling back, on the sets of isomorphism classes of bundles. This tells us that K -theory is a "homotopy invariant" of topological spaces and continuous maps between them. More precisely, the results of the next chapter will imply the following important properties of K - theory.

Theorem 3.19. *Let $f : X \to Y$ and $g : X \to Y$ be homotopic maps. then the pull back homomorphisms are equal*

$$f^* = g^* : K(Y) \to K(X)$$

and

$$f^* = g^* : KO(Y) \to KO(X).$$

This can be expressed in categorical language as follows: (Notice the similarity of role K - theory plays in the following theorem to cohomology theory.)

Theorem 3.20. *The assignments $X \to K(X)$ and $X \to KO^*(X)$ are contravariant functors from the category of topological spaces and homotopy classes of continuous maps to the category of rings and ring homomorphisms.*

3.3.3 Differential Forms

In the next two sections we describe certain differentiable constructions on bundles over smooth manifolds that are basic in geometric analysis. We begin by recalling some "multilinear algebra".

Let V be a vector space over a field k. Let $T(V)$ be the associated tensor algebra

$$T(V) = \oplus_{n \geq 0} V^{\otimes n}$$

where $V^0 = k$. The algebra structure is comes from the natural pairings

$$V^{\otimes n} \otimes V^{\otimes m} \xrightarrow{=} V^{\otimes (n+m)}.$$

Recall that the *exterior algebra*

$$\Lambda(V) = T(V)/\mathcal{A}$$

where $\mathcal{A} \subset T(V)$ is the two sided ideal generated by $\{a \otimes b + b \otimes a : a, b \in V\}$.

The algebra $\Lambda(V)$ inherits the grading from the tensor algebra, $\Lambda(V) = \oplus_{n \geq 0} \Lambda^k(V)$, and the induced multiplication is called the "wedge product", $u \wedge v$. Recall that if V is an n - dimensional vector space, $\Lambda^k(V)$ is an $\binom{n}{k}$ - dimensional vector space.

Assume now that V is a real vector space. An element of the dual space, $(V^{\otimes n})^* = Hom(V^{\otimes n}, \mathbb{R})$ is a multilinear form $V \times \cdots \times V \to \mathbb{R}$. An element of the dual space $(\Lambda^k(V))^*$ is an alternating form, i.e a multilinear function θ so that

$$\theta(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) = sgn(\sigma)\theta(v_1, \cdots, v_k)$$

where $\sigma \in \Sigma_k$ is any permutation.

Let $\mathcal{A}^k(V) = (\Lambda^k(V))^*$ be the space of alternating k - forms. Let $U \subset \mathbb{R}^n$ be an open set. Recall the following definition.

Definition 3.17. *A differential k - form on the open set $U \subset \mathbb{R}^n$ is a smooth function*

$$\omega : U \to \mathcal{A}^k(\mathbb{R}^n).$$

By convention, 0 -forms are just smooth functions, $f : U \to \mathbb{R}$. Notice that given such a smooth function, its differential, df assigns to a point $x \in U \subset \mathbb{R}^n$ a linear map on tangent spaces, $df(x) : \mathbb{R}^n = T_x\mathbb{R}^n \to T_{f(x)}\mathbb{R} = \mathbb{R}$. That is, $df : U \to (\mathbb{R}^n)^*$, and hence is a one form on U.

Let $\Omega^k(U)$ denote the space of k - forms on the open set U. Recall that any k -form $\omega \in \Omega^k(U)$ can be written in the form

$$\omega(x) = \sum_I f_I(x) dx_I \tag{3.4}$$

where the sum is taken over all sequences of length k of integers from 1 to n, $I = (i_1, \cdots, i_k)$, $f_I : U \to \mathbb{R}$ is a smooth function, and where

$$dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Here dx_i denotes the differential of the function $x_i : U \subset \mathbb{R}^n \to \mathbb{R}$ which is the projection onto the i^{th} - coordinate.

Recall also that there is an exterior derivative,

$$d : \Omega^k(U) \to \Omega^{k+1}(U)$$

defined by

$$d(f dx_I) = df \wedge dx_I == \sum_{j=1}^{k} \frac{\partial f}{\partial x_j} dx_j \wedge dx_I$$

A simple calculation shows that $d^2(\omega) = d(d\omega) = 0$, using the symmetry of second order partial derivatives.

These constructions can be extended to arbitrary manifolds in the following way. Given an n - dimensional smooth manifold M, let $\Lambda^k(T(M))$ be the $\binom{n}{k}$ - dimensional vector bundle whose fiber at $x \in M$ is the k - fold exterior product, of the tangent space, $\Lambda^k(T_xM)$.

Exercise.

Define clutching functions of $\Lambda^k(T(M))$ in terms of clutching functions of the tangent bundle, $T(M)$

Definition 3.18. *A differential k-form on M is a section of the dual bundle,*

$$\Lambda^k(T(M))^* \cong \Lambda^k(T^*(M)) \cong Hom(\Lambda^k(T(M)), \epsilon_1).$$

That is, the space of k -forms is given by the space of sections,

$$\Omega^k(M) = \Gamma(\Lambda^k(T^*(M))).$$

So a k -form $\omega \in \Omega^k(M)$ assigns to $x \in M$ an alternating k form on its tangent space,

$$\omega(x) : T_xM \times \cdots \times T_xM \to \mathbb{R}.$$

and hence given a local chart with a local coordinate system, then locally ω can be written in the form (3.4).

Since differentiation is a local operation, we may extend the definition of the exterior derivative of forms on open sets in $\mathbb{R}^n$ to all n - manifolds,

$$d : \Omega^k(M) \to \Omega^{k+1}(M).$$

In particular, the zero forms are the space of functions, $\Omega^0(M) = C^\infty(M;\mathbb{R})$, and for $f \in \Omega^0(M)$, then $df \in \Omega^1(M) = \Gamma(T(M)^*)$ is the 1 -form defined by the differential,

$$df(x) : T_xM \to T_{f(x)}\mathbb{R} = \mathbb{R}.$$

Now as above, $d^2(\omega) = 0$ for any form ω. Thus we have a cochain complex, called the *deRham* complex,

$$\begin{array}{l}\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{k-1}(M) \xrightarrow{d} \Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M) \\ \qquad \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \xrightarrow{d} 0.\end{array} \tag{3.5}$$

Recall that a k - form ω with $d\omega = 0$ is called a *closed* form. A k - form ω in the image of d, i.e $\omega = d\eta$ for some $\eta \in \Omega^{k-1}(M)$ is called an *exact* form. The quotient vector space of closed forms modulo exact forms defined the "deRham cohomology" group:

Definition 3.19.

$$H^k_{deRham}(M) = \{closed\ k\ \text{-}\ forms\}/\{exact\ k\ \text{-}\ forms\}.$$

The famous *de Rham theorem* asserts that these cohomology groups are isomorphic to singular cohomology with $\mathbb{R}$ - coefficients. To see the relationship, let $C_k(M)$ be the space of k - dimensional singular chains on M, (i.e the free abelian group generated by smooth singular simplices $\sigma : \Delta^k \to M$), and let

$$C^k(M;\mathbb{R}) = Hom(C_k(M), \mathbb{R})$$

be the space of real valued singular cochains. Notice that a k -form ω gives rise to a k - dimensional singular cochain in that it acts on a singular simplex $\sigma : \Delta^k \to M$ by

$$\langle \omega, \sigma \rangle = \int_\sigma \omega.$$

This defines a homomorphism

$$\gamma : \Omega^k(M) \to C^k(M;\mathbb{R})$$

for each k.

Exercise. Prove that γ is a map of cochain complexes. That is,

$$\gamma(d\omega) = \delta\gamma(\omega)$$

where $\delta : C^k(M;\mathbb{R}) \to C^{k+1}(M;\mathbb{R})$ is the singular coboundary operator.
Hint. Use Stokes' theorem.

We refer the reader to [12] for a proof of the *deRham Theorem:*

Theorem 3.21. *The map of cochain complexes,*

$$\gamma : \Omega^*(M) \to C^*(M;\mathbb{R})$$

is a chain homotopy equivalence. Therefore it induces an isomorphism in cohomology

$$H^*_{deRham}(M) \xrightarrow{\cong} H^*(M;\mathbb{R}).$$

3.3.4 Lie Groups

Lie groups play a central role in bundle theory and in differential topology and geometry. In this section we give a basic description of Lie groups, their actions on manifolds (and other spaces), as well as their their principal bundles.

Definition 3.20. *A Lie group is a topological group G which has the structure of a differentiable manifold. Moreover the multiplication map*

$$G \times G \to G$$

and the inverse map

$$\begin{aligned} G &\to G \\ g &\to g^{-1} \end{aligned}$$

are required to be differentiable maps.

The following is an important basic property of the differential topology of Lie groups.

Theorem 3.22. *Let G be a Lie group. Then G is parallelizable. That is, its tangent bundle TG is trivial.*

Proof. Let $1 \in G$ denote the identity element, and T_1G the tangent space of G at 1. If G is an n - dimensional manifold, T_1G is an n- dimensional vector space. We define a bundle isomorphism of the tangent bundle TG with the trivial bundle $G \times T_1(G)$, which, on the total space level is given by a map

$$\phi : G \times T_1G \longrightarrow TG$$

defined as follows. Let $g \in G$. Then multiplication by g on the right is a diffeomorphism

$$\begin{aligned} \times g : G &\to G \\ x &\to xg \end{aligned}$$

Since $\times g$ is a diffeomorphism, its derivative is a linear isomorphism at every point:

$$Dg(x) : T_xG \xrightarrow{\cong} T_{xg}G.$$

We can now define

$$\phi : G \times T_1G \to TG$$

by

$$\phi(g, v) = Dg(1)(v) \in T_gG.$$

Clearly ϕ is a bundle isomorphism. □

If G is a Lie group and M is a smooth manifold with a right G - action. We say that the action is *smooth* if the homomorphism μ defined above factors through a homomorphism

$$\mu : G \to Diffeo(M)$$

where $Diffeo(M)$ is the group of diffeomorphisms of M.

The following result is originally due to A. Gleason [36], and its proof can be found in Steenrod's book [90]. It is quite helpful in studying free group actions.

Theorem 3.23. *Let E be a smooth manifold, having a free, smooth G - action, where G is a compact Lie group. Then the action has slices. In particular, the projection map*

$$p : E \to E/G$$

defines a principal G - bundle.

The following was one of the early theorems in fiber bundle theory, appearing originally in H. Samelson's thesis. [82]

Corollary 3.24. *Let G be a Lie group, and let $H < G$ be a compact subgroup. Then the projection onto the orbit space*

$$p : G \to G/H$$

is a principal H - bundle.

3.3.5 Connections and Curvature

In modern geometry, differential topology, and geometric analysis, one often needs to study not only smooth functions on a manifold, but more generally, spaces of smooth sections of a vector bundle $\Gamma(\zeta)$. (Notice that sections of bundles are indeed a generalization of smooth functions in that the space of sections of the n - dimensional trivial bundle over a manifold M, $\Gamma(\epsilon_n) = C^\infty(M; \mathbb{R}^n) = \oplus_n C^\infty(M; \mathbb{R})$.) Similarly, one needs to study differential forms that take values in vector bundles. These are defined as follows.

Definition 3.21. *Let ζ be a smooth bundle over a manifold M. A differential k - form with values in ζ is defined to be a smooth section of the bundle of homomorphisms, $Hom(\Lambda^k(T(M)), \zeta) = \Lambda^k(T(M)^*) \otimes \zeta$.*

We write the space of k -forms with values in ζ as

$$\Omega^k(M;\zeta) = \Gamma(\Lambda^k(T(M)^* \otimes \zeta).$$

The zero forms are simply the space of sections, $\Omega^0(M;\zeta) = \Gamma(\zeta)$. Notice that if ζ is the trivial bundle $\zeta = \epsilon_n$, then one gets standard forms,

$$\Omega^k(M;\epsilon_n) = \Omega^k(M) \otimes \mathbb{R}^n = \oplus_n \Omega^k(M).$$

Even though spaces of forms with values in a bundle are easy to define, there is no canonical analogue of the exterior derivative. There do however exist differential operators

$$D : \Omega^k(M;\zeta) \to \Omega^{k+1}(M;\zeta)$$

that satisfy familiar product formulas. These operators are called *covariant derivatives* (or *connections*) and are related to the notion of a connection on a principal bundle, which we now define and study.

Let G be a compact Lie group. Recall that the tangent bundle TG has a canonical trivialization

$$\begin{aligned} \psi : G \times T_1G &\to TG \\ (g,v) &\to D(\ell_g)(v) \end{aligned}$$

where for any $g \in G$, $\ell_g : G \to G$ is the map given by left multiplication by g, and $D(\ell_g) : T_hG \to T_{gh}G$ is its derivative. r_g and $D(r_g)$ will denote the analogous maps corresponding to right multiplication.

The differential of right multiplication on G defines a right action of G on the tangent bundle TG. We claim that the trivialization ψ is equivariant with respect to this action, if we take as the right action of G on T_1G to be the *adjoint action:*

$$\begin{aligned} T_1G \times G &\to T_1G \\ (v,g) &\to D(\ell_{g^{-1}})(v)D(r_g). \end{aligned}$$

Exercise. Verify this claim.

As is standard, we identify T_1G with the Lie algebra $\mathfrak{g}$. This action is referred to as the *adjoint representation* of the Lie group G on its Lie algebra $\mathfrak{g}$. Now let

$$p : P \to M$$

be a smooth principal G -bundle over a manifold M. This adjoint representation induces a vector bundle $ad(P)$,

$$ad(P) : P \times_G \mathfrak{g} \to M. \tag{3.6}$$

This bundle has the following relevance. Let $p^*(TM) : p^*(TM) \to P$ be the pull - back over the total space P of the tangent bundle of M. We have a surjective map of bundles

$$TP \to p^*(TM).$$

Define $T_F P$ to be the kernel bundle of this map. So the fiber of $T_F P$ at a point $y \in P$ is the kernel of the surjective linear transformation $Dp(y) : T_y P \to T_{p(y)} M$. Notice that the right action of G on the total space of the principal bundle P defines an action of G on the tangent bundle TP, which restricts to an action of G on $T_F P$. Furthermore, by recognizing that the fibers are equivariantly homeomorphic to the Lie group G, the following is a direct consequence of the above considerations:

Proposition 3.25. *$T_F P$ is naturally isomorphic to the pull - back of the adjoint bundle,*

$$T_F P \cong p^*(ad(P)).$$

Thus we have an exact sequence of G - equivariant vector bundles over P:

$$0 \to p^*(ad(P)) \to TP \xrightarrow{Dp} p^*(TM) \to 0. \tag{3.7}$$

Recall that short exact sequences of bundles split as Whitney sums. A connection is a G - equivariant splitting of this sequence:

Definition 3.22. *A* **connection** on the principal bundle P is a G - equivariant splitting

$$\omega_A : TP \to p^*(ad(P))$$

of the above sequence of vector bundles. That is, ω_A defines a G - equivariant isomorphism

$$\omega_A \oplus Dp : TP \to p^*(ad(P)) \oplus p^*(TM).$$

The following is an important description of the space of connections on P, $\mathcal{A}(P)$.

Proposition 3.26. *The space of connections on the principal bundle P, $\mathcal{A}(P)$, is an affine space modeled on the infinite dimensional vector space of one forms on M with values in the bundle $ad(P)$, $\Omega^1(M; ad(P))$.*

Proof. Consider two connections ω_A and ω_B,

$$\omega_A, \omega_B : TP \to p^*(ad(P).$$

Since these are splittings of the exact sequence 3.7, they are both the identity when restricted to $p^*(ad(P)) \hookrightarrow TP$. Thus their difference, $\omega_A - \omega_B$ is zero when restricted to $p^*(ad(P))$. By the exact sequence it therefore factors as a composition

$$\omega_A - \omega_B : TP \to p^*(TM) \xrightarrow{\alpha} p^*(ad(P))$$

for some bundle homomorphism $\alpha : p^*(TM) \to p^*(ad(P))$. That is, for every $y \in P$, α defines a linear transformation

$$\alpha_y : p^*(TM)_y \to p^*(ad(P))_y.$$

Hence for every $y \in P$, α defines (and is defined by) a linear transformation

$$\alpha_y : T_{p(y)}M \to ad(P)_{p(y)}.$$

Furthermore, the fact that both ω_A and ω_B are *equivariant* splittings says that $\omega_A - \omega_B$ is equivariant, which translates to the fact that α_y only depends on the orbit of y under the G - action. That is,

$$\alpha_y = \alpha_{yg} : T_{p(y)}M \to ad(P)_{p(y)}$$

for every $g \in G$. Thus α_y only depends on $p(y) \in M$. Hence for every $x \in M$, α defnes, and is defined by, a linear transformation

$$\alpha_x : T_xM \to ad(P)_x.$$

Thus α may be viewed as a section of the bundle of homormorphisms, $Hom(TM, ad(P))$, and hence is a one form,

$$\alpha \in \Omega^1(M; ad(P)).$$

Thus any two connections on P differ by an element in $\Omega^1(M; ad(P))$ in this sense.

Now reversing the procedure, an element $\beta \in \Omega^1(M; ad(P))$ defines an equivariant homomorphism of bundles over P,

$$\beta : p^*(TM) \to p^*(ad(P)).$$

By adding the composition

$$TP \xrightarrow{Dp} p^*(TM) \xrightarrow{\beta} p^*(ad(P))$$

to any connection (equivariant splitting)

$$\omega_A : TP \to p^*(ad(P))$$

one produces a new equivariant splitting of TP, and hence a new connection. The proposition follows. □

Remark. Even though the space of connections $\mathcal{A}(P)$ is affine, it is not, in general a vector space. There is no "zero" in $\mathcal{A}(P)$ since there is no pre-chosen, canonical connection. The one exception to this, of course, is when P is the trivial G - bundle,

$$P = M \times G \to M.$$

In this case there is an obvious equvariant splitting of TP, which serves as the "zero" in $\mathcal{A}(P)$. Moreover in this case the adjoint bundle $ad(P)$ is also trivial,

$$ad(P) = M \times \mathfrak{g} \to M.$$

Hence there is a canonical identification of the space of connections on the trivial bundle with $\Omega^1(M; \mathfrak{g}) = \Omega^1(M) \otimes \mathfrak{g}$.

Let $p : P \to M$ be a principal G - bundle and let $\omega_A \in \mathcal{A}(P)$ be a connection.

The curvature F_A of ω_A is a two form

$$F_A \in \Omega^2(M; ad(P))$$

which measures to what extent the splitting ω_A commutes with the braket operation on vector fields. More precisely, let X and Y be vector fields on M. The connection ω_A defines an equivariant splitting of TP and hence defines a "horizontal" lifting of these vector fields, which we denote by $\tilde{X}$ and $\tilde{Y}$ respectively.

Definition 3.23. *The curvature $F_A \in \Omega^2(M; ad(P))$ is defined by*

$$F_A(X, Y) = \omega_A[\tilde{X}, \tilde{Y}].$$

For those unfamiliar with the bracket operation on vector fields, we refer you to [89]

Another important construction with connections is the associated *covariant derivative* which is defined as follows.

Definition 3.24. *The covariant derivative induced by the connection ω_A*

$$D_A : \Omega^0(M; ad(P)) \to \Omega^1(M; ad(P))$$

is defined by

$$D_A(\sigma)(X) = [\tilde{X}, \sigma].$$

where X is a vector field on M.

The notion of covariant derivative, and hence connection, extends to vector bundles as well. Let $\zeta : p : E^\zeta \to M$ be a finite dimensional vector bundle over M.

Definition 3.25. *A connection on ζ (or a covariant derivative) is a linear transformation*

$$D_A : \Omega^0(M;\zeta) \to \Omega^1(M;\zeta)$$

that satisfies the Leibnitz rule

$$D_A(f\phi) = df \otimes \phi + f D_A(\phi) \tag{3.8}$$

for any $f \in C^\infty(M;\mathbb{R})$ and any $\phi \in \Omega^0(M;\zeta)$.

Now we can model the space of connections on a vector bundle, $\mathcal{A}(\zeta)$ similarly to how we modeled the space of connections on a principal bundle $\mathcal{A}(P)$. Namely, given any two connections D_A and D_B on ζ and a function $f \in C^\infty(M;\mathbb{R})$, one can take the convex combination

$$f \cdot D_A + (1-f) \cdot D_B$$

and obtain a new connection. From this it is not difficult to see the following. We leave the proof as an exercise to the reader.

Proposition 3.27. *The space of connections on the vector bundle ζ, $\mathcal{A}(\zeta)$ is an affine space modeled on the vector space of one forms $\Omega^1(M;End(\zeta))$, where $End(\zeta)$ is the bundle of endomorphisms of ζ.*

Let X be a vector field on M and D_A a connection on the vector bundle ζ. The covariant derivative in the direction of X, which we denote by $(D_A)_X$ is an operator on the space of sections of ζ,

$$(D_A)_X : \Omega^0(M;\zeta) \to \Omega^0(M;\zeta)$$

defined by

$$(D_A)_X(\sigma) = \langle D_A(\phi); X\rangle.$$

One can then define the curvature $F_A \in \Omega^2(M;End(\zeta))$ by defining its action on a pair of vector fields X and Y to be

$$F_A(X,Y) = (D_A)_X(D_A)_Y - (D_A)_Y(D_A)_X - (D_A)_{[X,Y]}. \tag{3.9}$$

To interpret this formula notice that a - priori $F_A(X,Y)$ is a second order differential operator on the space of sections of ζ. However a direct calculation shows that for $f \in C^\infty(M;\mathbb{R})$ and $\sigma \in \Omega^0(M;\zeta)$, then

$$F_A(X,Y)(f\sigma) = fF_A(X,Y)(\sigma)$$

and hence $F_A(X,Y)$ is in fact a zero - order operator on $\Omega^0(M;\zeta)$. But a zero order operator on the space of sections of ζ is a section of the endomorphism bundle $End(\zeta)$. Thus F_A assigns to any pair of vector fields X and Y a section of $End(\zeta)$. Moreover it is straightforward to check that this assignment is tensorial in X and Y (i.e $F_A(fX,Y) = F_A(X,fY) = fF_A(X,Y)$). Thus F_A is an element of $\Omega^2(M; End(\zeta))$. The curvature measures the lack of commutativity in second order partial covariant derivatives.

Given a connection on a bundle ζ the linear mapping $D_A : \Omega^0(M;\zeta) \to \Omega^1(M;\zeta)$ extends to a *deRham* type sequence,

$$\Omega^0(M;\zeta) \xrightarrow{D_A} \Omega^1(M;\zeta) \xrightarrow{D_A} \Omega^2(M;\zeta) \xrightarrow{D_A} \cdots$$

where for $\sigma \in \Omega^p(M;\zeta)$, $D_A(\sigma)$ is the $p+1$ -form defined by the formula

$$\begin{aligned} D_A(\sigma)(X_0, \cdots, X_p) = &\sum_{j=0}^{p} (-1)^j (D_A)_{X_j}(\sigma(X_0, \cdots, \hat{X}_j, \cdots, X_p)) \\ &+ \sum_{i<j} (-1)^{i+j} \sigma([X_i, X_j], X_0, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_p). \end{aligned} \tag{3.10}$$

We observe that unlike with the standard deRham exterior derivative (which can be viewed as a connection on the trivial line bundle), it is not generally true that $D_A \circ D_A = 0$. In fact we have the following, whose proof is a direct calculation that we leave to the reader.

Proposition 3.28.

$$D_A \circ D_A = F_A : \Omega^0(M;\zeta) \to \Omega^2(M;\zeta)$$

where in this context the curvature F_A is interpreted as a assigning to a section $\sigma \in \Omega^0(M;\zeta)$ the 2 - form $F_A(\sigma)$ which associates to vector fields X and Y the section $F_A(X,Y)(\sigma)$ as defined in (3.9).

Thus the curvature of a connection F_A can also be viewed as measuring the extent to which the covariant derivative D_A fails to form a cochain complex on the space of differential forms with values in the bundle ζ. However it is always true that the covariant derivative of the curvature tensor is zero. This is the well known *Bianchi identity* (see [89] for a complete discussion).

Theorem 3.29. *Let A be a connection on a vector bundle ζ. Then*

$$D_A F_A = 0.$$

We end this section by observing that if P is a principal G - bundle with a connection ω_A, then any representation of G on a finite dimensional vector space V induces a connection on the corresponding vector bundle

$$P \times_G V \to M.$$

We refer the reader to [43] and [89] for thorough discussions of the various ways of viewing connections. [7] has a nice, brief discussion of connections on principal bundles, and [32] and [55] have similarly concise discussions of connections on vector bundles.

3.3.6 The Levi - Civita Connection

Let M be a manifold equipped with a Riemannian structure. Recall that this is a Euclidean structure on its tangent bundle. In this section we will show how this structure induces a connection, or covariant derivative, on the tangent bundle. This connection is called the *Levi - Civita* connection associated to the Riemannian structure. Our treatment of this topic follows that of Milnor and Stasheff [74]

Let $D_A : \Omega^0(M;\zeta) \to \Omega^1(M;\zeta)$ be a connection (or covariant derivative) on an n - dimensional vector bundle ζ. Its curvature is a two- form with values in the endomorphism bundle

$$F_A \in \Omega^2(M; End(\zeta))$$

The endomorphism bundle can be described alternatively as follows. Let E_ζ be the principal $GL(n,\mathbb{R})$ bundle associated to ζ. Then of course $\zeta = E_\zeta \otimes_{GL(n,\mathbb{R})} \mathbb{R}^n$. The endomorphism bundle can then be described as follows. The proof is an easy exercise that we leave to the reader.

Proposition 3.30.

$$End(\zeta) \cong ad(\zeta) = E_\zeta \times_{GL(n,\mathbb{R})} M_n(\mathbb{R})$$

where $GL(n,\mathbb{R})$ acts on $M_n(\mathbb{R})$ by conjugation,

$$A \cdot B = ABA^{-1}.$$

Let ω be a differential p - form on M with values in $End(\zeta)$,

$$\omega \in \Omega^p(M; End(\zeta)) \cong \Omega^p(M; ad(\zeta)) = \Omega^p(M; E_\zeta \times_{GL(n,\mathbb{R})} M_n(\mathbb{R})).$$

Then on a coordinate chart $U \subset M$ with local trivialization $\psi : \zeta_{|_U} \cong U \times \mathbb{C}^n$ for ζ, (and hence the induced coordinate chart and local trivialization for $ad(\zeta)$), ω can be viewed as an $n \times n$ matrix of p -forms on M. We write

$$\omega = (\omega_{i,j}).$$

Of course this description depends on the coordinate chart and local trivialization chosen, but at any $x \in U$, then by the above proposition, two trivializations yield conjugate matrices. That is, if $(\omega_{i,j}(x))$ and $(\omega'_{i,j}(x))$ are two matrix descriptions of $\omega(x)$ defined by two different local trivializations of $\zeta_{|_U}$, then there exists an $A \in GL(n,\mathbb{C})$ with

$$A(\omega_{i,j}(x))A^{-1} = (\omega'_{i,j}(x)).$$

Now suppose the bundle ζ is equipped with a Euclidean structure. As seen earlier in this chapter this is equivalent to its associated principal $GL(n,\mathbb{R})$ - bundle E_ζ having a reduction to the structure group $O(n)$. We let $E_{O(n)} \to M$ denote this principal $O(n)$ - bundle.

Now the Lie algebra $\mathfrak{o}(n)$ of $O(n)$ (i.e the tangent space $T_1(O(n))$) is a subspace of the Lie algebra of $GL(n,\mathbb{R})$, i.e

$$\mathfrak{o}(n) \subset M_n(\mathbb{R}).$$

The following is well known (see, for example[83])

Proposition 3.31. *The Lie algebra $\mathfrak{o}(n) \subset M_n(\mathbb{R})$ is the subspace consisting of skew symmetric $n \times n$ - matrices. That is, $A \in \mathfrak{o}(n)$ if and only if*

$$A^t = -A$$

where A^t is the transpose.

So if ζ has a Euclidean structure, we can form the adjoint bundle

$$ad^O(\zeta) = E_{O(n)} \times_{O(n)} \mathfrak{o}(n) \subset E_\zeta \times_{GL(n,\mathbb{R})} M_n(\mathbb{R}) = ad(\zeta)$$

where, again $O(n)$ acts on $\mathfrak{o}(n)$ by conjugation.

Now suppose D_A is an orthogonal connection on ζ. That is, it is induced by a connection on the principal $O(n)$ - bundle $E_{O(n)} \to M$. The following is fairly clear, and we leave its proof as an exercise.

Corollary 3.32. *If D_A is an orthogonal connection on a Euclidean bundle ζ, then the curvature F_A lies in the space of $\mathfrak{o}(n)$ valued two forms*

$$F_A \in \Omega^2(M; ad^O(\zeta)) \subset \Omega^2(M; ad(\zeta)) = \Omega^2(M; End(\zeta)).$$

Furthermore, on a coordinate chart $U \subset M$ with local trivialization $\psi : \zeta_{|_U} \cong U \times \mathbb{C}^n$ that preserves the Euclidean structure, we may write the form F_A as a skew - symmetric matrix of two forms,

$$F_{A_{|_U}} = (\omega_{i,j}) \quad i,j = 1, \cdots, n$$

where each $\omega_{i,j} \in \Omega^2(M)$ and $\omega_{i,j} = -\omega_{j,i}$. In fact the connection D_A itself can be written as skew symmetric matrix of one forms

$$D_{A_{|_U}} = (\alpha_{i,j})$$

where each $\alpha_{i,j} \in \Omega^1(M)$.

We now describe the notion of a "symmetric" connection on the cotangent bundle of a manifold, and then show that if the manifold is equipped with a Riemannian structure (i.e there is a Euclidean structure on the (co) - tangent bundle), then there is a unique symmetric, orthogonal connection on the cotangent bundle.

Definition 3.26. *A connection D_A on the cotangent bundle T^*M is symmetric* (or *torsion free*) if the composition

$$\Gamma(T^*) = \Omega^0(M;T^*) \xrightarrow{D_A} \Omega^1(M;T^*) = \Gamma(T^* \otimes T^*) \xrightarrow{\wedge} \Gamma(\Lambda^2 T^*)$$

is equal to the exterior derivative d.

In terms of local coordinates $x_1, \cdots, x_n$, if we write

$$D_A(dx_k) = \sum_{i,j} \Gamma^k_{i,j} dx_i \otimes dx_j \tag{3.11}$$

(the functions $\Gamma^k_{i,j}$ are called the "Christoffel symbols"), then the requirement that D_A is symmetric is that the image $\sum_{i,j} \Gamma^k_{i,j} dx_i \otimes dx_j$ be equal to the exterior derivative $d(dx_k) = 0$. This implies that the Christoffel symbols $\Gamma^k_{i,j}$ must be symmetric in i and j. The following is straightforward to verify.

Lemma 3.33. *A connection D_A on T^* is symmetric if and only if the covariant derivative of the differential of any smooth function*

$$D_A(df) \in \Gamma(T^* \otimes T^*)$$

is a symmetric tensor. That is, if $\psi_1, \cdots, \psi_n$ form a local basis of sections of T^*, and we write the corresponding local expression

$$D_A(df) = \sum_{i,j} a_{i,j}\, \psi_i \otimes \psi_j$$

then $a_{i,j} = a_{j,i}$.

We now show that the (co)-tangent bundle of a Riemannian metric has a preferred connection.

Theorem 3.34. *The cotangent bundle T^*M of a Riemannian manifold has a unique orthogonal, symmetric connection. (It is orthogonal with respect to the Euclidean structure defined by the Riemannian metric.)*

Proof. Let U be an open neighborhood in M with a trivialization

$$\psi : U \times \mathbb{R}^n :\to T^*_{|U}$$

which preserves the Euclidean structure. ψ defines n orthonormal sections of $T^*_{|U}$, $\psi_1, \cdots, \psi_n$. The ψ_j's constitute an orthonormal basis of one forms on M. We will show that there is one and only one skew-symmetric matrix $(\alpha_{i,j})$ of one forms such that

$$d\psi_k = \sum \alpha_{k,j} \wedge \psi_j.$$

We can then define a connection D_A on $T^*_{|U}$ by requiring that

$$D_A(\psi_k) = \sum \alpha_{k,j} \otimes \psi_j.$$

It is then clear that D_A is the unique symmetric connection which is compatible with the metric. Since the local connections are unique, they glue together to yield a unique global connection with this property.

In order to prove the existence and uniqueness of the skew symmetric matrix of one forms $(\alpha_{i,j})$ we need the following combinatorial observation.

Any $n \times n \times n$ array of real valued functions $A_{i,j,k}$ can be written uniquely as the sum of an array $B_{i,j,k}$ which is symmetric in i, j, and an array $C_{i,j,k}$ which is skew symmetric in j, k. To see this, consider the formulas

$$B_{i,j,k} = \frac{1}{2}(A_{i,j,k} + A_{j,i,k} - A_{k,i,j} - A_{k,j,i} + A_{j,k,i} + A_{i,k,j})$$
$$C_{i,j,k} = \frac{1}{2}(A_{i,j,k} - A_{j,i,k} + A_{k,i,j} + A_{k,j,i} - A_{j,k,i} - A_{i,k,j})$$

Uniqueness would follow since if an array $D_{i,j,k}$ were both symmetric in i, j and skew symmetric in j, k, then one would have

$$D_{i,j,k} = D_{j,i,k} = -D_{j,k,i} = -D_{k,j,i} = D_{k,i,j} = D_{i,k,j} = -D_{i,j,k}$$

and hence all the entries are zero.

Now choose functions $A_{i,j,k}$ such that

$$d\psi_k = \sum A_{i,j,k}\, \psi_i \wedge \psi_j$$

and set $A_{i,j,k} = B_{i,j,k} + C_{i,j,k}$ as above. It then follows that

$$d\psi_k = \sum C_{i,j,k}\, \psi_i \wedge \psi_j$$

by the symmetry of the $B_{i,j,k}$'s. Then we define the one forms

$$\alpha_{k,j} = \sum C_{i,j,k}\, \psi_i.$$

They clearly form the unique skew symmetric matrix of one forms with $d\psi_k = \sum \alpha_{k,j} \wedge \psi_j$. This proves the lemma. □

This preferred connection on the (co)tangent bundle of a Riemannian metric is called the Levi - Civita connection. Statements about the curvature of a metric on a manifold are actually statements about the curvature form of the Levi - Civita connection associated to the Riemannian metric. For example, a "flat metric" on a manifold is a Riemannian structure whose corresponding Levi-Civita connection has zero curvature form. As is fairly clear, these connections form a central object of study in Riemannian geometry.

4

Classification of Bundles

In this chapter we prove Steenrod's classification theorem of principal G - bundles, and the corresponding classification theorem of vector bundles. This theorem states that for every group G, there is a "classifying space" BG with a well defined homotopy type so that the homotopy classes of maps from a space X, $[X, BG]$, is in bijective correspondence with the set of isomorphism classes of principal G - bundles, $Prin_G(X)$. We then describe various examples and constructions of these classifying spaces, and use them to study structures on principal bundles, vector bundles, and manifolds.

4.1 The homotopy invariance of fiber bundles

The goal of this section is to prove the following theorem, and to examine certain applications such as the classification of principal bundles over spheres in terms of the homotopy groups of Lie groups.

Theorem 4.1. *Let $p : E \to B$ be a fiber bundle with fiber F, and let $f_0 : X \to B$ and $f_1 : X \to B$ be homotopic maps. Then the pull - back bundles are isomorphic,*

$$f_0^*(E) \cong f_1^*(E).$$

The main step in the proof of this theorem is the basic *Covering Homotopy Theorem* for fiber bundles which we now state and prove.

Theorem 4.2. Covering Homotopy theorem. *Let $p_0 : E \to B$ and $q : Z \to Y$ be fiber bundles with the same fiber, F, where B is normal and locally compact. Let h_0 be a bundle map*

$$\begin{array}{ccc} E & \xrightarrow{\tilde{h}_0} & Z \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ B & \xrightarrow[h_0]{} & Y \end{array}$$

Let $H : B \times I \to Y$ be a homotopy of h_0 (i.e $h_0 = H_{|_{B \times \{0\}}}$.) Then there exists a covering of the homotopy H by a bundle map

$$\begin{array}{ccc} E \times I & \xrightarrow{\tilde{H}} & Z \\ {\scriptstyle p\times 1}\downarrow & & \downarrow{\scriptstyle q} \\ B \times I & \xrightarrow[H]{} & Y. \end{array}$$

Proof. We prove the theorem here when the base space B is compact. The natural extension is to when B has the homotopy type of a CW - complex. The proof in full generality can be found in Steenrod's book [90].

The idea of the proof is to decompose the homotopy H into homotopies that take place in local neighborhoods where the bundle is trivial. The theorem is obviously true for trivial bundles, and so the homotopy H can be covered on each local neighborhood. One then must be careful to patch the coverings together so as to obtain a global covering of the homotopy H.

Since the space X is compact, we may assume that the pull - back bundle $H^*(Z) \to B \times I$ has locally trivial neighborhoods of the form $\{U_\alpha \times I_j\}$, where $\{U_\alpha\}$ is a locally trivial covering of B (i.e there are local trivializations $\phi_{\alpha,\beta} : U_\alpha \times F \to p^{-1}(U_\alpha)$), and $I_1, \cdots, I_r$ is a finite sequence of open intervals covering $I = [0, 1]$, so that each I_j meets only I_{j-1} and I_{j+1} nontrivially. Choose numbers

$$0 = t_0 < t_1 < \cdots < t_r = 1$$

so that $t_j \in I_j \cap I_{j+1}$. We assume inductively that the covering homotopy $\tilde{H}(x, t)$ has been defined $E \times [0, t_j]$ so as to satisfy the theorem over this part.

For each $x \in B$, there is a pair of neighborhoods (W, W') such that for $x \in W$, $\bar{W} \subset W'$ and $\bar{W}' \subset U_\alpha$ for some U_α. Choose a finite number of such pairs (W_i, W_i'), $(i = 1, \cdots, s)$ covering B. Then the Urysohn lemma implies there is a map $u_i : B \to [t_j, t_{j+1}]$ such that $u_i(\bar{W}_i) = t_{j+1}$ and $u_j(B - W_i') = t_j$. Define $\tau_0(x) = t_j$ for $x \in B$, and

$$\tau_i(x) = max(u_1(x), \cdots, u_i(x)), \quad x \in B, \quad i = 1, \cdots, s.$$

Then

$$t_j = \tau_0(x) \leq \tau_1(x) \leq \cdots \leq t_s(x) = t_{j+1}.$$

Define B_i to be the set of pairs (x, t) such that $t_j \leq t \leq \tau_i(x)$. Let E_i be the part of $E \times I$ lying over B_i. Then we have a sequence of total spaces of bundles

$$E \times t_j = E_0 \subset E_1 \subset \cdots \subset E_s = E \times [t_j, t_{j+1}].$$

We suppose inductively that $\tilde{H}$ has been defined on E_{i-1} and we now define its extension over E_i.

By the definition of the τ's, the set $B_i - B_{i-1}$ is contained in $W_i' \times [t_j, t_{j+1}]$;

and by the definition of the W's, $\bar{W'}_i \times [t_j, t_{j+1}] \subset U_\alpha \times I_j$ which maps via H to a locally trivial neighborhood, say V_k, for $q : Z \to Y$. Say $\phi_k : V_k \times F \to q^{-1}(V_k)$ is a local trivialization. In particular we can define $\rho_k : q^{-1}(V_k) \to F$ to be the inverse of ϕ_k followed by the projection onto F. We now define

$$\tilde{H}(e,t) = \phi_k(H(x,t), \rho(\tilde{H}(e, \tau_{i-1}(x)))$$

where $(e,t) \in E_i - E_{i-1}$ and $x = p(e) \in B$.

It is now a straightforward verification that this extension of $\tilde{H}$ is indeed a bundle map on E_i. This then completes the inductive step. □

We now prove theorem 4.1 using the covering homotopy theorem.

Proof. Let $p : E \to B$, and $f_0; X \to B$ and $f_1 : X \to B$ be as in the statement of the theorem. Let $H : X \times I \to B$ be a homotopy with $H_0 = f_0$ and $H_1 = f_1$. Now by the covering homotopy theorem there is a covering homotopy $\tilde{H} : f_0^*(E) \times I \to E$ that covers $H : X \times I \to B$. By definition this defines a map of bundles over $X \times I$, that by abuse of notation we also call $\tilde{H}$,

$$\begin{array}{ccc} f_0^*(E) \times I & \xrightarrow{\tilde{H}} & H^*(E) \\ \downarrow & & \downarrow \\ X \times I & \xrightarrow[=]{} & X \times I. \end{array}$$

This is clearly a bundle isomorphism since it induces the identity map on both the base space and on the fibers. Restricting this isomorphism to $X \times \{1\}$, and noting that since $H_1 = f_1$, we get a bundle isomorphism

$$\begin{array}{ccc} f_0^*(E) & \xrightarrow[\cong]{\tilde{H}} & f_1^*(E) \\ \downarrow & & \downarrow \\ X \times \{1\} & \xrightarrow[=]{} & X \times \{1\}. \end{array}$$

This proves theorem 4.1 □

We now derive certain consequences of this theorem.

Corollary 4.3. *Let $p : E \to B$ be a principal G - bundle over a connected space B. Then for any space X the pull back construction gives a well defined map from the set of homotopy classes of maps from X to B to the set of isomorphism classes of principal G - bundles,*

$$\rho_E : [X, B] \to Prin_G(X).$$

Definition 4.1. *A principal G - bundle $p : EG \to BG$ is called* universal if the pull back construction

$$\rho_{EG} : [X, BG] \to Prin_G(X)$$

is a bijection for every space X of the homotopy type of a CW complex. The base space of the universal bundle BG is called a *classifying space* for G (or for principal G - bundles).

The main goal of this chapter is to show that universal bundles exist for every group G, and that the classifying spaces are unique up to homotopy type.

Applying theorem 4.1 to vector bundles gives the following, which was claimed at the end of chapter 1.

Corollary 4.4. *If $f_0 : X \to Y$ and $f_1 : X \to Y$ are homotopic, they induce the same homomorphism of abelian monoids,*

$$\begin{aligned} f_0^* = f_1^* : Vect^*(Y) &\to Vect^*(X) \\ Vect^*_{\mathbb{R}}(Y) &\to Vect^*_{\mathbb{R}}(X) \end{aligned}$$

and hence of K theories

$$\begin{aligned} f_0^* = f_1^* : K(Y) &\to K(X) \\ KO(Y) &\to KO(X) \end{aligned}$$

Corollary 4.5. *If $f : X \to Y$ is a homotopy equivalence, then it induces isomorphisms*

$$\begin{array}{ccc} f^* : Prin_G(Y) & \xrightarrow{\cong} & Prin_G(X) \\ Vect^*(Y) & \xrightarrow{\cong} & Vect^*(X) \\ K(Y) & \xrightarrow{\cong} & K(X) \end{array}$$

Corollary 4.6. *Any fiber bundle over a contractible space is trivial.*

Proof. If X is contractible, it is homotopy equivalent to a point. Apply the above corollary. □

The following result is a classification theorem for bundles over spheres. It begins to describe why understanding the homotopy type of Lie groups is so important in Topology.

Theorem 4.7. *There is a bijective correspondence between principal bundles and homotopy groups*

$$Prin_G(S^n) \cong \pi_{n-1}(G)$$

where as a set $\pi_{n-1}G = [S^{n-1}, x_0; G, \{1\}]$*, which refers to (based) homotopy classes of basepoint preserving maps from the sphere* S^{n-1} *with basepoint* $x_0 \in S^{n-1}$*, to the group* G *with basepoint the identity* $1 \in G$*.*

Proof. Let $p : E \to S^n$ be a G - bundle. Write S^n as the union of its upper and lower hemispheres,

$$S^n = D^n_+ \cup_{S^{n-1}} D^n_-.$$

Since D^n_+ and D^n_- are both contractible, the above corollary says that E restricted to each of these hemispheres is trivial. Morever if we fix a trivialization of the fiber of E at the basepoint $x_0 \in S^{n-1} \subset S^n$, then we can extend this trivialization to both the upper and lower hemispheres. We may therefore write

$$E = (D^n_+ \times G) \cup_\theta (D^n_- \times G)$$

where θ is a clutching function defined on the equator, $\theta : S^{n-1} \to G$. That is, E consists of the two trivial components, $(D^n_+ \times G)$ and $(D^n_- \times G)$ where if $x \in S^{n-1}$, then $(x, g) \in (D^n_+ \times G)$ is identified with $(x, \theta(x)g) \in (D^n_- \times G)$. Notice that since our original trivializations extended a common trivialization on the basepoint $x_0 \in S^{n-1}$, then the trivialization $\theta : S^{n-1} \to G$ maps the basepoint x_0 to the identity $1 \in G$. The assignment of a bundle its clutching function, will define our correspondence

$$\Theta : Prin_G(S^n) \to \pi_{n-1}G.$$

To see that this correspondence is well defined we need to check that if E_1 is isomorphic to E_2, then the corresponding clutching functions θ_1 and θ_2 are homotopic. Let $\Psi : E_1 \to E_2$ be an isomorphism. We may assume this isomorphism respects the given trivializations of these fibers of these bundles over the basepoint $x_0 \in S^{n-1} \subset S^n$. Then the isomorphism Ψ determines an isomorphism

$$(D^n_+ \times G) \cup_{\theta_1} (D^n_- \times G) \xrightarrow[\cong]{\Psi} (D^n_+ \times G) \cup_{\theta_2} (D^n_- \times G).$$

By restricting to the hemispheres, the isomorphism Ψ defines maps

$$\Psi_+ : D^n_+ \to G$$

and

$$\Psi_- : D^n_- \to G$$

which both map the basepoint $x_0 \in S^{n-1}$ to the identity $1 \in G$, and furthermore have the property that for $x \in S^{n-1}$,

$$\Psi_+(x)\theta_1(x) = \theta_2(x)\Psi_-(x),$$

or, $\Psi_+(x)\theta_1(x)\Psi_-(x)^{-1} = \theta_2(x) \in G$. Now by considering the linear homotopy $\Psi_+(tx)\theta_1(x)\Psi_-(tx)^{-1}$ for $t \in [0,1]$, we see that $\theta_2(x)$ is homotopic to $\Psi_+(0)\theta_1(x)\Psi_-(0)^{-1}$, where the two zeros in this description refer to the origins of D^n_+ and D^n_- respectively, i.e the north and south poles of the sphere S^n. Now since Ψ_+ and Ψ_- are defined on connected spaces, their images lie in a connected component of the group G. Since their image on the basepoint $x_0 \in S^{n-1}$ are both the identity, there exist paths $\alpha_+(t)$ and $\alpha_-(t)$ in S^n that start when $t = 0$ at $\Psi_+(0)$ and $\Psi_-(0)$ respectively, and both end at $t = 1$ at the identity $1 \in G$. Then the homotopy $\alpha_+(t)\theta_1(x)\alpha_-(t)^{-1}$ is a homotopy from the map $\Psi_+(0)\theta_1(x)\Psi_-(0)^{-1}$ to the map $\theta_1(x)$. Since the first of these maps is homotopic to $\theta_2(x)$, we have that θ_1 is homotopic to θ_2, as claimed. This implies that the map $\Theta : Prin_G(S^n) \to \pi_{n-1}G$ is well defined.

The fact that Θ is surjective comes from the fact that every map $S^{n-1} \to G$ can be viewed as the clutching function of the bundle

$$E = (D^n_+ \times G) \cup_\theta (D^n_- \times G)$$

as seen in our discussion of clutching functions in chapter 1.

We now show that Θ is injective. That is, suppose E_1 and E_2 have homotopic clutching functions, $\theta_1 \simeq \theta_2 : S^{n-1} \to G$. We need to show that E_1 is isomorphic to E_2 As above we write

$$E_1 = (D^n_+ \times G) \cup_{\theta_1} (D^n_- \times G)$$

and

$$E_2 = (D^n_+ \times G) \cup_{\theta_2} (D^n_- \times G).$$

Let $H : S^{n-1} \times [-1,1] \to G$ be a homotopy so that $H_1 = \theta_1$ and $H_1 = \theta_2$. Identify the closure of an open neighborhood $\mathcal{N}$ of the equator S^{n-1} in S^n with $S^{n-1} \times [-1,1]$ Write $\mathcal{D}_+ = D^2_+ \cup \bar{\mathcal{N}}$ and $\mathcal{D}_- = D^2_- \cup \bar{\mathcal{N}}$ Then $\mathcal{D}_+$ and $\mathcal{D}_-$ are topologically closed disks and hence contractible, with

$$\mathcal{D}_+ \cap \mathcal{D}_- = \bar{\mathcal{N}} \cong S^{n-1} \times [-1,1].$$

Thus we may form the principal G - bundle

$$E = \mathcal{D}_+ \times G \cup_H \mathcal{D}_- \times G$$

where by abuse of notation, H refers to the composition

$$\bar{\mathcal{N}} \cong S^{n-1} \times [-1,1] \xrightarrow{H} G\cdot$$

We leave it to the interested reader to verify that E is isomorphic to both E_1 and E_2. This completes the proof of the theorem. □

4.2 Universal bundles and classifying spaces

The goal of this section is to study universal principal G - bundles, the resulting classification theorem, and the corresponding classifying spaces. We will discuss several examples including the universal bundle for any subgroup of the general linear group. We postpone the proof of the existence of universal bundles for all groups until the next section.

In order to identify universal bundles, we need to recall the following definition from homotopy theory.

Definition 4.2. *A space X is said to be* aspherical if all of its homotopy groups are trivial,

$$\pi_n(X) = 0 \quad \text{for all } n \geq 0.$$

Equivalently, a space X is aspherical if every map from a sphere $S^n \to X$ can be extended to a map of its bounding disk, $D^{n+1} \to X$.

Note. A famous theorem of J.H.C. Whitehead states that if X has the homotopy type of a CW - complex, then X being aspherical is equivalent to X being contractible (see [103]).

The following is the main result of this section. It identifies when a principal bundle is universal.

Theorem 4.8. *Let $p : E \to B$ be a principal G - bundle, where the total space E is aspherical. Then this bundle is* universal in the sense that if X is any space of the homotopy type of a CW-complex, the induced pull-back map

$$\psi : [X, B] \to Prin_G(X)$$
$$f \to f^*(E)$$

is a bijective correspondence.

For the purposes of these notes we will prove the theorem in the setting where the action of G on the total space E is *cellular*. That is, there is a CW - decomposition of the space E which, in an appropriate sense, is respected by the group action. In practical terms there is not much loss in making these assumptions, since the actions of compact Lie groups on manifolds, and algebraic actions on projective varieties satisfy this property. For the proof of the theorem in its full generality we refer the reader to Steenrod's book [90], and for a full reference on equivariant CW - complexes and how they approximate a wide range of group actions, we refer the reader to [56]

In order to make the notion of cellular action precise, we need to define the notion of an *equivariant CW - complex*, or a G - CW - complex. The idea is the following. Recall that a CW - complex is a space that is made up out of disks of various dimensions whose interiors are disjoint. In particular it can be built up skeleton by skeleton, and the $(k+1)^{st}$ skeleton $X^{(k+1)}$ is constructed out of the k^{th} skeleton $X^{(k)}$ by attaching $(k+1)$ - dimensional disks via "attaching maps", $S^k \to X^{(k)}$.

A "G - CW - complex" is one that has a group action so that the orbits of the points on the interior of a cell are uniform in the sense that each point in a cell D^k has the same isotropy subgroup, say H, and the orbit of a cell itself is of the form $G/H \times D^k$. This leads to the following definition.

Definition 4.3. *A G - CW - complex is a space with G -action X which is topologically the direct limit of G - invariant subspaces $\{X^{(k)}\}$ called the equivariant skeleta,*

$$X^{(0)} \subset X^{(1)} \subset \cdots \subset X^{(k-1)} \subset X^{(k)} \subset \cdots X$$

where for each $k \geq 0$ there is a countable collection of k dimensional disks, subgroups of G, and maps of boundary spheres

$$\{D^k_j, \, H_j < G, \, \phi_j : \partial D^k_j \times G/H_j = S^{k-1}_j \times G/H_j \to X^{(k-1)} \quad j \in I_k\}$$

so that

1. *Each "attaching map" $\phi_j : S^{k-1}_j \times G/H_j \to X^{(k-1)}$ is G -equivariant, and*

2. $$X^{(k)} = X^{k-1)} \bigcup_{\phi_j \; j \in I_j} (D^k_j \times G/H_j).$$

This notation means that each " disk orbit " $D^k_j \times G/H_j$ is attached to $X^{(k-1)}$ via the map $\phi_j : S^{k-1}_j \times G/H_j \to X^{(k-1)}$.

We leave the following as an exercise to the reader.

Exercise. Prove that when X is a G - CW complex the orbit space X/G has the an induced structure of a (non-equivariant) CW - complex.

Note. Observe that in a G -CW complex X with a *free* G action, all disk orbits are of the form $D^k \times G$, since all isotropy subgroups are trivial.

We now prove the above theorem under the assumption that the principal bundle $p : E \to B$ has the property that with respect to group action of G on E, then E has the structure of a G - CW - complex. The basespace is then given the induced CW - structure. The spaces X in the statement of the theorem are assumed to be of the homotopy type of CW - complexes.

Proof. We first prove that the pull - back map

$$\psi : [X, B] \to Prin_G(X)$$

is surjective. So let $q : P \to X$ be a principal G - bundle, with P a G - CW - complex. We prove there is a G - equivariant map $h : P \to E$ that maps each orbit pG homeomorphically onto its image, $h(y)G$. We prove this by induction on the equivariant skeleta of P. So assume inductively that the map h has been constructed on the $(k-1)$ - skeleton,

$$h_{k-1} : P^{(k-1)} \to E.$$

Since the action of G on P is free, all the k - dimensional disk orbits are of the form $D^k \times G$. Let $D^k_j \times G$ be a disk orbit in the G-CW - structure of the k - skeleton $P^{(k)}$. Consider the disk $D^k_j \times \{1\} \subset D^k_j \times G$. Then the map h_{k-1} extends to $D^k_j \times \{1\}$ if and only if the composition

$$S^{k-1}_j \times \{1\} \subset S^{k-1}_j \times G \xrightarrow{\phi_j} P^{(k-1)} \xrightarrow{h_{k-1}} E$$

is null homotopic. But since E is aspherical, any such map is null homotopic and extends to a map of the disk, $\gamma : D^k_j \times \{1\} \to E$. Now extend γ equivariantly to a map $h_{k,j} : D^k_j \times G \to E$. By construction $h_{k,j}$ maps the orbit of each point $x \in D^k_j$ equivariantly to the orbit of $\gamma(x)$ in E. Since both orbits are isomorphic to G (because the action of G on both P and E are free), this map is a homeomorphism on orbits. Taking the collection of the extensions $h_{k,j}$ together then gives an extension

$$h_k : P^{(k)} \to E$$

with the required properties. This completes the inductive step. Thus we may conclude we have a G - equivariant map $h : P \to E$ that is a homeomorphism on the orbits. Hence it induces a map on the orbit space $f : P/G = X \to E/G = B$ making the following diagram commute

$$\begin{array}{ccc} P & \xrightarrow{h} & E \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[f]{} & B \end{array}$$

Since h induces a homeomorphism on each orbit, the maps h and f determine a homeomorphism of principal G - bundles which induces an equivariant isomorphism on each fiber. This implies that h induces an isomorphism of principal bundles to the pull - back

$$\begin{array}{ccc} P & \xrightarrow[\cong]{h} & f^*(E) \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[=]{} & X. \end{array}$$

Thus the isomorphism class $[P] \in Prin_G(X)$ is given by $f^*(E)$. That is, $[P] = \psi(f)$, and hence

$$\psi : [X, B] \to Prin_G(X)$$

is surjective.

We now prove ψ is injective. To do this, assume $f_0 : X \to B$ and $f_1 : X \to B$ are maps so that there is an isomorphism

$$\Phi : f_0^*(E) \xrightarrow{\cong} f_1^*(E).$$

We need to prove that f_0 and f_1 are homotopic maps. Now by the cellular approximation theorem (see [88]) we can find cellular maps homotopic to f_0 and f_1 respectively. We therefore assume without loss of generality that f_0 and f_1 are cellular. This, together with the assumption that E is a G - CW complex, gives the pull back bundles $f_0^*(E)$ and $f_1^*(E)$ the structure of G -CW complexes.

Define a principal G - bundle $\mathcal{E} \to X \times I$ by

$$\mathcal{E} = f_0^*(E) \times [0, 1/2] \cup_\Phi f_1^*(E) \times [1/2, 1]$$

where $v \in f_0^*(E) \times \{1/2\}$ is identified with $\Phi(v) \in f_1^*(E) \times \{1/2\}$. $\mathcal{E}$ also has the structure of a G - CW - complex.

Now by the same kind of inductive argument that was used in the surjectivity argument above, we can find an equivariant map $H : \mathcal{E} \to E$ that induces a homeomorphism on each orbit, and that extends the obvious maps $f_0^*(E) \times \{0\} \to E$ and $f_1^*(E) \times \{1\} \to E$. The induced map on orbit spaces

$$F : \mathcal{E}/G = X \times I \to E/G = B$$

is a homotopy between f_0 and f_1. This proves the correspondence Ψ is injective, and completes the proof of the theorem. □

The following result establishes the homotopy uniqueness of universal bundles.

Theorem 4.9. *Let $E_1 \to B_1$ and $E_2 \to B_2$ be universal principal G - bundles. Then there is a bundle map*

$$\begin{array}{ccc} E_1 & \xrightarrow{\tilde{h}} & E_2 \\ \downarrow & & \downarrow \\ B_1 & \xrightarrow[h]{} & B_2 \end{array}$$

so that h is a homotopy equivalence.

Proof. The fact that $E_2 \to B_2$ is a universal bundle means, by 4.8 that there is a "classifying map" $h : B_1 \to B_2$ and an isomorphism $\tilde{h} : E_1 \to h^*(E_2)$. Equivalently, $\tilde{h}$ can be thought of as a bundle map $\tilde{h} : E_! \to E_2$ lying over $h : B_1 \to B_2$. Similarly, using the universal property of $E_1 \to B_1$, we get a classifying map $g : B_2 \to B_1$ and an isomorphism $\tilde{g} : E_2 \to g^*(E_1)$, or equivalently, a bundle map $\tilde{g} : E_2 \to E_1$. Notice that the composition

$$g \circ f : B_1 \to B_2 \to B_1$$

is a map whose pull back,

$$\begin{aligned}(g \circ f)^*(E_1) &= g^*(f^*(E_1)) \\ &\cong g^*(E_2) \\ &\cong E_1.\end{aligned}$$

That is, $(g \circ f)^*(E_1) \cong id^*(E_1)$, and hence by 4.8 we have $g \circ f \simeq id : B_1 \to B_1$. Similarly, $f \circ g \simeq id : B_2 \to B_2$. Thus f and g are homotopy inverses of each other. □

Because of this theorem, the basespace of a universal principal G - bundle has a well defined homotopy type. We denote this homotopy type by BG, and refer to it as the *classifying space* of the group G. We also use the notation EG to denote the total space of a universal G - bundle.

We have the following immediate result about the homotopy groups of the classifying space BG.

Corollary 4.10. *For any group G, there is an isomorphism of homotopy groups,*

$$\pi_{n-1} G \cong \pi_n(BG).$$

Proof. By considering 4.7 and 4.8 we see that both of these homotopy groups are in bijective correspondence with the set of principal bundles $Prin_G(S^n)$. To realize this bijection by a group homomorphism, consider the "suspension" of the group G, ΣG obtained by attaching two cones on G along the equator. That is,

$$\Sigma G = G \times [-1, 1] / \sim$$

where all points of the form $(g, 1)$, $(h, -1)$, or $(1, t)$ are identified to a single point.

Notice that this suspension construction can be applied to any space with a basepoint, and in particular $\Sigma S^{n-1} \cong S^n$.

Consider the principal G bundle E over ΣG defined to be trivial on both cones with clutching function $id : G \times \{0\} \xrightarrow{=} G$ on the equator. That is, if $C_+ = G \times [0,1]/\sim \subset \Sigma G$ and $C_- = G \times [-1,0] \subset \Sigma E$ are the upper and lower cones, respectively, then

$$E = (C_+ \times G) \cup_{id} (C_- \times G)$$

where $((g,0),h) \in C_+ \times G$ is identified with $((g,0)gh \in C_- \times G$. Then by 4.8 there is a classifying map

$$f : \Sigma G \to BG$$

such that $f^*(EG) \cong E$.

Now for any space X, let ΩX be the *loop space* of X,

$$\Omega X = \{\gamma : [-1,1] \to X \text{ such that } \gamma(-1) = \gamma(1) = x_0 \in X\}$$

where $x_0 \in X$ is a fixed basepoint. Then the map $f : \Sigma G \to BG$ determines a map (its adjoint)

$$\bar{f} : G \to \Omega BG$$

defined by $\bar{f}(g)(t) = f(g,t)$. But now the loop space ΩX of any connected space X has the property that $\pi_{n-1}(\Omega X) = \pi_n(X)$ (see the exercise below). We then have the induced group homomorphism

$$\pi_{n-1}(G) \xrightarrow{\bar{f}_*} \pi_{n-1}(\Omega BG) \xrightarrow{\cong} \pi_n(BG)$$

which induces the bijective correspondence described above. □

Exercises. 1. Let X and Y be connected spaces equipped with basepoints. Prove that there is a bijection

$$[\Sigma X, Y] \cong [X, \Omega Y].$$

Here the notation $[-,-]$ denotes the set of homotopy classes of basepoint preserving maps. As a special case, conclude that $\pi_n(Y, y_o) \cong \pi_{n-1}(\Omega Y, \epsilon_0)$, where $\epsilon_0 : S^1 \to Y$ is the constant map at the basepoint y_0.

2. Let G be a topological group, and consider the map $f : G \to \Omega BG$ defined in the above proof of Corollary 4.10. Prove that f induces an isomorphism in homotopy groups (in all degrees). Such a map is called a "weak homotopy equivalence".

3. Prove that the composition

$$\pi_{n-1}(G) \xrightarrow{\bar{f}_*} \pi_{n-1}(\Omega BG) \xrightarrow{\cong} \pi_n(BG)$$

yields the bijection associated with identifying both $\pi_{n-1}(G)$ and $\pi_n(BG)$ with $Prin_G(S^n)$.

We recall the following definition from homotopy theory.

Definition 4.4. *An Eilenberg - MacLane space of type* (G, n) *is a space* X *such that*

$$\pi_k(X) = \begin{cases} G & \text{if } k = n \\ 0 & \text{otherwise} \end{cases}$$

We write $K(G, n)$ for an Eilenberg - MacLane space of type (G, n). Recall that for $n \geq 2$, the homotopy groups $\pi_n(X)$ are abelian groups, so in this $K(G, n)$ only exists

Corollary 4.11. *Let* π *be a discrete group. Then the classifying space* $B\pi$ *is an Eilenberg - MacLane space* $K(\pi, 1)$.

Examples.

- $\mathbb{R}$ has a free, cellular action of the integers $\mathbb{Z}$ by

  $$(t, n) \to t + n \quad t \in \mathbb{R}, n \in \mathbb{Z}.$$

  Since $\mathbb{R}$ is contractible, $\mathbb{R}/\mathbb{Z} = S^1 = B\mathbb{Z} = K(\mathbb{Z}, 1)$.

- The inclusion $S^n \subset S^{n+1}$ as the equator is clearly null homotopic since the inclusion obviously extends to a map of the disk. Hence the direct limit space

  $$\varinjlim_n S^n = \cup_n S^n = S^\infty$$

  is aspherical. Now $\mathbb{Z}_2$ acts freely on each S^n by the antipodal map, and the inclusions $S^n \subset S^{n+1}$ are equivariant with respect to these actions. Hence there is an induced free action of $\mathbb{Z}_2$ on S^∞. Thus the projection map

  $$S^\infty \to S^\infty/\mathbb{Z}_2 = \mathbb{RP}^\infty$$

  is a universal principal $\mathbb{Z}_2 = O(1)$ - bundle, and so

  $$\mathbb{RP}^\infty = BO(1) = B\mathbb{Z}_2 = K(\mathbb{Z}_2, 1)$$

  .

- Similarly, the inclusion of the unit sphere in $\mathbb{C}^n$ into the unit sphere in $\mathbb{C}^{n+1}$ gives an the inclusion $S^{2n-1} \subset S^{2n+1}$ which is null homotopic. It is also equivariant with respect to the free $S^1 = U(1)$ - action given by (complex) scalar multiplication. Then the limit $S^\infty = \cup_n S^{2n+1}$ is aspherical with a free S^1 action. We therefore have that the projection

  $$S^\infty \to S^\infty/S^1 = \mathbb{CP}^\infty$$

  is a principal $S^1 = U(1)$ bundle. Hence we have

  $$\mathbb{CP}^\infty = BS^1 = BU(1).$$

  Moreover since S^1 is a $K(\mathbb{Z}, 1)$, then we have that

  $$\mathbb{CP}^\infty = K(\mathbb{Z}, 2).$$

- The cyclic groups $\mathbb{Z}_n$ are subgroups of $U(1)$ and so they act freely on S^∞ as well. Thus the projection maps
$$S^\infty \to S^\infty/\mathbb{Z}_n$$
is a universal principal $\mathbb{Z}_n$ bundle. The quotient space $S^\infty/\mathbb{Z}_n$ is denoted $L^\infty(n)$ and is referred to as the infinite $\mathbb{Z}_n$ - lens space.

These examples allow us to give the following description of line bundles and their relation to cohomology. We first recall a well known theorem in homotopy theory. This theorem will be discussed further in chapter 4. We refer the reader to [101] for details.

Theorem 4.12. *Let G be an abelian group. Then there is a natural isomorphism*
$$\phi : H^n(K(G,n);G) \xrightarrow{\cong} Hom(G,G).$$

Let $\iota \in H^n(K(G,n);G)$ be $\phi^{-1}(id)$. This is called the fundamental class. Then if X has the homotopy type of a CW - complex, the mapping
$$[X, K(G,n)] \to H^n(X;G)$$
$$f \to f^*(\iota)$$
is a bijective correspondence.

With this we can now prove the following:

Theorem 4.13. *There are bijective correspondences which allow us to classify complex line bundles,*
$$Vect^1(X) \cong Prin_{U(1)}(X) \cong [X, BU(1)] = [X, \mathbb{CP}^\infty] \cong [X, K(\mathbb{Z},2)] \cong H^2(X;\mathbb{Z})$$
where the last correspondence takes a map $f : X \to \mathbb{CP}^\infty$ to the class
$$c_1 = f^*(c) \in H^2(X),$$
where $c \in H^2(\mathbb{CP}^\infty)$ is the generator. In the composition of these correspondences, the class $c_1 \in H^2(X)$ corresponding to a line bundle $\zeta \in Vect^1(X)$ is called the first Chern class of ζ (or of the corresponding principal $U(1)$ - bundle).

Proof. These correspondences follow directly from the above considerations, once we recall that $Vect^1(X) \cong Prin_{GL(1,\mathbb{C})}(X) \cong [X, BGL(1,\mathbb{C})]$, and that $\mathbb{CP}^\infty$ is a model for $BGL(1,\mathbb{C})$ as well as $BU(1)$. This is because, we can express $\mathbb{CP}^\infty$ in its homogeneous form as
$$\mathbb{CP}^\infty = \varinjlim_n (\mathbb{C}^{n+1} - \{0\})/GL(1,\mathbb{C}),$$
and that $\varinjlim_n (\mathbb{C}^{n+1} - \{0\})$ is an aspherical space with a free action of $GL(1,\mathbb{C}) = \mathbb{C}^*$. □

There is a similar theorem classifying real line bundles:

Theorem 4.14. *There are bijective correspondences*

$$Vect^1_{\mathbb{R}}(X) \cong Prin_{O(1)}(X) \cong [X, BO(1)] = [X, \mathbb{RP}^\infty] \cong [X, K(\mathbb{Z}_2, 1)] \cong H^1(X; \mathbb{Z}_2)$$

where the last correspondence takes a map $f : X \to \mathbb{RP}^\infty$ to the class

$$w_1 = f^*(w) \in H^1(X; \mathbb{Z}_2),$$

where $w \in H^1(\mathbb{RP}^\infty; \mathbb{Z}_2)$ is the generator. In the composition of these correspondences, the class $w_1 \in H^1(X; \mathbb{Z}_2)$ corresponding to a line bundle $\zeta \in Vect^1_{\mathbb{R}}(X)$ is called the first Stiefel - Whitney class of ζ (or of the corresponding principal $O(1)$ - bundle).

More Examples.

- Let $V_n(\mathbb{C}^N)$ be the Stieflel - manifold described in Chapter 2. We claim that the inclusion of vector spaces $\mathbb{C}^N \subset \mathbb{C}^{2N}$ as the first N - coordinates induces an inclusion $V_n(\mathbb{C}^N) \hookrightarrow V_n(\mathbb{C}^{2N})$ which is null homotopic. To see this, let $\iota : \mathbb{C}^n \to \mathbb{C}^{2N}$ be a fixed linear embedding, whose image lies in the last N - coordinates in $\mathbb{C}^{2N}$. Then given any $\rho \in V_n(\mathbb{C}^N) \subset V_n(\mathbb{C}^{2N})$, then $t \cdot \iota + (1-t) \cdot \rho$ for $t \in [0, 1]$ defines a one parameter family of linear embeddings of $\mathbb{C}^n$ in $\mathbb{C}^{2N}$, and hence a contraction of the image of $V_n(\mathbb{C}^N)$ onto the element ι. Hence the limiting space $V_n(\mathbb{C}^\infty)$ is aspherical with a free $GL(n, \mathbb{C})$ - action. Therefore the projection

  $$V_n(\mathbb{C}^\infty) \to V_n(\mathbb{C}^\infty)/GL(n, \mathbb{C}) = Gr_n(\mathbb{C}^\infty)$$

  is a universal $GL(n, \mathbb{C})$ - bundle. Hence the infinite Grassmannian is the classifying space

  $$Gr_n(\mathbb{C}^\infty) = BGL(n, \mathbb{C})$$

  and so we have a classification

  $$Vect^n(X) \cong Prin_{GL(n,\mathbb{C})}(X) \cong [X, BGL(n, \mathbb{C})] \cong [X, Gr_n(\mathbb{C}^\infty)]. \quad (4.1)$$

- A simlar argument shows that the infinite unitary Stiefel manifold, $V_n^U(\mathbb{C}^\infty)$ is aspherical with a free $U(n)$ - action. Thus the projection

  $$V_n^U(\mathbb{C}^\infty) \to V_n(\mathbb{C}^\infty)/U(n) = Gr_n(\mathbb{C}^\infty)$$

  is a universal principal $U(n)$ - bundle. Hence the infinite Grassmanian $Gr_n(\mathbb{C}^\infty)$ is the classifying space for $U(n)$ bundles as well,

  $$Gr_n(\mathbb{C}^\infty) = BU(n).$$

  The fact that this Grassmannian is both $BGL(n, \mathbb{C})$ and $BU(n)$ reflects the fact that every n - dimensional complex vector bundle has a $U(n)$ - structure, and that structure is unique up to homotopy.

- We have similar universal $GL(n,\mathbb{R})$ and $O(n)$ - bundles:

  $$V_n(\mathbb{R}^\infty) \to V_n(\mathbb{R}^\infty)/GL(n,\mathbb{R}) = Gr_n(\mathbb{R}^\infty)$$

  and

  $$V_n^O(\mathbb{R}^\infty) \to V_n^O(\mathbb{R}^\infty)/O(n) = Gr_n(\mathbb{R}^\infty).$$

  Thus we have

  $$Gr_n(\mathbb{R}^\infty) = BGL(n,\mathbb{R}) = BO(n)$$

  and so this infinite dimensional Grassmannian classifies real n - dimensional vector bundles as well as principal $O(n)$ - bundles.

Now suppose $p : EG \to EG/G = BG$ is a universal G - bundle. Suppose further that $H < G$ is a subgroup. Then H acts freely on EG as well, and hence the projection

$$EG \to EG/H$$

is a universal H - bundle. Hence $EG/H = BH$. Using the infinite dimensional Stiefel manifolds described above, this observation gives us models for the classifying spaces for any subgroup of a general linear group. So for example if we have a subgroup (i.e a faithful representation) $H \subset GL(n,\mathbb{C})$, then

$$BH = V_n(\mathbb{C}^\infty)/H.$$

This observation also leads to the following useful fact.

Proposition 4.15. *. Let $p : EG \to BG$ be a universal principal G - bundle, and let $H < G$. Then there is a fiber bundle*

$$BH \to BG$$

with fiber the orbit space G/H.

Proof. This bundle is given by

$$G/H \to EG \times_G G/H \to EG/G = BG$$

together with the observation that $EG \times_G G/H = EG/H = BH$. □

4.3 Classifying gauge groups

In this section we describe the classifying space of the group of automorphisms of a principal G - bundle, or the *gauge group* of the bundle. We describe the classifying space in two different ways: in terms of the space of connections on the bundle, and in terms of the mapping space of the base manifold to the classifying space BG. These constructions are important in Yang - Mills theory, and we refer the reader to [7] and [27] for more details.

Let A be a connection on a principal bundle $P \longrightarrow M$ where M is a closed manifold equipped with a Riemannian metric. The Yang - Mills functional applied to A, $\mathcal{YM}(A)$ is the square of the L^2 norm of the curvature,

$$\mathcal{YM}(A) = \frac{1}{2}\int_M \|F_A\|^2 \, d(vol).$$

We view $\mathcal{YM}$ as a mapping $\mathcal{YM} : \mathcal{A}(P) \longrightarrow \mathbb{R}$. The relevance of the gauge group in Yang - Mills theory is that $\mathcal{YM}$ preserves this group of symmetries.

Definition 4.5. *The gauge group $\mathcal{G}(P)$ of the principal bundle P is the group of bundle automorphisms of $P \longrightarrow M$. That is, an element $\phi \in \mathcal{G}(P)$ is a bundle isomorphism of P with itself lying over the identity:*

$$\begin{array}{ccc} P & \xrightarrow[\cong]{\phi} & P \\ \downarrow & & \downarrow \\ M & \xrightarrow{=} & M. \end{array}$$

Equivalently, $\mathcal{G}(P)$ is the group $\mathcal{G}(P) = Aut_G(P)$ of G - equivariant diffeomorphisms of the space P, inducing the identity map on the orbit space $P/G = M$.

The gauge group $\mathcal{G}(P)$ can be thought of in several equivalent ways. The following one is particularly useful.

Consider the conjugation action of the Lie group G on itself,

$$\begin{aligned} G \times G &\longrightarrow G \\ (g, h) &\longrightarrow ghg^{-1}. \end{aligned}$$

This left action defines a fiber bundle

$$Ad(P) = P \times_G G \longrightarrow P/G = M$$

with fiber G. We leave the following as an exercise for the reader.

Proposition 4.16. *The gauge group of a principal bundle $P \longrightarrow M$ is naturally isomorphic (as topological groups) the the group of sections of $Ad(P)$, $C^\infty(M; Ad(P))$.*

The gauge group $\mathcal{G}(P)$ acts on the space of connections $\mathcal{A}(P)$ by the pull-back construction. More generally, if $f : P \to Q$ is any smooth map of principal G - bundles and A is a connection on Q, then there is a natural pull back connection $f^*(A)$ on Q, defined by pulling back the equivariant splitting of the tangent bundle TQ to an equivariant splitting of TP in the obvious way. The pull - back construction for automorphisms $\phi : P \longrightarrow P$ defines an action of $\mathcal{G}(P)$ on $\mathcal{A}(P)$.

We leave the proof of the following is an exercise for the reader.

Proposition 4.17. *Let P be the trivial bundle $M \times G \to M$. Then the gauge group $\mathcal{G}(P)$ is given by the function space from M to G,*

$$\mathcal{G}(P) \cong C^\infty(M; G).$$

Furthermore if $\phi : M \to G$ is identified with an element of $\mathcal{G}(P)$, and $A \in \Omega^1(M; \mathfrak{g})$ is identified with an element of $\mathcal{A}(G)$, then the induced action of ϕ on G is given by

$$\phi^*(A) = \phi^{-1} A \phi + \phi^{-1} d\phi.$$

It is not difficult to see that in general the gauge group $\mathcal{G}(P)$ does not act freely on the space of connections $\mathcal{A}(P)$. However there is an important subgroup $\mathcal{G}_0(P) < \mathcal{G}(P)$ that does. This is the group of *based* gauge transformations. To define this group, let $x_0 \in M$ be a fixed basepoint, and let P_{x_0} be the fiber of P at x_0.

Definition 4.6. *The based gauge group $\mathcal{G}_0(P)$ is a subgroup of the group of bundle automorphisms $\mathcal{G}(P)$ which pointwise fix the fiber P_{x_0}. That is,*

$$\mathcal{G}_0(P) = \{\phi \in \mathcal{G}(P) : \text{ if } \quad v \in P_{x_0} \text{ then } \quad \phi(v) = v\}.$$

Theorem 4.18. *The based gauge group $\mathcal{G}_0(P)$ acts freely on the space of connections $\mathcal{A}(P)$.*

Proof. Suppose that $A \in \mathcal{A}(P)$ is a fixed point of $\phi \in \mathcal{G}_0(P)$. That is, $\phi^*(A) = A$. We need to show that $\phi = 1$.

The equivariant splitting ω_A given by a connection A defines a notion of parallel transport in P along curves in M (see [43]). It is not difficult to see that the statement $\phi^*(A) = A$ implies that application of the automorphism ϕ commutes with parallel transport. Now let $w \in P_x$ be a point in the fiber of an element $x \in M$. Given curve γ in M between the basepoint x_0 and x one sees that

$$\phi(w) = T_\gamma(\phi(T_{\gamma^{-1}}(w))$$

where T_γ is parallel transport along γ. But since $T_{\gamma^{-1}}(w) \in P_{x_0}$ and $\phi \in \mathcal{G}_0(P)$,

$$\phi(T_{\gamma^{-1}}(w)) = w.$$

Hence $\phi(w) = w$, that is, $\phi = 1$. □

Remark. Notice that this argument actually says that if $A \in \mathcal{A}(P)$ is the fixed point of *any* gauge transformation $\phi \in \mathcal{G}(P)$, then ϕ is determined by its action on a single fiber.

Let $\mathcal{B}(P)$ and $\mathcal{B}_0(P)$ be the orbit spaces of connections on P up to guage and based gauge equivalence respectively,

$$\mathcal{B}(P) = \mathcal{A}(P)/\mathcal{G}(P) \quad \mathcal{B}_0(P) = \mathcal{A}(P)/\mathcal{G}_0(P).$$

Now it is straightforward to check directly that the Yang - Mills functional in invariant under gauge transformations. Thus it yields maps

$$\mathcal{YM} : \mathcal{B}(P) \to \mathbb{R} \quad \text{and} \quad \mathcal{YM} : \mathcal{B}_0(P) \to \mathbb{R}.$$

It is therefore important to understand the homotopy types of these orbit spaces. Because of the freeness of the action of $\mathcal{G}_0(P)$, the homotopy type of the orbit space $\mathcal{G}_0(P)$ is easier to understand.

We end this section with a discussion of its homotopy type. Since the space of connections $\mathcal{A}(P)$ is affine, it is contractible. Moreover it is possible to show that the free action of the based gauge group $\mathcal{G}_0(P)$ defines a principal bundle $\mathcal{A}(P) \to \mathcal{A}(P)/\mathcal{G}_0(P) = \mathcal{B}_0(P)$ (See [27]). Thus $\mathcal{B}_0(P)$ the classifying space of the based gauge group,

$$\mathcal{B}_0(P) = B\mathcal{G}_0(P).$$

But the classifying spaces of the gauge groups are relatively easy to understand. (see [7].)

Theorem 4.19. *Let $G \longrightarrow EG \longrightarrow BG$ be a universal principal bundle for the Lie group G (so that EG is aspherical). Let $y_0 \in BG$ be a fixed basepoint. Then there are homotopy equivalences*

$$B\mathcal{G}(P) \simeq Map^P(M, BG) \quad \textit{and} \quad \mathcal{B}_0(P) \simeq B\mathcal{G}_0(P) \simeq Map_0^P(M, BG)$$

where $Map(M, BG)$ is the space of all continuous maps from M to BG and $Map_0(M, BG)$ is the space of those maps that preserve the basepoints. The superscript P denotes the path component of these mapping spaces consisting of the homotopy class of maps that classify the principal G - bundle P.

Proof. Consider the space of all G - equivariant maps from P to EG, $Map^G(P, EG)$. The gauge group $\mathcal{G}(P) \cong Aut^G(P)$ acts freely on the left of this space by composition. It is easy to see that $Map^G(P, EG)$ is aspherical, and its orbit space is given by the space of maps from the G - orbit space of P $(= M)$ to the G - orbit space of EG (= BG),

$$Map^G(P, EG)/\mathcal{G}(P) \cong Map^P(M, BG).$$

This proves that $Map(M, BG) = B\mathcal{G}(P)$. Similarly $Map_0^G(P, EG)$, the space of G - equivariant maps that send the fiber P_{x_0} to the fiber EG_{y_0}, is an aspherical space with a free $\mathcal{G}_0(P)$ action, whose orbit space is $Map_0^P(M, BG)$. Hence $Map_0^P(M, BG) = B\mathcal{G}_0(P)$. □

4.4 Existence of universal bundles: the Milnor join construction and the simplicial classifying space

In the last section we proved a "recognition principle" for universal principal G bundles. Namely, if the total space of a principal G - bundle $p : E \to B$ is aspherical, then it is universal. We also proved a homotopy uniqueness theorem, stating among other things that the homotopy type of the base space of a universal bundle, i.e the classifying space BG, is well defined. We also described many examples of universal bundles, and particular have a model for the classifying space BG, using Stiefel manifolds, for every subgroup of a general linear group.

The goal of this section is to prove the general existence theorem. Namely, for every group G, there is a universal principal G - bundle $p : EG \to BG$. We will give two constructions of the universal bundle and the corresponding classifying space. One, due to Milnor [72] involves taking the "infinite join" of a group with itself. The other is an example of a simplicial space, called the simplicial bar construction. It is originally due to Eilenberg and MacLane [28]. These constructions are essentially equivalent when G has a CW-structure, and they both yield G - CW - complexes. Since they are so useful in algebraic topology and combinatorics, we will also take this opportunity to introduce the notion of a general simplicial space and show how these classifying spaces are important examples.

4.4.1 The join construction

One can think of the "join" of two spaces X and Y, written $X * Y$ as the space consisting of points that lie on a line that connects a point in X to a point in Y. The following is a more precise definition:

Definition 4.7. *The join $X * Y$ is defined by*

$$X * Y = X \times I \times Y / \sim$$

where $I = [0,1]$ is the unit interval and the equivalence relation is given by $(x, 0, y_1) \sim (x, 0, y_2)$ for any two points y_1, $y_2 \in Y$, and similarly $(x_1, 1, y) \sim (x_2, 1, y)$ for any two points x_1, $x_2 \in X$.

A point $(x, t, y) \in X * Y$ can be viewed as a point on the line connecting x to y. Here are some examples.

Examples.

- Let y be a single point. Then $X * y$ is the cone $CX = X \times I / X \times \{1\}$.
- Let $Y = \{y_1, y_2\}$ be the space consisting of two distinct points. Then $X * Y$ is the suspension ΣX discussed earlier. Notice that the suspension can be viewed as the union of two cones, with vertices y_1 and y_2 respectively, attached along the equator.
- **Exercise.** Prove that the join of two spheres, is another sphere,

$$S^n * S^m \cong S^{n+m+1}.$$

- Let $\{x_0, \cdots, x_k\}$ be a collection of $k+1$ - distinct points. Then the k - fold join $x_0 * x_1 * \cdots * x_k$ is the convex hull of these points and hence is the k - dimensional simplex Δ^k with vertices $\{x_0, \cdots, x_k\}$.

Observe that the space X sits naturally as a subspace of the join $X * Y$ as endpoints of line segments,

$$\begin{aligned} \iota : X &\hookrightarrow X * Y \\ x &\to (x, 0, y). \end{aligned}$$

Notice that this formula for the inclusion makes sense and does not depend on the choice of $y \in Y$. There is a similar embedding

$$\begin{aligned} j : Y &\hookrightarrow X * Y \\ y &\to (x, 1, y). \end{aligned}$$

Lemma 4.20. *The inclusions $\iota : X \hookrightarrow X * Y$ and $j : Y \hookrightarrow X * Y$ are null homotopic.*

Proof. Pick a point $y_0 \in Y$. By definition, the embedding $\iota : X \to X * Y$ factors as the composition

$$\begin{aligned} \iota : X &\hookrightarrow X * y_0 \subset X * Y \\ x &\to (x, 0, y_0). \end{aligned}$$

But as observed above, the join $X * y_0$ is the cone on X and hence contractible. This means that ι is null homotopic, as claimed. The fact that $j : Y \hookrightarrow X * Y$ is null homotopic is proved in the same way. □

Now let G be a group and consider the iterated join

$$G^{*(k+1)} = G * G * \cdots * G$$

where there are $k + 1$ copies of the group element. This space has a free G action given by the diagonal action

$$g \cdot (g_0, t_1, g_1, \cdots, t_k, g_k) = (gg_0, t_1, gg_1, \cdots, t_k, gg_k).$$

Exercise. 1. Prove that there is a natural G - equivariant map

$$\Delta^k \times G^{k+1} \to G^{*(k+1)}$$

which is a homeomorphism when restricted to $\tilde{\Delta}^k \times G^{k+1}$ where $\tilde{\Delta}^k \subset \Delta^k$ is the interior. Here G acts on $\Delta^k \times G^{k+1}$ trivially on the simplex Δ^k and diagonally on G^{k+1}.
2. Use exercise 1 to prove that if G is a CW complex, the iterated join $G^{*(k+1)}$ has the structure of a G - CW - complex.

Define $\mathcal{J}(G)$ to be the infinite join

$$\mathcal{J}(G) = \lim_{k \to \infty} G^{*(k+1)}$$

where the limit is taken over the embeddings $\iota : G^{*(k+1)} \hookrightarrow G^{*(k+2)}$ Since these embedding maps are G -equivariant, we have an induced G - action on $\mathcal{J}(G)$.

Theorem 4.21. *If G is a CW-complex, the projection map*

$$p : \mathcal{J}(G) \to \mathcal{J}(G)/G$$

is a universal principal G - bundle.

Proof. By the above exercise the space $\mathcal{J}(G)$ has the structure of a G - CW - complex with a free G - action. Therefore by the results of the last section the projection $p : \mathcal{J}(G) \to \mathcal{J}(G)/G$ is a principal G - bundle. To see that $\mathcal{J}(G)$ is aspherical, notice that since S^n is compact, any map $\alpha : S^n \to \mathcal{J}(G)$ is homotopic to one that factors through a finite join (that by abuse of notation we still call α), $\alpha : S^n \to G^{*(n+1)} \hookrightarrow \mathcal{J}(G)$. But by the above lemma the inclusion $G^{*(n+1)} \subset \mathcal{J}(G)$ is null homotopic, and hence so is α. Thus $\mathcal{J}(G)$ is aspherical. By the results of last section, this means that the projection $\mathcal{J}(G) \to \mathcal{J}(G)/G$ is a universal G - bundle. □

4.4.2 Simplicial spaces and classifying spaces

We therefore now have a universal bundle for every topological group G with a CW-structure. We actually know a fair amount about the geometry of the total space $EG = \mathcal{J}(G)$ which, by the above exercise can be described as the union of simplices, where the k - simplices are parameterized by $k+1$ -tuples of elements of G,

$$EG = \mathcal{J}(G) = \bigcup_k \Delta^k \times G^{k+1} / \sim$$

and so the classifying space can be described by

$$BG = \mathcal{J}(G)/G \cong \bigcup_k \Delta^k \times G^k / \sim$$

It turns out that in these constructions, the simplices are glued together along faces, and these gluings are parameterized by the $k+1$ - product maps $\partial_i : G^{k+2} \to G^{k+1}$ given by multiplying the i^{th} and $(i+1)^{st}$ coordinates.

Having this type of data (parameterizing spaces of simplices as well as gluing maps) is an example of an object known as a *"simplicial set"* which is an important combinatorial object in topology. We now describe this notion in more detail and show how these universal G - bundles and classifying spaces can be viewed in these terms.

Good references for this theory are [26], [61].

The idea of simplicial sets is to provide a combinatorial technique to study cell complexes built out of simplices; i.e simplicial complexes. A simplicial complex X is built out of a union of simplices, glued along faces. Thus if X_n denotes the indexing set for the n - dimensional simplices of X, then we can write

$$X = \bigcup_{n \geq 0} \Delta^n \times X_n / \sim$$

where Δ^n is the standard n - simplex in $\mathbb{R}^n$;

$$\Delta^n = \{(t_1, \cdots, t_n) \in \mathbb{R}^n : 0 \le t_j \le 1, \text{ and } \sum_{i=1}^n t_i \le 1\}.$$

The gluing relation in this union can be encoded by set maps among the X_n's that would tell us for example how to identify an $n-1$ simplex indexed by an element of X_{n-1} with a particular face of an n - simplex indexed by an element of X_n. Thus in principal simplicial complexes can be studied purely combinatorially in terms of the sets X_n and set maps between them. The notion of a *simplicial set* is a generalization of simplicial complex that makes this idea precise.

Definition 4.8. *A simplicial set* X_* is a collection of sets

$$X_n, \quad n \ge 0$$

together with set maps

$$\partial_i : X_n \longrightarrow X_{n-1} \quad \text{and} \quad s_j : X_n \longrightarrow X_{n+1}$$

for $0 \le i, j \le n$ called **face** and **degeneracy** maps respectively. These maps are required to satisfy the following compatibility conditions

$$\begin{aligned} \partial_i \partial_j &= \partial_{j-1} \partial_i \quad \text{for } i < j \\ s_i s_j &= s_{j+1} s_i \quad \text{for } i < j \end{aligned}$$

and

$$\partial_i s_j = \begin{cases} s_{j-1} \partial_i & \text{for } i < j \\ 1 & \text{for } i = j, \, j+1 \\ s_j \partial_{i-1} & \text{for } i > j+1 \end{cases}$$

As mentioned above, the maps ∂_i and s_j encode the combinatorial information necessary for gluing the simplices together. To say precisely how this works, consider the following maps between the standard simplices:

$$\delta_i : \Delta^{n-1} \longrightarrow \Delta^n \quad \text{and} \quad \sigma_j : \Delta^{n+1} \longrightarrow \Delta^n$$

for $0 \le i, j \le n$ defined by the formulae

$$\delta_i(t_1, \cdots, t_{n-1}) = \begin{cases} (t_1, \cdots, t_{i-1}, 0, t_i, \cdots, t_{n-1}) & \text{for } i \ge 1 \\ (1 - \sum_{q=1}^{n-1} t_q, t_1, \cdots, t_{n-1}) & \text{for } i = 0 \end{cases}$$

and

$$\sigma_j(t_1, \cdots, t_{n+1}) = \begin{cases} (t_1, \cdots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \cdots, t_{n+1}) & \text{for } i \geq 1 \\ (t_2, \cdots, t_{n+1}) & \text{for } i = 0\ . \end{cases}$$

δ_i includes Δ^{n-1} in Δ^n as the i^{th} face, and σ_j projects, in a linear fashion, Δ^{n+1} onto its j^{th} face.

We can now define the space associated to the simplicial set X_* as follows.

Definition 4.9. *The geometric realization* of a simplicial set X_* is the space

$$\|X_*\| = \bigcup_{n \geq 0} \Delta^n \times X_n / \sim$$

where if $t \in \Delta^{n-1}$ and $x \in X_n$, then

$$(t, \partial_i(x)) \sim (\delta_i(t), x)$$

and if $t \in \Delta^{n+1}$ and $x \in X_n$ then

$$(t, s_j(x)) \sim (\sigma_j(t), x).$$

In the topology of $\|X_*\|$, each X_n is assumed to have the discrete topology, so that $\Delta^n \times X_n$ is a discrete set of n - simplices.

Thus $\|X_*\|$ has one n - simplex for every element of X_n, glued together in a way determined by the face and degeneracy maps.

Example. Consider the simplicial set $\mathbf{S}_*$ defined as follows. The set of n - simplices is given by

$$\mathbf{S}_n = \mathbb{Z}/(n+1), \text{ generated by an element } \tau_n.$$

The face maps are given by

$$\partial_i(\tau_n^r) = \begin{cases} \tau_{n-1}^r & \text{if } r \leq i \leq n \\ \tau_{n-1}^{r-1} & \text{if } 0 \leq i \leq r-1. \end{cases}$$

The degeneracies are given by

$$s_i(\tau_n^r) = \begin{cases} \tau_{n+1}^r & \text{if } r \leq i \leq n \\ \tau_{n+1}^{r+1} & \text{if } 0 \leq i \leq r-1. \end{cases}$$

Notice that there is one zero simplex, two one simplices, one of them the image of the degeneracy $s_0 : \mathbf{S}_0 \longrightarrow \mathbf{S}_1$, and the other nondegenerate (i.e not in the image of a degeneracy map). Notice also that all simplices in dimensions larger than one are in the image of a degeneracy map. Hence we have that the geometric realization

$$\|\mathbf{S}_*\| = \Delta^1/0 \sim 1 = S^1.$$

Let X_* be any simplicial set. There is a particularly nice and explicit way for computing the homology of the geometric realization, $H_*(\|X_*\|)$.

Consider the following chain complex. Define $C_n(X_*)$ to be the free abelian group generated by the set of n - simplices X_n. Define the homomorphism

$$d_n : C_n(X_*) \longrightarrow C_{n-1}(X_*)$$

by the formula

$$d_n([x]) = \sum_{i=0}^{n} (-1)^i \partial_i([x])$$

where $x \in X_n$.

Proposition 4.22. *The homology of the geometric realization $H_*(\|X_*\|)$ is the homology of the chain complex*

$$\longrightarrow \cdots \xrightarrow{d_{n+1}} C_n(X_*) \xrightarrow{d_n} C_{n-1}(X_*) \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_0} C_0(X_*).$$

Proof. It is straightforward to check that the geometric realization $\|X_*\|$ is a CW - complex and that this is the associated cellular chain complex. □

Besides being useful computationally, the following result establishes the fact that all CW complexes can be studied simplicially.

Theorem 4.23. *Every CW complex has the homotopy type of the geometric realization of a simplicial set.*

Proof. Let X be a CW complex. Define the singular simplicial set of X , $\mathcal{S}(X)_*$ as follows. The n simplices $\mathcal{S}(X)_n$ is the set of singular n - simplices,

$$\mathcal{S}(X)_n = \{c : \Delta^n \longrightarrow X\}.$$

The face and degeneracy maps are defined by

$$\partial_i(c) = c \circ \delta_i : \Delta^{n-1} \longrightarrow \Delta^n \longrightarrow X$$

and

$$s_j(c) = c \circ \sigma_i : \Delta^{n+1} \longrightarrow \Delta^n \longrightarrow X.$$

Notice that the associated chain complex to $\mathcal{S}(X)_*$ as in 4.22 is the singular chain complex of the space X. Hence by 4.22 we have that

$$H_*(\|\mathcal{S}(X)\|) \cong H_*(X).$$

This isomorphism is actually realized by a map of spaces

$$E : \|\mathcal{S}(X)_*\| \longrightarrow X$$

defined by the natural evaluation maps

$$\Delta^n \times \mathcal{S}(X)_n \longrightarrow X$$

given by

$$(t, c) \longrightarrow c(t).$$

It is straightforward to check that the map E does induce an isomorphism in homology. In fact it induces an isomorphism in homotopy groups. We will not prove this here; it is more technical and we refer the reader to [61] for details. Note that it follows from the homological isomorphism by the Hurewicz theorem if we knew that X was simply connected. A map between spaces that induces an isomorphism in homotopy groups is called a *weak homotopy equivalence*. Thus *any* space is weakly homotopy equivalent to a CW - complex (i.e the geometric realization of its singular simplicial set). But by the Whitehead theorem, two CW complexes that are weakly homotopy equivalent are homotopy equivalent. Hence X and $\|\mathcal{S}(X)_*\|$ are homotopy equivalent. □

We next observe that the notion of simplicial set can be generalized as follows. We say that X_* is a **simplicial space** if it is a simplicial set (i.e it satisfies definition 4.8) with the extra data that the sets X_n have the structure of a compactly-generated topological space, and the face and degeneracy maps

$$\partial_i : X_n \longrightarrow X_{n-1} \quad \text{and} \quad s_j : X_n \longrightarrow X_{n+1}$$

are continuous maps. The definition of the geometric realization of a simplicial space X_*, $\|X_*\|$, is the same as in 4.9 with the proviso that the topology of each $\Delta^n \times X_n$ is the product topology. Notice that since the "set of n - simplices" X_n is actually a space, it is not necessarily true that $\|X_*\|$ is a CW complex. However if in fact each X_n is a CW complex and the face and degeneracy maps are cellular, then $\|X_*\|$ does have a natural CW structure induced by the product CW - structures on $\Delta^n \times X_n$.

Notice that this simplicial notion generalizes even further. For example a **simplicial group** would be defined similarly, where each X_n would be a group and the face and degeneracy maps are group homomorphisms. Simplicial vector spaces, modules, etc. are defined similarly. The categorical nature of these definitions should by now be coming clear. Indeed more generally one can define a **simplicial object in a category** $\mathcal{C}$ using the above definition where now the X_n's are assumed to be objects in the category and the face and degenarcies are assumed to be morphisms. If the category $\mathcal{C}$ is a subcategory of the category of compactly-generated topological spaces, then geometric realizations can be defined as in Definition 4.9. For example the geometric realization of a simplicial (abelian) group turns out to be a topological (abelian) group. (Try to verify this for yourself!)

The notion of a simplicial object in a category $\mathcal{C}$ can be formalized somewhat in the following way.

Let Δ denote the *simplex category*. The objects of Δ are nonempty, linearly ordered sets of the form $[n] = \{0, 1, \cdots, n\}$. A morphism $\phi : [n] \to [m]$ is a non-strictly order-preserving set map. Important examples of such morphisms are "*coface* maps" δ_i, $i = 0, \cdots, n : [n-1] \to [n]$, where δ_i is defined to be the unique injective, order preserving set map from $[n-1]$ to $[n]$ whose image does not contain i. There are also "*codegeneracy* maps" σ_j, $j = 0, \cdots, n : [n+1] \to [n]$, where σ_j is the unique surjective order preserving set map $[n+1] \to [n]$ such that j is in the image of two elements.

We can then define a simplicial object $\mathbf{X}$ in a category $\mathcal{C}$ to be a contravariant functor

$$\mathbf{X} : \Delta \to \mathcal{C}.$$

Given such a simplicial object $\mathbf{X}$, the p simplicies are given by $X_p = \mathbf{X}([\mathrm{p}])$, and the face and degeneracy maps

$$\partial_i : X_n \longrightarrow X_{n-1} \quad \text{and} \quad s_j : X_n \longrightarrow X_{n+1}$$

are given by $\mathbf{X}(\delta_\mathrm{i})$ and $\mathbf{X}(\sigma_\mathrm{j})$, respectively.

Exercise: The two definitions of a simplicial object in a category $\mathcal{C}$ given above are equivalent.

If $\mathcal{C}$ is a subcategory of the category of compactly-generated topological spaces, then the *geometric realization*, $\|\mathbf{X}\|$ of a simplicial object in $\mathcal{C}$, can be defined as in Definition 4.9. Observe that $\|\mathbf{X}\|$ is then object in $\mathcal{C}$.

We now use this simplicial theory to construct universal principal G - bundles and classifying spaces.

Let G be a topological group and let $\mathcal{E}G_*$ be the simplicial space defined as follows. The space of n - simplices is given by the $n+1$ - fold cartesian product

$$\mathcal{E}G_n = G^{n+1}.$$

The face maps $\partial_i : G^{n+1} \longrightarrow G^n$ are given by the formula

$$\partial_i(g_0, \cdots, g_n) = (g_0, \cdots, \hat{g}_i, \cdots, g_n).$$

The degeneracy maps $s_j : G^{n+1} \longrightarrow G^{n+2}$ are given by the formula

$$s_j(g_0, \cdots, g_n) = (g_0, \cdots, g_j, g_j, \cdots, g_n).$$

Exercise. Show that the geometric realization $\|\mathcal{E}G_*\|$ is aspherical.

Hint. Let $\|\mathcal{E}G_*\|^{(n)}$ be the n^{th} - skeleton,

$$\|\mathcal{E}G_*\|^{(n)} = \bigcup_{p=0}^{n} \Delta^p \times G^{p+1}.$$

Then show that the inclusion of one skeleton in the next $\|\mathcal{E}G_*\|^{(n)} \hookrightarrow \|\mathcal{E}G_*\|^{(n+1)}$ is null - homotopic. One way of doing this is to establish a homeomorphism between $\|\mathcal{E}G_*\|^{(n)}$ and n - fold join $G * \cdots * G$.

Notice that the group G acts freely on the right of $\|\mathcal{E}G_*\|$ by the rule

$$\|\mathcal{E}G_*\| \times G = \left(\bigcup_{p \geq 0} \Delta^p \times G^{p+1} \right) \times G \longrightarrow \|\mathcal{E}G_*\| \tag{4.2}$$
$$(t; (g_0, \cdots, g_p)) \times g \longrightarrow (t; (g_0 g, \cdots, g_p g)).$$

Thus we can define $EG = \|\mathcal{E}G_*\|$. The projection map

$$p : EG \to EG/G = BG$$

is principal G-bundles whose total space is aspherical. Therefor it is universal principal G - bundle.

This description gives the classfiying space BG an induced simplicial structure described as follows.

Let $\mathcal{B}G_*$ be the simplicial space whose n - simplices are the cartesian product

$$\mathcal{B}G_n = G^n. \tag{4.3}$$

The face and degeneracy maps are given by

$$\partial_i(g_1, \cdots, g_n) = \begin{cases} (g_2, \cdots, g_n) & \text{for } i = 0 \\ (g_1, \cdots, g_i g_{i+1}, \cdots g_n) & \text{for } 1 \leq i \leq n-1 \\ (g_1, \cdots, g_{n-1}) & \text{for } i = n. \end{cases}$$

The degeneracy maps are given by

$$s_j(g_1, \cdots, g_n) = \begin{cases} (1, g_1, \cdots, g_n) & \text{for } j = 0 \\ (g_1, \cdots g_j, 1, g_{j+1}, \cdots, g_n) & \text{for } j \geq 1. \end{cases}$$

The simplicial projection map

$$p : \mathcal{E}G_* \longrightarrow \mathcal{B}G_*$$

defined on the level of n - simplicies by

$$p(g_0, \cdots, g_n) = (g_0 g_1^{-1}, g_1 g_2^{-1}, \cdots, g_{n-1} g_n^{-1})$$

is easily checked to commute with face and degeneracy maps and so induces a map on the level of geometric realizations

$$p : EG = \|\mathcal{E}G_*\| \longrightarrow \|\mathcal{B}G_*\|$$

which induces a homemorphism

$$BG = EG/G \xrightarrow{\cong} \|\mathcal{B}G_*\|.$$

Thus for any topological group this construction gives a simplicial space model for its classifying space. This is referred to as the **simplicial bar construction.** Notice that when G is discrete the bar construction is a CW complex for the classifying space $BG = K(G, 1)$ and 4.22 gives a particularly nice complex for computing its homology. (The homology of a $K(G, 1)$ is referred to as the homology of the group G.)

The n - chains are the group ring

$$C_n(\mathcal{B}G_*) = \mathbb{Z}[G^n] \cong \mathbb{Z}[G]^{\otimes n}$$

and the boundary homomorphisms

$$d_n : \mathbb{Z}[G]^{\otimes n} \longrightarrow \mathbb{Z}[G]^{\otimes n-1}$$

are given by

$$d_n(a_1 \otimes \cdots \otimes a_n) = (a_2 \otimes \cdots \otimes a_n) + \sum_{i=1}^{n-1} (-1)^i (a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n) + (-1)^n (a_1 \otimes \cdots \otimes a_{n-1}).$$

This complex is called the **bar complex** for computing the homology of a group and was discovered by Eilenberg and MacLane in the mid 1950's.

We end this chapter by observing that the bar construction of the classifying space of a group did not use the full group structure. It only used the existence of an associative multiplication with unit. That is, it did not use the existence of inverse. So in particular one can study the classifying space BA of a monoid A. Indeed one can define the classifying space $B\mathcal{C}$ of any "small category" $\mathcal{C}$ in a similar way. (A "small" category is one whose objects and morphisms are sets.) These are important construction in algebraic - K - theory as well as homotopy theory.

4.5 Some Applications

In a sense much of what we will study in the next chapter are applications of the classification theorem for principal bundles. In this section we describe a few immediate applications.

4.5.1 Line bundles over projective spaces

By the classification theorem we know that the set of isomorphism classes of complex line bundles over the projective space $\mathbb{CP}^n$ is given by

$$\begin{aligned} Vect^1(\mathbb{CP}^n) \cong Prin_{GL(1,\mathbb{C})}(\mathbb{CP}^n) \cong Prin_{U(1)}(\mathbb{CP}^n) \cong [\mathbb{CP}^n, BU(1)] = [\mathbb{CP}^n, \mathbb{CP}^\infty] \\ = [\mathbb{CP}^n, K(\mathbb{Z}, 2)] \cong H^2(\mathbb{CP}^n, \mathbb{Z}) \cong \mathbb{Z} \end{aligned}$$

Theorem 4.24. *Under the above isomorphism,*

$$Vect^1(\mathbb{CP}^n) \cong \mathbb{Z}$$

the n - fold tensor product of the universal line bundle $\gamma_1^{\otimes n}$ corresponds to the integer $n \geq 0$.

Proof. The classification theorem says that every line bundle ζ over $\mathbb{CP}^n$ is the pull back of the universal line bundle via a map $f_\zeta : \mathbb{CP}^n \to \mathbb{CP}^\infty$. That is,

$$\zeta \cong f_\zeta^*(\gamma_1).$$

The cohomology class corresponding to ζ, *the first chern class* $c_1(\zeta)$, is given by

$$c_1(\zeta) = f_\zeta^*(c) \in H^2(\mathbb{CP}^n) \cong \mathbb{Z}$$

where $c \in H^2(\mathbb{CP}^\infty) \cong \mathbb{Z}$ is the generator. Clearly $\iota^*(c) \in H^2(\mathbb{CP}^n)$ is the generator, where $\iota : \mathbb{CP}^n \hookrightarrow \mathbb{CP}^\infty$ is natural inclusion. But $\iota^*(\gamma_1) = \gamma_1 \in Vect^1(\mathbb{CP}^n)$. Thus $\gamma_1 \in Vect^1(\mathbb{CP}^n) \cong \mathbb{Z}$ corresponds to the generator.

To see the effect of taking tensor products, consider the following "tensor product map"

$$BU(1) \times \cdots \times BU(1) \xrightarrow{\otimes} BU(1)$$

defined to be the unique map (up to homotopy) that classifies the external tensor product $\gamma_1 \otimes \cdots \otimes \gamma_1$ over $BU(1) \times \cdots \times BU(1)$. Using $\mathbb{CP}^\infty \cong Gr_1(\mathbb{C}^\infty)$ as our model for $BU(1)$, this tensor product map is given by taking k lines $\ell_1, \cdots, \ell_k$ in $\mathbb{C}^\infty$ and considering the tensor product line

$$\ell_1 \otimes \cdots \otimes \ell_k \subset \mathbb{C}^\infty \otimes \cdots \otimes \mathbb{C}^\infty \xrightarrow[\psi]{\cong} \mathbb{C}^\infty$$

where $\psi : \mathbb{C}^\infty \otimes \cdots \otimes \mathbb{C}^\infty \cong \mathbb{C}^\infty$ is a fixed isomorphism. The induced map

$$\tau : \mathbb{CP}^\infty \times \cdots \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty \cong K(\mathbb{Z}, 2)$$

is determined up to homotopy by its effect on H^2. Clearly the restriction to each factor is the identity map and so

$$\tau^*(c) = c_1 + \cdots + c_k \in H^2(\mathbb{CP}^\infty \times \cdots \times \mathbb{CP}^\infty) = H^2(\mathbb{CP}^\infty) \oplus \cdots \oplus H^2(\mathbb{CP}^\infty) \cong \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$$

where c_i denotes the generator of H^2 of the i^{th} factor in the product. Therefore the composition

$$t_k : \mathbb{CP}^\infty \xrightarrow{\Delta} \mathbb{CP}^\infty \times \cdots \times \mathbb{CP}^\infty \xrightarrow{\tau} \mathbb{CP}^\infty$$

has the property that $t_k^*(c) = kc \in H^2(\mathbb{CP}^\infty)$. But also we have that on the bundle level,

$$t_k^*(\gamma_1) = \gamma_1^{\otimes k} \in Vect^1(\mathbb{CP}^\infty).$$

The theorem now follows. □

We have a similar result for real line vector bundles over real projective spaces.

Theorem 4.25. *The only nontrivial real line bundle over $\mathbb{RP}^n$ is the canonical line bundle γ_1.*

Proof. We know that γ_1 is nontrivial because its restriction to $S^1 = \mathbb{RP}^1 \subset \mathbb{RP}^n$ is the Moebeus strip line bundle, which is nonorientable, and hence nontrivial. On the other hand, by the classification theorem,

$$Vect^1_{\mathbb{R}}(\mathbb{RP}^n) \cong [\mathbb{RP}^n, BGL(1,\mathbb{R})] = [\mathbb{RP}^n, \mathbb{RP}^\infty] = [\mathbb{RP}^n, K(\mathbb{Z}_2, 1)] \cong H^1(\mathbb{RP}^n, \mathbb{Z}_2) \cong \mathbb{Z}_2.$$

Hence there is only one nontrivial line bundle over $\mathbb{RP}^n$. □

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4.5.2 Structures on bundles and homotopy liftings

The following theorem is a direct consequence of the classification theorem. We leave its proof as an exercise.

Theorem 4.26. *. Let $p : E \to B$ be a principal G - bundle classified by a map $f : B \to BG$. Let $H < G$ be a subgroup. By the naturality of the construction of classifying spaces, this inclusion induces a map (well defined up to homotopy) $\iota : BH \to BG$. Then the bundle $p : E \to B$ has an H -*

structure (i.e a reduction of its structure group to H) if and only if there is a map

$$\tilde{f} : B \to BH$$

so that the composition

$$B \xrightarrow{\tilde{f}} BH \xrightarrow{\iota} BG$$

is homotopic to $f : B \to BG$. In particular if $\tilde{p} : \tilde{E} \to B$ is the principal H - bundle classified by $\tilde{f}$, then there is an isomorphism of principal G bundles,

$$\tilde{E} \times_H G \cong E.$$

The map $\tilde{f} : B \to BH$ is called a "lifting" of the classifying map $f : B \to BG$. It is called a lifting because, as we saw at the end of the last section, the map $\iota : BH \to BG$ can be viewed as a fiber bundle, by taking our model for BH to be $BH = EG/H$. Then ι is the projection for the fiber bundle

$$G/H \to EG/H = BH \xrightarrow{\iota} EG/G = BG.$$

This bundle structure will allow us to analyze in detail what the obstructions are to obtaining a lift $\tilde{f}$ of a classifying map $f : B \to BG$. We will study this is chapter 4.

Examples.

- An orientation of a bundle classified by a map $f : B \to BO(k)$ is a lifting $\tilde{f} : B \to BSO(k)$. Notice that the map $\iota : BSO(k) \to BO(k)$ can be viewed as a two - fold covering map

$$\mathbb{Z}_2 = O(k)/SO(k) \to BSO(k) \xrightarrow{\iota} BO(k).$$

- An almost complex structure on a bundle classified by a map $f : B \to BO(2n)$ is a lifting $\tilde{f} : B \to BU(n)$. Notice we have a bundle

$$O(2n)/U(n) \to BU(n) \to BO(2n).$$

The following example will be particularly useful in the next chapter when we define characteristic classes and do calculations with them.

Theorem 4.27. *A complex bundle vector bundle ζ classified by a map $f : B \to BU(n)$ has a nowhere zero section if and only if f has a lifting $\tilde{f} : B \to BU(n-1)$. Similarly a real vector bundle η classified by a map $f : B \to BO(n)$ has a nowhere zero section if and only if f has a lifting $\tilde{f} : B \to BO(n-1)$. Notice we have the following bundles:*

$$S^{2n-1} = U(n)/U(n-1) \to BU(n-1) \to BU(n)$$

and

$$S^{n-1} = O(n)/O(n-1) \to BO(n-1) \to BO(n).$$

This theorem says that $BU(n-1)$ forms a sphere bundle (S^{2n-1}) over $BU(n)$, and similarly, $BO(n-1)$ forms a S^{n-1} - bundle over $BO(n)$. We identify these sphere bundles as follows.

Corollary 4.28. *The sphere bundles*

$$S^{2n-1} \to BU(n-1) \to BU(n)$$

and

$$S^{n-1} \to BO(n-1) \to BO(n)$$

are isomorphic to the unit sphere bundles of the universal vector bundles γ_n over $BU(n)$ and $BO(n)$ respectively.

Proof. We consider the complex case. The real case is proved in the same way. Notice that the model for the sphere bundle in the above theorem is the projection map

$$p : BU(n-1) = EU(n)/U(n-1) \to EU(n)/U(n) = BU(n).$$

But γ_n is the vector bundle $EU(n) \times_{U(n)} \mathbb{C}^n \to BU(n)$ which therefore has unit sphere bundle

$$S(\gamma_n) = EU(n) \times_{U(n)} S^{2n-1} \to BU(n) \tag{4.4}$$

where $S^{2n-1} \subset \mathbb{C}^n$ is the unit sphere with the induced $U(n)$ - action. But $S^{2n-1} \cong U(n)/U(n-1)$ and this diffeomorphism is equivariant with respect to this action. Thus the unit sphere bundle is given by

$$S(\gamma_n) = EU(n) \times_{U(n)} U(n)/U(n-1) \cong EU(n)/U(n-1) = BU(n-1)$$

as claimed. □

We observe that by using the Grassmannian models for $BU(n)$ and $BO(n)$, then their relation to the sphere bundles can be seen explicitly in the following way. This time we work in the real case.

Consider the embedding

$$\iota : Gr_{n-1}(\mathbb{R}^N) \hookrightarrow Gr_n(\mathbb{R}^N \times \mathbb{R}) = Gr_n(\mathbb{R}^{N+1})$$

defined by

$$(V \subset \mathbb{R}^N) \to (V \times \mathbb{R} \subset \mathbb{R}^N \times \mathbb{R}).$$

Clearly as $N \to \infty$ this map becomes a model for the inclusion $BO(n-1) \hookrightarrow BO(n)$. Now for $V \in Gr_{n-1}(\mathbb{R}^N)$ consider the vector $(0,1) \in V \times \mathbb{R} \subset \mathbb{R}^N \times \mathbb{R}$. This is a unit vector, and so is an element of the fiber of the unit sphere bundle $S(\gamma_n)$ over $V \times \mathbb{R}$. Hence this association defines a map

$$j : Gr_{n-1}(\mathbb{R}^N) \to S(\gamma_n)$$

which lifts $\iota : Gr_{n-1}(\mathbb{R}^N) \hookrightarrow Gr_n(\mathbb{R}^{N+1})$. By taking a limit over N we get a map $j : BO(n-1) \to S(\gamma_n)$.

To define a homotopy inverse $\rho : S(\gamma_n) \to BO(n-1)$, we again work on the finite Grassmannian level.

Let $(W, w) \in S(\gamma_n)$, the unit sphere bundle over $Gr_n(\mathbb{R}^K)$. Thus $W \subset \mathbb{R}^K$ is an n -dimensional subspace and $w \in W$ is a unit vector. Let $W_w \subset W$ denote the orthogonal complement to the vector w in W. Thus $W_w \subset W \subset \mathbb{R}^K$ is an $n-1$ - dimensional subspace. This association defines a map

$$\rho : S(\gamma_n) \to Gr_{n-1}(\mathbb{R}^K)$$

and by taking the limit over K, defines a map $\rho : S(\gamma_n) \to BO(n-1)$. We leave it to the reader to verify that $j : BO(n-1) \to S(\gamma_n)$ and $\rho : S(\gamma_n) \to BO(n-1)$ are homotopy inverse to each other.

4.5.3 Embedded bundles and K -theory

The classification theorem for vector bundles says that for every n - dimensional complex vector bundle ζ over X, there is a classifying map $f_\zeta : X \to BU(n)$ so that ζ is isomorphic to pull back, $f^*(\gamma_n)$ of the universal vector bundle. A similar statement holds for real vector bundles. Using the Grassmannian models for these classifying spaces, we obtain the following as a corollary.

Theorem 4.29. *Every n - dimensional complex bundle ζ over a space X can be embedded in a trivial infinite dimensional bundle, $X \times \mathbb{C}^\infty$. Similarly, every n - dimensional real bundle η over X can be embedded in the trivial bundle $X \times \mathbb{R}^\infty$.*

Proof. Let $f_\zeta : X \to Gr_n(\mathbb{C}^\infty) = BU(n)$ classify ζ. So $\zeta \cong f^*(\gamma_n)$. But recall that

$$\gamma_n = \{(V, v) \in Gr_n(\mathbb{C}^\infty) \times \mathbb{C}^\infty \quad \text{such that}\, v \in V.\}$$

Hence γ_n is naturally embedded in the trivial bundle $Gr_n(\mathbb{C}^\infty) \times \mathbb{C}^\infty$. Thus $\zeta \cong f^*(\gamma_n)$ is naturally embedded in $X \times \mathbb{C}^\infty$. The real case is proved similarly. □

Notice that because of the direct limit topology on $Gr_n(\mathbb{C}^\infty) = \varinjlim Gr_n(\mathbb{C}^N)$, then if X is a compact space, any map $f : X \to Gr_n(\mathbb{C}^\infty)$ has image that lies in $Gr_n(\mathbb{C}^N)$ for some finite N. But notice that over this finite Grassmannian, $\gamma_n \subset Gr_n(\mathbb{C}^N) \times \mathbb{C}^N$. The following is then an immediate corollary. This result was used in chapter one in our discussion about K -theory.

Corollary 4.30. *If X is a compact space of the homotopy type of a CW-complex, then every n - dimensional complex bundle zeta can be embedded in a trivial bundle $X \times \mathbb{C}^N$ for some N. The analogous result also holds for real vector bundles.*

Let $f : X \to BU(n)$ classify the n - dimensional complex vector bundle ζ. Then clearly the composition $f : X \to BU(n) \hookrightarrow BU(n+1)$ classifies the $n+1$ dimensional vector bundle $\zeta \oplus \epsilon_1$, where as before, ϵ_1 is the one dimensional trivial line bundle. This observation leads to the following.

Proposition 4.31. *Let ζ_1 and ζ_2 be two n -dimensional vector bundles over X classified by f_1 and $f_2 : X \to BU(n)$ respectively. Then if we add trivial bundles, we get an isomorphism*

$$\zeta_1 \oplus \epsilon_k \cong \zeta_2 \oplus \epsilon_k$$

if and only if the compositions,

$$f_1,\, f_2 : X \to BU(n) \hookrightarrow BU(n+k)$$

are homotopic.

Now recall from the discussion of K - theory in the last chapter that the set of *stable* isomorphism classes of vector bundles $\mathcal{S}Vect(X)$ is isomorphic to the reduced K - theory, $\tilde{K}(X)$, when X is compact. This proposition then implies the following important result, which displays how in the case of compact spaces, computing K -theory reduces to a specific homotopy theory calculation.

Definition 4.10. *Let BU be the limit of the spaces*

$$BU = \varinjlim_n BU(n).$$

Similarly,

$$BO = \varinjlim_n BO(n).$$

Theorem 4.32. *For X compact there are isomorphisms (bijective correspondences)*

$$\tilde{K}(X) \cong \mathcal{S}Vect(X) \cong [X, BU]$$

and

$$\tilde{KO}(X) \cong \mathcal{S}Vect_{\mathbb{R}}(X) \cong [X, BO].$$

4.5.4 Representations and flat connections

Recall the following classification theorem for covering spaces.

Theorem 4.33. . *Let X be a connected space. Then the set of isomorphism classes of connected covering spaces, $p : E \to X$ is in bijective correspondence with conjugacy classes of normal subgroups of $\pi_1(X)$. This correspondence sends a covering $p : E \to B$ to the image $p_*(\pi_1(E)) \subset \pi_1(X)$.*

Let $\pi = \pi_1(X)$ and let $p : E \to X$ be a connected covering space with $\pi_1(E) = N \lhd \pi$. Then the group of deck transformations of E is the quotient group π/N, and so can be thought of as a principal π/N - bundle. Viewed this way it is classified by a map $f_E : X \to B(\pi/N)$, which on the level of fundamental groups,

$$f_* : \pi = \pi_1(X) \to \pi_1(B(\pi/N)) = \pi/N \tag{4.5}$$

is just the projection on to the quotient space. In particular the universal cover $\tilde{X} \to X$ is the unique simply connected covering space. It is classified by a map

$$\gamma_X : X \to B\pi$$

which induces an isomorphism on the fundamental group.

Now let $\theta : \pi \to G$ be any group homomorphism. By the naturality of classifying spaces this induces a map on classifying spaces,

$$B\theta : B\pi \to BG.$$

This induces a principal G - bundle over X classified by the composition

$$X \xrightarrow{\gamma_X} B\pi \xrightarrow{B\theta} BG.$$

The bundle this map classifies is given by

$$\tilde{X} \times_\pi G \to X$$

where π acts on G via the homomorphism $\theta : \pi \to G$.

This construction defines a map

$$\rho : Hom(\pi_1(X), G) \to Prin_G(X).$$

Now if X is a smooth manifold then its universal cover $p : \tilde{X} \to X$ induces an isomorphism on tangent spaces,

$$Dp(x) : T_x\tilde{X} \to T_{p(x)}X$$

for every $x \in \tilde{X}$. Thus, viewed as a principal π - bundle, it has a canonical connection. Notice furthermore that this connection is *flat*, i.e its curvature is zero.

Exercise.

Check this claim! That is, show that the canonical connection on a covering space is flat.

Now notice that any bundle of the form $\tilde{X} \times_\pi G \to X$ has an induced flat connection. In particular the image of $\rho : Hom(\pi_1(X), G) \to Prin_G(X)$ consists of principal bundles equipped with flat connections.

Notice furthermore that by taking $G = GL(n, \mathbb{C})$ the map ρ assigns to an n - dimensional representation an n - dimensional vector bundle with flat connection

$$\rho : Rep_n(\pi_1(X)) \to Vect_n(X).$$

By taking the sum over all n and passing to the Grothendieck group completion,we get a homomorphism of rings from the representation ring to K - theory,

$$\rho : R(\pi_1(X)) \to K(X).$$

An important question is what is the image of this map of rings. Again we know the image is contained in the classes represented by bundles that have flat connections. For $X = B\pi$, for π a finite group, the following is a famous theorem of Atiyah and Segal:

Let

$$\epsilon : R(\pi) \to \mathbb{Z} \quad \text{and} \quad \epsilon : K(B\pi) \to \mathbb{Z}$$

be the augmentation maps induced by sending a representation or a vector bundle to its dimension. Let $I \subset R(\pi)$ and $I \subset K(B\pi)$ denote the kernels of these augmentations, i.e the "augmentation ideals". Finally let $\bar{R}(\pi)$ and $\bar{K}(B\pi)$ denote the completions of these rings with respect to these ideals. That is,

$$\bar{R}(\pi) = \varprojlim_n R(\pi)/I^n \quad \text{and} \quad \bar{K}(B\pi) = \varprojlim_n K(B\pi)/I^n$$

where I^n is the product of the ideal I with itself n - times.

Theorem 4.34. *(Atiyah and Segal) [9] For π a finite group, the induced map on the completions of the rings with respect ot the augmentation ideals,*

$$\rho : \bar{R}(\pi) \to \bar{K}(B\pi)$$

is an isomorphism.

5

Characteristic Classes

In this chapter we define and calculate characteristic classes for principal bundles and vector bundles. Characteristic classes are the basic cohomological invariants of bundles and have a wide variety of applications throughout topology and geometry. Characteristic classes were introduced originally by E. Stiefel in Switzerland and H. Whitney in the United States in the mid 1930's. Stiefel, who was a student of H. Hopf intoduced in his thesis certain "characteristic homology classes" determined by the tangent bundle of a manifold. At about the same time Whitney studied general sphere bundles, and later introduced the general notion of a characteristic cohomology class coming from a vector bundle, and proved the product formula for their calculation.

In the early 1940's, L. Pontrjagin, in Moscow, introduced new characteristic classes by studying the Grassmannian manifolds, using work of C. Ehresmann from Switzerland. In the mid 1940's, after just arriving in Princeton from China, S.S Chern defined characteristic classes for complex vector bundles using differential forms and his calculations led a great clarification of the theory.

Much of the modern view of characteristic classes has been greatly influenced by the highly influential book of Milnor and Stasheff. This book was originally circulated as lecture notes written in 1957 and finally published in 1974. This book is one of the great textbooks in modern mathematics. These notes follow, in large part, their treatment of the subject. The reader is encouraged to consult their book for further details.

5.1 Preliminaries

Definition 5.1. *Let G be a topological group (possibly with the discrete topology). Then a characteristic class* for principal G - bundles is an assignment to each principal G - bundle $p : P \to B$ a cohomology class

$$c(P) \in H^*(B)$$

satisfying the following naturality condition. If

$$\begin{array}{ccc} P_1 & \xrightarrow{\bar{f}} & P_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ B_1 & \xrightarrow[f]{} & B_2 \end{array}$$

is a map of principal G - bundles inducing an equivariant homeomorphism on fibers, then

$$f^*(c(P_2)) = c(P_1) \in H^*(B_1).$$

Remarks. 1. In this definition cohomology could be taken with any coefficients, including, for example, DeRham cohomology, which has coefficients in the real numbers $\mathbb{R}$. The particular cohomology theory used is referred to as the "values" of the characteristic classes.

2. The same definition of characteristic classes applies to real or complex vector bundles as well as principal bundles.

The following is an easy consequence of the definition.

Lemma 5.1. *Let c be a characteristic class for principal G - bundles so that c takes values in $H^q(-)$, for $q \geq 1$. Then if ϵ is the trivial G bundle,*

$$\epsilon = X \times G \to X$$

then $c(\epsilon) = 0$.

Proof. The trivial bundle ϵ is the pull - back of the constant map to the one point space $e : X \to pt$ of the bundle $\nu = G \to pt$. Thus $c(\epsilon) = e^*(c(\nu))$. But $c(\nu) \in H^q(pt) = 0$ when $q > 0$. □

The following observation is also immediate from the definition.

Lemma 5.2. *Characteristic classes are invariant under isomorphism. More specifically, Let c be a characteristic class for principal G - bundles. Also let $p_1 : E_1 \to X$ and $p_2 : E_2 \to X$ be isomorphic principal G - bundles. Then*

$$c(E_1) = c(E_2) \in H^*(X).$$

Thus for a given space X, a characteristic class c can be viewed as a map

$$c : Prin_G(X) \to H^*(X).$$

3. The naturality property in the definition can be stated in more functorial terms in the following way.

Cohomology (with any coefficients) $H^*(-)$ is a contravariant functor from the category $ho\mathcal{T}op$ of topological spaces and homotopy classes of maps, to the category $\mathcal{A}b$ of abelian groups. By the results of chapter 2, the set of principal G - bundles $Prin_G(-)$ can be viewed as a contravariant functor from the category $ho\mathcal{T}op$ to the category of sets $\mathcal{S}ets$.

Definition 5.2. (Alternative) A characteristic class is a natural transformation c between the functors $Prin_G(-)$ and $H^*(-)$:

$$c : Prin_G(-) \rightsquigarrow H^*(-)$$

Examples.

1. The *first Chern class* $c_(\zeta)$ is a characteristic class on principal $U(1)$ - bundles, or equivalently, complex line bundles. If ζ is a line bundle over X, then $c_1(\zeta) \in H^2(X;\mathbb{Z})$. As we saw in the last chapter, c_1 is a complete invariant of line bundles. That is to say, the map

$$c_1 : Prin_{U(1)}(X) \to H^2(X;\mathbb{Z})$$

is an isomorphism.

2. The *first Stiefel - Whitney class* $w_1(\eta)$ is a characteristic class of two fold covering spaces (i.e a principal $\mathbb{Z}_2 = O(1)$ - bundles) or of real line bundles. If η is a real line bundle over a space X, then $w_1(\eta) \in H^1(X;\mathbb{Z}_2)$. Moreover, as we saw in the last chapter, the first Stiefel - Whitney class is a complete invariant of line bundles. That is, the map

$$w_1 : Prin_{O(1)}(X) \to H^1(X;\mathbb{Z}_2)$$

is an isomorphism.

We remark that the first Stiefel - Whitney class can be extended to be a characteristic class of real n - dimensional vector bundles (or principal $O(n)$ - bundles) for any n. To see this, consider the subgroup $SO(n) < O(n)$. As we saw in the last chapter, a bundle has an $SO(n)$ structure if and only if it is orientable. Moreover the induced map of classifying spaces gives a 2 - fold covering space or principal $O(1)$ - bundle,

$$\mathbb{Z}_2 = O(1) = O(n)/SO(n) \to BSO(n) \to BO(n).$$

This covering space defines, via its classifying map $w_1 : BO(n) \to BO(1) = \mathbb{RP}^\infty$ an element $w_1 \in H^1(BO(n);\mathbb{Z}_2)$ which is the first Stiefel - Whitney class of this covering space.

Now let η be any n - dimensional real vector bundle over X, and let

$$f_\eta : X \to BO(n)$$

be its classifying map.

Definition 5.3. *The first Stiefel - Whitney class* $w_1(\eta) \in H^1(X;\mathbb{Z}_2)$ *is defined to be*

$$w_1(\eta) = f_\eta^*(w_1) \in H^1(X;\mathbb{Z}_2)$$

The first Chern class c_1 *of an* n *- dimensional complex vector bundle* ζ *over* X *is defined similarly, by pulling back the first Chern class of the principal* $U(1)$ *- bundle*

$$U(1) \cong U(n)/SU(n) \to BSU(n) \to BU(n)$$

via the classifying map $f_\zeta : X \to BU(n)$.

The following is an immediate consequence of the above lemma and the meaning of $SO(n)$ and $SU(n)$ - structures.

Theorem 5.3. *Given a complex* n *- dimensional vector bundle* ζ *over* X*, then* $c_1(\zeta) \in H^2(X)$ *is zero if and only if* ζ *has an* $SU(n)$ *-structure.*

Furthermore, given a real n *- dimensional vector bundle* η *over* X*, then* $w_1(\eta) \in H^1(X;\mathbb{Z}_2)$ *is zero if and only if the bundle* η *has an* $SO(n)$ *- structure, which is equivalent to* η *being orientable.*

We now use the classification theorem for bundles to describe the set of characteristic classes for principal G - bundles.

Let R be a commutative ring and let $Char_G(R)$ be the set of all characteristic classes for principal G bundles that take values in $H^*(-;R)$. Notice that the sum (in cohomology) and the cup product of characteristic classes is again a characteristic class. This gives $Char_G$ the structure of a ring. (Notice that the unit in this ring is the constant characteristic class $c(\zeta) = 1 \in H^0(X)$.

Theorem 5.4. *There is an isomorphism of rings*

$$\rho : Char_G(R) \xrightarrow{\cong} H^*(BG;R)$$

Proof. Let $c \in Char_G(R)$. Define

$$\rho(c) = c(EG) \in H^*(BG;R)$$

where $EG \to BG$ is the universal G - bundle over BG. By definition of the ring structure of $Char_G(R)$, ρ is a ring homomorphism.

Now let $\gamma \in H^q(BG;R)$. Define the characteristic class c_γ as follows. Let $p : E \to X$ be a principal G - bundle classified by a map $f_E : X \to BG$. Define

$$c_\gamma(E) = f_E^*(\gamma) \in H^q(X;R)$$

where $f_E^* : H^*(BG : R) \to H^*(X;R)$ is the cohomology ring homomorphism induced by f_E. This association defines a map

$$c : H^*(BG;R) \to Char_G(R)$$

which immediately seen to be inverse to ρ. □

5.2 Chern Classes and Stiefel - Whitney Classes

In this section we compute the rings of unitary characteristic classes $Char_{U(n)}(\mathbb{Z})$ and $\mathbb{Z}_2$ - valued orthogonal characteristic classes $Char_{O(n)}(\mathbb{Z}_2)$. These are the characteristic classes of complex and real vector bundles and as such have a great number of applications. By Theorem 5.4 computing these rings of characteristic classes reduces to computing the cohomology rings $H^*(BU(n); \mathbb{Z})$ and $H^*(BO(n); \mathbb{Z}_2)$. The following is the main theorem of this section.

Theorem 5.5. *a. The ring of $U(n)$ characteristic classes is a polynomial algebra on n - generators,*

$$Char_{U(n)}(\mathbb{Z}) \cong H^*(BU(n); \mathbb{Z}) \cong \mathbb{Z}[c_1, c_2, \cdots, c_n]$$

where $c_i \in H^{2i}(BU(n); \mathbb{Z})$ is known as the i^{th} - Chern class.

b. The ring of $\mathbb{Z}_2$ - valued $O(n)$ characteristic classes is a polynomial algebra on n - generators,

$$Char_{O(n)}(\mathbb{Z}_2) \cong H^*(BO(n); \mathbb{Z}_2) \cong \mathbb{Z}_2[w_1, w_2, \cdots, w_n]$$

where $w_i \in H^i(BO(n); \mathbb{Z}_2)$ is known as the i^{th} - Stiefel - Whitney class.

This theorem will be proven by induction on n. For $n = 1$ $BU(1) = \mathbb{CP}^\infty$ and $BO(1) = \mathbb{RP}^\infty$ and so the theorem describes the ring structure in the cohomology of these projective spaces. To complete the inductive step we will study the sphere bundles

$$S^{n-1} \to BO(n-1) \to BO(n)$$

and

$$S^{2n-1} \to BU(n-1) \to BU(n)$$

described in the last chapter. In particular recall from Corollary 4.28 that in these fibrations, $BO(n-1)$ and $BU(n-1)$ are the unit sphere bundles $S(\gamma_n)$ of the universal bundle γ_n over $BO(n)$ and $BU(n)$ respectively. Let $D(\gamma_n)$ be the unit disk bundles of the universal bundles. That is, in the complex case,

$$D(\gamma_n) = EU(n) \times_{U(n)} D^{2n} \to BU(n)$$

and in the real case,

$$D(\gamma_n) = EO(n) \times_{O(n)} D^n \to BO(n)$$

where $D^{2n} \subset \mathbb{C}^n$ and $D^n \subset \mathbb{R}^n$ are the unit disks, and therefore have the induced unitary and orthogonal group actions.

Here is one easy observation about these disk bundles.

Proposition 5.6. *The projection maps*

$$p : D(\gamma_n) = EU(n) \times_{U(n)} D^{2n} \to BU(n)$$

and

$$D(\gamma_n) = EO(n) \times_{O(n)} D^n \to BO(n)$$

are homotopy equivalences.

Proof. Both of these bundles have zero sections $\mathcal{Z} : BU(n) \to D(\gamma_n)$ and $\mathcal{Z} : BO(n) \to D(\gamma_n)$. In both the complex and real cases, we have $p \circ \mathcal{Z} = 1$. To see that $\mathcal{Z} \circ p \simeq 1$ consider the homotopy $H : D(\gamma_n) \times I \to D(\gamma_n)$ defined by $H(v, t) = tv$. □

We will use this result when studying the cohomology exact sequence of the pair $(D(\gamma_n), S(\gamma_n))$:

$$\begin{aligned} \cdots \to H^{q-1}(S(\gamma_n)) &\xrightarrow{\delta} H^q(D(\gamma_n), S(\gamma_n)) \to H^q(D(\gamma_n)) \to H^q(S(\gamma_n)) \\ &\xrightarrow{\delta} H^{q+1}(D(\gamma_n), S(\gamma_n)) \to H^{q+1}(D(\gamma_n)) \to \cdots \end{aligned} \tag{5.1}$$

Using the above proposition and Corollary 4.28 we can substitute $H^*(BU(n))$ for $H^*(D(\gamma_n))$, and $H^*(BU(n-1))$ for $H^*(S(\gamma_n))$ in this sequence to get the following exact sequence

$$\begin{aligned} \cdots \to H^{q-1}(BU(n-1)) &\xrightarrow{\delta} H^q(D(\gamma_n), S(\gamma_n)) \to H^q(BU(n)) \\ &\xrightarrow{\iota} H^q(BU(n-1)) \xrightarrow{\delta} H^{q+1}(D(\gamma_n), S(\gamma_n)) \to H^{q+1}(BU(n)) \to \cdots . \end{aligned} \tag{5.2}$$

and we get a similar exact sequence in the real case

$$\begin{aligned} \cdots \to H^{q-1}(BO(n-1); \mathbb{Z}_2) &\xrightarrow{\delta} H^q(D(\gamma_n), S(\gamma_n); \mathbb{Z}_2) \to H^q(BO(n); \mathbb{Z}_2) \\ &\xrightarrow{\iota} H^q(BO(n-1); \mathbb{Z}_2) \xrightarrow{\delta} H^{q+1}(D(\gamma_n), S(\gamma_n); \mathbb{Z}_2) \to H^{q+1}(BO(n); \mathbb{Z}_2) \to \cdots \end{aligned} \tag{5.3}$$

These exact sequences will be quite useful for inductively computing the cohomology of these classifying spaces, but to do so we need a method for computing $H^*(D(\gamma_n), S(\gamma_n))$, or more generally, $H^*(D(\zeta), S(\zeta))$, where ζ is any Euclidean vector bundle and $D(\zeta)$ and $S(\zeta)$ are the associated unit disk bundles and sphere bundles respectively. The quotient space,

$$T(\zeta) = D(\zeta)/S(\zeta) \tag{5.4}$$

is called the *Thom space* of the bundle ζ. As the name suggests, this construction was first studied by R. Thom [94], and has been quite useful in both bundle theory and cobordism theory. Notice that on each fiber (say at $x \in X$) of the n - dimensional disk bundle ζ, the Thom space construction takes the unit n - dimensional disk modulo its boundary $n-1$ - dimensional sphere which therefore yields an n - dimensional sphere, with marked basepoint, say $\infty_x \in S^n(\zeta_x) = D^n(\zeta_x)/S^{n-1}(\zeta_x)$. The Thom space construction then identifies all the basepoints ∞_x to a single point. Notice that for a bundle over a point $\mathbb{R}^n \to pt$, the Thom space $T(\mathbb{R}^n) = D^n/S^{n-1} = S^n \cong \mathbb{R}^n \cup \infty$. More generally, notice that when the basespace X is compact, then the Thom space is simply the one point compactification of the total space of the vector bundle ζ,

$$T(\zeta) \cong \zeta^+ = \zeta \cup \infty \tag{5.5}$$

where we think of the extra point in this compactification as the common point at infinity assigned to each fiber. In order to compute with the above exact sequences, we will need to study the cohomology of Thom spaces. But before we do we examine the topology of the Thom spaces of product bundles. For this we introduce the "smash product" construction.

Let X and Y be spaces with basepoints $x_0 \in X$ and $y_0 \in Y$.

Definition 5.4. *The wedge* $X \vee Y$ is the "one point union",

$$X \vee Y = X \times y_0 \cup x_0 \times Y \subset X \times Y.$$

The *smash product* $X \wedge Y$ is given by

$$X \wedge Y = X \times Y/X \vee Y.$$

Observations. 1. The k be a field. Then the Kunneth formula gives

$$\tilde{H}^*(X \wedge Y; k) \cong \tilde{H}^*(X; k) \otimes \tilde{H}^*(Y; k).$$

2. Let V and W be vector spaces, and let V^+ and W^+ be their one point compactifications. These are spheres of the same dimension as the respective vector spaces. Then

$$V^+ \wedge W^+ = (V \times W)^+.$$

So in particular,

$$S^n \wedge S^m = S^{n+m}.$$

Proposition 5.7. *Let ζ be an n - dimensional vector bundle over a space X, and let η be an m - dimensional bundle over X. Let $\zeta \times \eta$ be the product $n+m$ - dimensional vector bundle over $X \wedge Y$. Then the Thom space of $\zeta \times \eta$ is given by*

$$T(\zeta \times \eta) \cong T(\zeta) \wedge T(\eta).$$

Proof. Notice that the disk bundle is given by

$$D(\zeta \times \eta) \cong D(\zeta) \times D(\eta)$$

and its boundary sphere bundle is given by

$$S(\zeta \times \eta) \cong S(\zeta) \times D(\eta) \cup D(\zeta) \times S(\eta).$$

Thus

$$\begin{aligned} T(\zeta \times \eta) = D(\zeta \times \eta)/S(\zeta \times \eta) &\cong D(\zeta) \times D(\eta)/\left(S(\zeta) \times D(\eta) \cup D(\zeta) \times S(\eta)\right) \\ &\cong D(\zeta)/S(\zeta) \wedge D(\eta)/S(\eta) \\ &\cong T(\zeta) \wedge T(\eta). \end{aligned}$$

□

We now proceed to study the cohomology of Thom spaces.

5.2.1 The Thom Isomorphism Theorem

We begin by describing a cohomological notion of orientability of an vector bundle ζ over a space X.

Consider the 2 - fold cover over X defined as follows. Let E_ζ be the principal $GL(n,\mathbb{R})$ bundle associated to ζ. Also let Gen_n be the set of generators of $H^n(S^n) \cong \mathbb{Z}$. So Gen_n is a set with two elements. Moreover the general linear group $GL(n,\mathbb{R})$ acts on $S^n = \mathbb{R}^n \cup \infty$ by the usual linear action on $\mathbb{R}^n$ extended to have a fixed point at $\infty \in S^n$. By looking at the induced map on cohomology, there is an action of $GL(n,\mathbb{R})$ on Gen_n. We can then define the double cover

$$\mathcal{G}(\zeta) = E_\zeta \times_{GL(n,\mathbb{R})} Gen_n \longrightarrow E_\zeta/GL(n,\mathbb{R}) = X.$$

Lemma 5.8. *The double covering $\mathcal{G}(\zeta)$ is isomorphic to the orientation double cover $Or(\zeta)$.*

Proof. Recall from chapter 1 that the orientation double cover $Or(\zeta)$ is given by

$$Or(\zeta) = E_\zeta \times_{GL(n,\mathbb{R})} Or(\mathbb{R}^n)$$

where $Or(\mathbb{R}^n)$ is the two point set consistingof orientations of the vector space $\mathbb{R}^n$. A matrix $A \in GL(n, \mathbb{R})$ acts on this set trivially if and only if the determinant $detA$ is positive. It acts nontrivially (i.e permutes the two elements) if and only if $detA$ is negative. Now the same is true of the action of $GL(n, \mathbb{R})$ on Gen_n. This is because $A \in GL(n, \mathbb{R})$ induces multiplication by the sign of $detA$ on $H^n(S^n)$.

Exercise. Verify this claim. That is, prove that $A \in GL(n, \mathbb{R})$ induces multiplication by the sign of $detA$ on $H^n(S^n)$.

Since $Or(\mathbb{R}^n)$ and Gen_n are both two point sets with the same action of $GL(n, \mathbb{R})$, the corresponding two fold covering spaces $Or(\zeta)$ and $\mathcal{G}(\zeta)$ are isomorphic. □

Corollary 5.9. *An orientation of an n - dimensional vector bundle ζ is equivalent to a section of $\mathcal{G}(\zeta)$ and hence defines a continuous family of generators*

$$u_x \in H^n(S^n(\zeta_x)) \cong \mathbb{Z}$$

for every $x \in X$. Here $S^n(\zeta_x)$ is the unit disk of the fiber ζ_x modulo its boundary sphere. $S^n(\zeta_x)$ is called the sphere at x.

Now recall that given a pair of spaces $A \subset Y$, there is a relative cup product in cohomology,

$$H^q(Y) \otimes H^r(Y, A) \xrightarrow{\cup} H^{q+r}(Y, A).$$

So in particular the relative cohomology $H^*((Y, A)$ is a (graded) module over the (graded) ring $H^*(Y)$.

In the case of a vector bundle ζ over a space X, we then have that $H^*(D(\zeta), S(\zeta)) = \tilde{H}^*(T(\zeta))$ is a module over $H^*(D(\zeta)) \cong H^*(X)$. So in particular, given any cohomology class in the Thom space, $\alpha \in H^r(T(\zeta))$ we get an induced homomorphism

$$H^q(X) \xrightarrow{\cup\alpha} H^{q+r}(T(\zeta)).$$

Our next goal is to prove the famous *Thom Isomorphism Theorem* which can be stated as follows.

Theorem 5.10. *Let ζ be an oriented n - dimensional real vector bundle over a connected space X. Let R be any commutative ring. The orientation gives* *generators* $u_x \in H^n(S^n(\zeta_x); R) \cong R$. *Then there is a unique class (called the Thom class) in the cohomology of the Thom space*

$$u \in H^n(T(\zeta); R)$$

so that for every $x \in X$, *if*

$$j_x : S^n(\zeta_x) \hookrightarrow D(\zeta)/S(\zeta) = T(\zeta)$$

is the natural inclusion of the sphere at x *in the Thom space, then under the induced homomorphism in cohomology,*

$$j_x^* : H^n(T(\zeta); R) \to H^n(S^n(\zeta_x); R) \cong R$$

$j_x^*(u) = u_x$.

Furthermore The induced cup product map

$$\gamma : H^q(X; R) \xrightarrow{\cup u} \tilde{H}^{q+n}(T(\zeta); R)$$

is an isomorphism for every $q \in \mathbb{Z}$. *So in particular* $\tilde{H}^r(T(\zeta); R) = 0$ *for* $r < n$.

If ζ *is not an orientable bundle over* X, *then the theorem remains true if we take* $\mathbb{Z}_2$ *coefficients,* $R = \mathbb{Z}_2$.

Proof. We prove the theorem for oriented bundles. We leave the nonorientable case (when $R = \mathbb{Z}_2$) to the reader. We also restrict our attention to the case $R = \mathbb{Z}$, since the theorem for general coefficients will follow immediately from this case using the universal coefficient theorem.

Case 1: ζ is the trivial bundle $X \times \mathbb{R}^n$.

In this case the Thom space $T(\zeta)$ is given by

$$T(\zeta) = X \times D^n / X \times S^{n-1}.$$

The projection of X to a point, $X \to pt$ defines a map

$$\pi : T(\zeta) = X \times D^n / X \times S^{n-1} \to D^n / S^{n-1} = S^n.$$

Let $u \in H^n(T(\zeta))$ be the image in cohomology of a generator,

$$\mathbb{Z} \cong H^n(S^n) \xrightarrow{\pi^*} H^n(T(\zeta)).$$

The fact that taking the cup product with this class

$$H^q(X) \xrightarrow{\cup u} H^{q+n}(T(\zeta)) = H^{q+n}(X \times D^n, X \times S^{n-1}) = H^{q+n}(X \times S^n, X \times pt)$$

is an isomorphism for every $q \in \mathbb{Z}$ follows from the universal coefficient theorem.

Case 2: X is the union of two open sets $X = X_1 \cup X_2$, where we know the Thom isomorphism theorem holds for the restrictions $\zeta_i = \zeta_{|_{X_i}}$ for $i = 1, 2$ and for $\zeta_{1,2} = \zeta_{|_{X_1 \cap X_2}}$.

We prove the theorem for X using the Mayer - Vietoris sequence for cohomology. Let $X_{1,2} = X_1 \cap X_2$.

$$\to H^{q-1}(T(\zeta_{1,2})) \to H^q(T(\zeta)) \to H^q(T(\zeta_1)) \oplus H^q(T(\zeta_2)) \to H^q((T(\zeta_{1,2})) \to \cdots$$

Looking at this sequence when $q < n$, we see that since

$$H^q(T(\zeta_{1,2})) = H^q(T(\zeta_1)) = H^q(T(\zeta_2)) = 0,$$

then by exactness we must have that $H^q(T(\zeta)) = 0$.

We now let $q = n$, and we see that by assumption, $H^n(T(\zeta_1)) \cong H^n(T(\zeta_2)) \cong H^n(T(\zeta_{1,2})) \cong \mathbb{Z}$, and that the Thom classes of each of the restriction maps $H^n(T(\zeta_1)) \to H^n(T(\zeta_{1,2}))$ and $H^n(T(\zeta_2)) \to H^n(T(\zeta_{1,2}))$ correspond. Moreover $H^{n-1}(T(\zeta_{1,2})) = 0$. Hence by the exact sequence, $H^n(T(\zeta)) \cong \mathbb{Z}$ and there is a class $u \in H^n(T(\zeta))$ that maps to the direct sum of the Thom classes in $H^n(T(\zeta_1)) \oplus H^n(T(\zeta_2))$.

Now for $q \geq n$ we compare the above Mayer - Vietoris sequence with the one of base spaces,

$$\to H^{q-1}(X_{1,2}) \to H^q(X) \to H^q(X_1) \oplus H^q(X_2) \to H^q(X_{1,2}) \to \cdots$$

This sequence maps to the one for Thom spaces by taking the cup product with the Thom classes. By assumption this map is an isomorphism on $H^*(X_i)$, $i = 1, 2$ and on $H^*(X_{1,2})$. Thus by the Five Lemma it is an isomorphism on $H^*(X)$. This proves the theorem in this case.

Case 3. X is covered by finitely many open sets X_i, $i = 1, \cdots, k$ so that the restrictions of the bundle to each X_i, ζ_i is trivial.

The proof in this case is an easy inductive argument (on the number of open sets in the cover), where the inductive step is completed using cases 1 and 2.

Notice that this case includes the situation when the basespace X is compact.

Case 4. General Case. We now know the theorem for compact spaces. However it is not necessarily true that the cohomology of a general space (i.e homotopy type of a C.W complex) is determined by the cohomology of its compact subspaces. However it is true that the homology of a space X is given by

$$H_*(X) \cong \varinjlim_K H_*(K)$$

where the limit is taken over the partially ordered set of compact subspaces $K \subset X$. Thus we want to first work in homology and then try to transfer our observations to cohomology.

To do this, recall that the construction of the cup product pairing actually comes from a map on the level of cochains,

$$C^q(Y) \otimes C^r(Y, A) \xrightarrow{\cup} C^{q+r}(Y, A)$$

and therefore has a dual map on the chain level

$$C_*(Y, A) \xrightarrow{\psi} C_*(Y) \otimes C_*(Y, A).$$

and thus induces a map in homology

$$\psi : H_k(Y, A) \to \oplus_{r \geq 0} H_{k-r}(Y) \otimes H_r(Y, A).$$

Hence given $\alpha \in H^r(Y, A)$ we have an induced map in homology (the "slant product")

$$/\alpha : H_k(Y, A) \to H_{k-r}(Y)$$

defined as follows. If $\theta \in H_k(Y, A)$ and

$$\psi(\theta) = \sum_j a_j \otimes b_j \in H_*(Y) \otimes H_*(Y, A)$$

then

$$/\alpha(\theta) = \sum_j \alpha(b_j) \cdot a_j$$

where by convention, if the degree of a homology class b_j is not equal to the degree of α, then $\alpha(b_j) = 0$.

Notice that this slant product is dual to the cup product map

$$H^q(Y) \xrightarrow{\cup \alpha} H^{q+r}(Y, A).$$

Again, by considering the pair $(D(\zeta), S(\zeta))$, and identifying $H_*(D(\zeta)) \cong H_*(X)$, we can apply the slant product operation to the Thom class, to define a map

$$/u : H_k(T(\zeta)) \to H_{k-n}(X).$$

which is dual to the Thom map $\gamma : H^q(X) \xrightarrow{\cup u} H^{q+n}(T(\zeta))$. Now since γ is an isomorphism in all dimensions when restricted to compact sets, then by the universal coefficient theorem, $/u : H_q(T(\zeta_{|K})) \to H_{q-n}(K)$ is an isomorphism for all q and for every compact subset $K \subset X$. By taking the limit over the partially ordered set of compact subsets of X, we get that

$$/u : H_q(T(\zeta)) \to H_{q-n}(X)$$

is an isomorphism for all q. Applying the universal coefficient theorem again, we can now conclude that

$$\gamma : H^k(X) \xrightarrow{\cup u} H^{k+n}(T(\zeta))$$

is an isomorphism for all k. This completes the proof of the theorem. □

We now observe that the Thom class of a product of two bundles is the appropriately defined product of the Thom classes.

Lemma 5.11. *Let ζ and η be an n and m dimensional oriented vector bundles over X and Y respectively. Then the Thom class $u(\zeta\times\eta)$ is given by the tensor product: $u(\zeta \times \eta) \in H^{n+m}(T(\zeta \times \eta))$ is equal to*

$$\begin{aligned} u(\zeta) \otimes u(\eta) &\in H^n(T(\zeta)) \otimes H^m(T(\eta)) \\ &\cong H^{n+m}(T(\zeta) \wedge T(\eta)) \\ &= H^{n+m}(T(\zeta \times \eta)). \end{aligned}$$

In this description, cohomology is meant to be taken with $\mathbb{Z}_2$ - coefficients if the bundles are not orientable.

Proof. $u(\zeta) \otimes u(\eta)$ restricts on each fiber $(x, y) \in X \times Y$ to

$$\begin{aligned} u_x \otimes u_y &\in H^n(S^n(\zeta_x)) \otimes H^m(S^m(\eta_y)) \\ &\cong H^{n+m}(S^n(\zeta_x) \wedge S^m(\eta_y)) \\ &= H^{n+m}(S^{n+m}(\zeta \times \eta)_{(x,y)})) \end{aligned}$$

which is the generator determined by the product orientation of $\zeta_x \times \eta_y$. The result follows by the uniqueness of the Thom class. □

We now use the Thom isomorphism theorem to define a characteristic class for oriented vector bundles, called the *Euler class.*

Definition 5.5. *The Euler class of an oriented, n dimensional bundle ζ, over a connected space X, is the n - dimensional cohomology class*

$$\chi(\zeta) \in H^n(X)$$

defined to be the image of the Thom class $u(\zeta) \in H^n(T(\zeta))$ under the composition

$$H^n(T(\zeta)) = H^n(D(\zeta), S(\zeta)) \to H^n(D(\zeta)) \cong H^n(X).$$

Again, if ζ is not orientable, cohomology is taken with $\mathbb{Z}_2$ - coefficients.

Exercise. Verify that the Euler class is a characteristic class according to our definition.

The following is then a direct consequence of Lemma 5.11.

Corollary 5.12. *Let ζ and η be as in 5.11. Then the Euler class of the product is given by*

$$\chi(\zeta \times \eta) = \chi(\zeta) \otimes \chi(\eta) \in H^n(X) \otimes H^m(Y) \hookrightarrow H^{n+m}(X \times Y).$$

We will also need the following observation.

Proposition 5.13. *Let η be an odd dimensional oriented vector bundle over a space X. Say dim $(\eta) = 2n+1$. Then its Euler class has order two:*

$$2\chi(\eta) = 0 \in H^{2n+1}(X).$$

Proof. Consider the bundle map

$$\begin{aligned} \nu : \eta &\to \eta \\ v &\to -v. \end{aligned}$$

Since η is odd dimensional, this bundle map is an orientation reversing automorphism of η. This means that $\nu^*(u) = -u$, where $u \in H^{2n+1}(T(\eta))$ is the Thom class. By the definition of the Euler class this in turn implies that $\nu^*(\chi(\eta)) = -\chi(\eta)$. But since the Euler class is a characteristic class and ν is a bundle map, we must have $\nu^*(\chi(\eta)) = \chi(\eta)$. Thus $\chi(\eta) = -\chi(\eta)$. □

5.2.2 The Gysin sequence

We now input the Thom isomorphism theorem into the cohomology exact sequence of the pair $D(\zeta), S(\zeta))$ in order to obtain an important calculational tool for computing the homology of vector bundles and sphere bundles.

Namely, let ζ be an oriented n - dimensional oriented vector bundle over a space X, and consider the exact sequence

$$\begin{aligned} \cdots \to H^{q-1}(S(\zeta)) &\xrightarrow{\delta} H^q(D(\zeta), S(\zeta)) \to H^q(D(\zeta)) \to H^q(S(\zeta)) \\ &\xrightarrow{\delta} H^{q+1}(D(\zeta), S(\zeta)) \to H^{q+1}(D(\zeta)) \to \cdots \end{aligned}$$

By identifying $H^*(D(\zeta), S(\zeta)) = \tilde{H}^*(T(\zeta))$ and $H^*(D(\zeta)) \cong H^*(X)$, this exact sequence becomes

$$\begin{aligned} \cdots \to H^{q-1}(S(\zeta)) &\xrightarrow{\delta} H^q(T(\zeta)) \to H^q(X) \to H^q(S(\zeta)) \\ &\xrightarrow{\delta} H^{q+1}(T(\zeta)) \to H^{q+1}(X) \to \cdots \end{aligned}$$

Finally, by inputting the Thom isomorphism, $H^{q-n}(X) \xrightarrow[\cong]{\cup u} H^q(T(\zeta))$ we get the following exact sequence known as the **Gysin sequence:**

$$\begin{aligned}\cdots \to H^{q-1}(S(\zeta)) &\xrightarrow{\delta} H^{q-n}(X) \xrightarrow{\chi} H^q(X) \to H^q(S(\zeta)) \\ &\xrightarrow{\delta} H^{q-n+1}(X) \xrightarrow{\chi} H^{q+1}(X) \to \cdots\end{aligned} \tag{5.6}$$

We now make the following observation about the homomorphism $\chi : H^q(X) \to H^{q+n}(X)$ in the Gysin sequence.

Proposition 5.14. *The homomorphism $\chi : H^q(X) \to H^{q+n}(X)$ is given by taking the cup product with the Euler class,*

$$\chi : H^q(X) \xrightarrow{\cup \chi} H^{q+n}(X).$$

Proof. The theorem is true for $q = 0$, by definition. Now in general, the map χ was defined in terms of the Thom isomorphism $\gamma : H^r(X) \xrightarrow{\cup u} H^{r+n}(T(\zeta))$, which, by definition is a homomorphism of graded $H^*(X)$ - modules. This will then imply that

$$\chi : H^q(X) \to H^{q+n}(X)$$

is a homomorphism of graded $H^*(X)$ - modules. Thus

$$\begin{aligned}\chi(\alpha) &= \chi(1 \cdot \alpha) \\ &= \chi(1) \cup \alpha \quad \text{since } \chi \text{ is an } H^*(X) \text{ - module homomorphism} \\ &= \chi(\zeta) \cup \alpha\end{aligned}$$

as claimed. □

5.2.3 Proof of theorem 5.5

the goal of this section is to use the Gysin sequence to prove 5.5, which we begin by restating:

Theorem 5.15. *a. The ring of $U(n)$ characteristic classes is a polynomial algebra on n - generators,*

$$Char_{U(n)}(\mathbb{Z}) \cong H^*(BU(n); \mathbb{Z}) \cong \mathbb{Z}[c_1, c_2, \cdots, c_n]$$

where $c_i \in H^{2i}(BU(n); \mathbb{Z})$ is known as the ith - Chern class.

b. The ring of $\mathbb{Z}_2$ - valued $O(n)$ characteristic classes is a polynomial algebra on n - generators,

$$Char_{O(n)}(\mathbb{Z}_2) \cong H^*(BO(n); \mathbb{Z}_2) \cong \mathbb{Z}_2[w_1, w_2, \cdots, w_n]$$

where $w_i \in H^i(BO(n); \mathbb{Z}_2)$ is known as the ith - Stiefel - Whitney class.

Proof. We start by considering the Gysin sequence, applied to the universal bundle γ_n over $BU(n)$. We input the fact that the sphere bundle $S(\gamma_n)$ is given by $BU(n-1)$ see 5.2:

$$\cdots \to H^{q-1}(BU(n-1)) \xrightarrow{\delta} H^{q-2n}(BU(n)) \xrightarrow{\cup\chi(\gamma_n)} H^q(BU(n)) \xrightarrow{\iota^*} H^q(BU(n-1)) \xrightarrow{\delta} H^{q-2n+1}(BU(n)) \xrightarrow{\cup\chi(\gamma_n)} H^{q+1}(BU(n)) \to \cdots \quad (5.$$

and we get a similar exact sequence in the real case

$$\cdots \to H^{q-1}(BO(n-1);\mathbb{Z}_2) \xrightarrow{\delta} H^{q-n}(BO(n);\mathbb{Z}_2) \xrightarrow{\cup\chi(\gamma_n)} H^q(BO(n);\mathbb{Z}_2) \xrightarrow{\iota^*} H^q(BO(n-1);\mathbb{Z}_2) \xrightarrow{\delta} H^{q-n+1}(BO(n));\mathbb{Z}_2) \xrightarrow{\cup\chi(\gamma_n)} H^{q+1}(BO(n);\mathbb{Z}_2) \to \cdots \quad (5.8)$$

We use these exact sequences to prove the above theorem by induction on n. For $n=1$ then sequence 5.7 reduces to the short exact sequences,

$$0 \to H^{q-2}(BU(1)) \xrightarrow[\cong]{\cup\chi(\gamma_1)} H^q(BU(1)) \to 0$$

for each $q \geq 2$. We let $c_1 \in H^2(BU(1)) = H^2(\mathbb{CP}^\infty)$ be the Euler class $\chi(\gamma_1)$. These isomorphisms imply that the ring structure of $H^*(BU(1))$ is that of a polynomial algebra on this single generator,

$$H^*(BU(1)) = H^*(\mathbb{CP}^\infty) = \mathbb{Z}[c_1]$$

which is the statement of the theorem in this case.

In the real case when $n=1$ the Gysin sequence 5.8 reduces to the short exact sequences,

$$0 \to H^{q-1}(BO(1);\mathbb{Z}_2) \xrightarrow[\cong]{\cup\chi(\gamma_1)} H^q(BO(1);\mathbb{Z}_2) \to 0$$

for each $q \geq 1$. We let $w_1 \in H^(BO(1);\mathbb{Z}_2) = H^1(\mathbb{RP}^\infty;\mathbb{Z}_2)$ be the Euler class $\chi(\gamma_1)$. These isomorphisms imply that the ring structure of $H^*(BO(1);\mathbb{Z}_2)$ is that of a polynomial algebra on this single generator,

$$H^*(BO(1);\mathbb{Z}_2) = H^*(\mathbb{RP}^\infty;\mathbb{Z}_2) = \mathbb{Z}_2[w_1]$$

which is the statement of the theorem in this case.

We now inductively assume the theorem is true for $n-1$. That is,

$$H^*(BU(n-1)) \cong \mathbb{Z}[c_1,\cdots,c_{n-1}] \quad \text{and} \quad H^*(BO(n-1);\mathbb{Z}_2) \cong \mathbb{Z}_2[w_1,\cdots,w_{n-1}].$$

We first consider the Gysin sequence 5.7, and observe that by exactness, for $q \leq 2(n-1)$, the homomorphism

$$\iota^* : H^q(BU(n)) \to H^q(BU(n-1))$$

is an isomorphism. That means there are unique classes, $c_1, \cdots, c_{n-1} \in H^*(BU(n))$ that map via ι^* to the classes of the same name in $H^*(BU(n-1))$. Furthermore, since ι^* is a ring homomorphism, every polynomial in $c_1, \cdots, c_{n-1}$ in $H^*(BU(n-1))$ is in the image under ι^* of the corresponding polynomial in the these classes in $H^*(BU(n))$. Hence by our inductive assumption,

$$\iota^* : H^*(BU(n)) \to H^*(BU(n-1)) = \mathbb{Z}[c_1, \cdots, c_{n-1}]$$

is a split surjection of rings. But by the exactness of the Gysin sequence 5.7 this implies that this long exact splits into short exact sequences,

$$0 \to H^{*-2n}(BU(n)) \xrightarrow{\cup\chi(\gamma_n)} H^*(BU(n)) \xrightarrow{\iota^*} H^*(BU(n-1)) \cong \mathbb{Z}[c_1, \cdots c_{n-1}] \to 0$$

Define $c_n \in H^{2n}(BU(n))$ to be the Euler class $\chi(\gamma_n)$. Then this sequence becomes

$$0 \to H^{*-2n}(BU(n)) \xrightarrow{\cup c_n} H^*(BU(n)) \xrightarrow{\iota^*} \mathbb{Z}[c_1, \cdots c_{n-1}] \to 0$$

which implies that $H^*(BU(n)) \cong \mathbb{Z}[c_1, \cdots, c_n]$. This completes the inductive step in this case.

In the real case now consider the Gysin sequence 5.8, and observe that by exactness, for $q < n-1$, the homomorphism

$$\iota^* : H^q(BO(n); \mathbb{Z}_2) \to H^q(BO(n-1); \mathbb{Z}_2)$$

is an isomorphism. That means there are unique classes, $w_1, \cdots, w_{n-2} \in H^*(BO(n); \mathbb{Z}_2)$ that map via ι^* to the classes of the same name in $H^*(BO(n-1); \mathbb{Z}_2)$.

In dimension $q = n-1$, the exactness of the Gysin sequence tells us that the homomorphism $\iota^* H^{n-1}(BO(n); \mathbb{Z}_2) \to H^{n-1}(BO(n-1); \mathbb{Z}_2)$ is injective. Also by exactness we see that ι^* is surjective if and only if $\chi(\gamma_n) \in H^n(BO(n); \mathbb{Z}_2)$ is nonzero. But to see this, by the universal property of γ_n, it suffices to prove that there exists some n -dimensional bundle ζ with Euler class $\chi(\zeta) \neq 0$. Now by 5.12, the Euler class of the product

$$\begin{aligned}\chi(\gamma_k \times \gamma_{n-k}) &= \chi(\gamma_k) \otimes \chi(\gamma_{n-k}) \in H^k(BO(k) \times BO(n-k); \mathbb{Z}_2) \\ &= w_k \otimes w_{n-k} \in H^*(BO(k); \mathbb{Z}_2) \otimes H^{n-k}(BO(n-k); \mathbb{Z}_2)\end{aligned}$$

which, by the inductive assumption is nonzero for $k \geq 1$. Thus $\chi(\gamma_n) \in H^n(BO(n); \mathbb{Z}_2)$ is nonzero, and we define it to be the n^{th} Stiefel - Whitney class

$$w_n = \chi(\gamma_n) \in H^n(BO(n); \mathbb{Z}_2).$$

As observed above, the nontriviality of $\chi(\gamma_n)$ implies that $\iota^* H^{n-1}(BO(n);\mathbb{Z}_2) \to H^{n-1}(BO(n-1);\mathbb{Z}_2)$ is an isomorphism, and hence there is a unique class $w_{n-1} \in H^{n-1}(BO(n-1);\mathbb{Z}_2)$ (as well as $w_1, \cdots w_{n-2}$) restricting to the inductively defined classes of the same names in $H^*(BO(n-1);\mathbb{Z}_2)$.

Furthermore, since ι^* is a ring homomorphism, every polynomial in $w_1, \cdots, w_{n-1}$ in $H^*(BO(n-1);\mathbb{Z}_2)$ is in the image under ι^* of the corresponding polynomial in the these classes in $H^*(BO(n);\mathbb{Z}_2)$. Hence by our inductive assumption,

$$\iota^* : H^*(BO(n);\mathbb{Z}_2) \to H^*(BO(n-1);\mathbb{Z}_2) = \mathbb{Z}_2[w_1, \cdots, w_{n-1}]$$

is a split surjection of rings. But by the exactness of the Gysin sequence 5.8 this implies that this long exact splits into short exact sequences,

$$0 \to H^{*-n}(BO(n);\mathbb{Z}_2) \xrightarrow{\cup w_n} H^*(BO(n);\mathbb{Z}_2) \xrightarrow{\iota^*} H^*(BO(n-1);\mathbb{Z}_2) \cong \mathbb{Z}_2[w_1, \cdots w_{n-1}] \to 0$$

which implies that $H^*(BO(n);\mathbb{Z}_2) \cong \mathbb{Z}_2[w_1, \cdots, w_n]$. This completes the inductive step and therefore the proof of the theorem. □

5.3 The product formula and the splitting principle

Perhaps the most important calculational tool for characteristic classes is the Whitney sum formula, which we now state and prove.

Theorem 5.16. *a. Let ζ and η be vector bundles over a space X. Then the Stiefel - Whitney classes of the Whitney sum bundle $\zeta \oplus \eta$ are given by*

$$w_k(\zeta \oplus \eta) = \sum_{j=0}^{k} w_j(\zeta) \cup w_{k-j}(\eta) \in H^k(X;\mathbb{Z}_2).$$

where by convention, $w_0 = 1 \in H^0(X;\mathbb{Z}_2)$.

b. If ζ and η are complex vector bundles, then the Chern classes of the Whitney sum bundle $\zeta \oplus \eta$ are given by

$$c_k(\zeta \oplus \eta) = \sum_{j=0}^{k} c_j(\zeta) \cup c_{k-j}(\eta) \in H^{2k}(X).$$

Again, by convention, $c_0 = 1 \in H^0(X)$.

Proof. We prove the formula in the real case. The complex case is done the same way.

Let ζ be an n - dimensional vector bundle over X, and let η be an m - dimensional bundle. Let $N = n + m$. Since we are computing $w_k(\zeta \oplus \eta)$, we may assume that $k \leq N$, otherwise this characteristic class is zero.

We prove the Whitney sum formula by induction on $N \geq k$. We begin with the case $N = k$. Since $\zeta \oplus \eta$ is a k - dimensional bundle, the k^{th} Stiefel - Whitney class, $w_k(\zeta \oplus \eta)$ is equal to the Euler class $\chi(\zeta \oplus \eta)$. We then have

$$\begin{aligned} w_k(\zeta \oplus \eta) &= \chi(\zeta \oplus \eta) \\ &= \chi(\zeta) \cup \chi(\eta) \quad \text{by } \ 5.12 \\ &= w_n(\zeta) \cup w_m(\eta). \end{aligned}$$

This is the Whitney sum formula in this case as one sees by inputting the fact that for a bundle ρ with $j > dim\,(\rho)$, $w_j(\rho) = 0$.

Now inductively assume that the Whitney sum formula holds for computing w_k for any sum of bundles whose sum of dimensions is $\leq N - 1 \geq k$. Let ζ have dimension n and η have dimension m with $n + m = N$. To complete the inductive step we need to compute $w_k(\zeta \oplus \eta)$.

Suppose ζ is classified by a map $f_\zeta : X \to BO(n)$, and η is classified by a map $f_\eta : X \to BO(m)$. Then $\zeta \oplus \eta$ is classified by the composition

$$f_{\zeta \oplus \eta} : X \xrightarrow{f_\zeta \times f_\eta} BO(n) \times BO(m) \xrightarrow{\ \mu\ } BO(n+m)$$

where μ is the map that classifies the product of the universal bundles $\gamma_n \times \gamma_m$ over $BO(n) \times BO(m)$. Equivalently, μ is the map on classifying spaces induced by the inclusion homomorphism of the subgroup $O(n) \times O(m) \hookrightarrow O(n+m)$. Thus to prove the theorem we must show that the map $\mu : BO(n) \times BO(m) \to BO(n+m)$ has the property that

$$\mu^*(w_k) = \sum_{j=0}^{k} w_j \otimes w_{k-j} \in H^*(BO(n); \mathbb{Z}_2) \otimes H^*(BO(m); \mathbb{Z}_2). \tag{5.9}$$

For a fixed $j \leq k$, let

$$p_j : H^k(BO(n) \times BO(m); \mathbb{Z}_2) \to H^j(BO(n); \mathbb{Z}_2) \otimes H^{k-j}(BO(m); \mathbb{Z}_2)$$

be the projection onto the summand. So we need to show that $p_j(\mu^*(w_k)) = w_j \otimes w_{k-j}$. Now since $n + m = N > k$, then either $j < n$ or $k - j < m$ (or both). We assume without loss of generality that $j < n$. Now by the proof of 5.5

$$\iota^* : H^j(BO(n); \mathbb{Z}_2) \to H^j(BO(j); \mathbb{Z}_2)$$

is an isomorophism. Moreover we have a commutative diagram:

$$\begin{array}{ccccc}
H^k(BO(N);\mathbb{Z}_2) & \xrightarrow{\mu^*} & H^k(BO(n)\times BO(m);\mathbb{Z}_2) & \xrightarrow{p_j} & H^j(BO(n);\mathbb{Z}_2)\otimes H^{k-j}(BO(m);\mathbb{Z}_2) \\
\downarrow{\iota^*} & & & & \downarrow{\iota^*\otimes 1} \\
H^k(BO(j+m);\mathbb{Z}_2) & \xrightarrow[\mu^*]{} & H^k(BO(j)\times BO(m);\mathbb{Z}_2) & \xrightarrow[p_j]{} & H^j(BO(j);\mathbb{Z}_2)\otimes H^{k-j}(BO(m);\mathbb{Z}_2).
\end{array}$$

Since $j < n$, $j + m < n + m = N$ and $\iota^*(w_k) = w_k \in H^k(BO(j+m);\mathbb{Z}_2)$. This fact and the commutativity of this diagram give,

$$\begin{aligned}
(\iota^*\otimes 1)\circ p_j\circ\mu^*(w_k) &= p_j\circ\mu^*\circ\iota^*(w_k) \\
&= p_j\circ\mu^*(w_k) \\
&= w_j\otimes w_{k-j} \quad \text{by the inductive assumption.}
\end{aligned}$$

Since $\iota^*\otimes 1$ is an isomorphism in this dimension, and since $\iota^*(w_j\otimes w_{k-j}) = w_j\otimes w_{k-j}$ we have that

$$p_j\circ\mu^*(w_k) = w_j\otimes w_{k-j}.$$

As remarked above, this suffices to complete the inductive step in the proof of the theorem. □

We can restate the Whitney sum formula in the following convenient way. For an n - dimensional bundle ζ, let

$$w(\zeta) = 1 + w_1(\zeta) + w_2(\zeta) + \cdots + w_n(\zeta) \in H^*(X;\mathbb{Z}_2)$$

This is called the *total Stiefel - Whitney class.* The total Chern class of a complex bundle is defined similarly.

The Whitney sum formula can be interpreted as saying these total characteristic classes have the "exponential property" that they take sums to products. That is, we have the following:

Corollary 5.17.

$$w(\zeta\oplus\eta) = w(\zeta)\cup w(\eta)$$

and

$$c(\zeta\oplus\eta) = c(\zeta)\cup c(\eta).$$

This implies that these characteristic classes are invariants of the stable isomorphism types of bundles:

Corollary 5.18. *If ζ and η are stably equivalent real vector bundles over a space X, then*

$$w(\zeta) = w(\eta) \in H^*(X;\mathbb{Z}_2),$$

Similarly if they are complex bundles,

$$c(\zeta) = c(\eta) \in H^*(X).$$

Proof. If ζ and η are stably equivalent, then

$$\zeta \oplus \epsilon^m \cong \eta \oplus \epsilon^r$$

for some m and r. So

$$w(\zeta \oplus \epsilon^m) = w(\eta \oplus \epsilon^r).$$

But by 5.17

$$w(\zeta \oplus \epsilon^m) = w(\zeta)w(\epsilon) = w(\zeta) \cdot 1 = w(\zeta).$$

Similarly $w(\eta \oplus \epsilon^r) = w(\eta)$. The statement follows. The complex case is proved in the same way. □

By our description of K - theory in chapter 3, we have that these characteristic classes define invariants of K - theory.

Theorem 5.19. *The Chern classes c_i and the Stiefel - Whitney classes w_i define natural transformations*

$$c_i : K(X) \to H^{2i}(X)$$

and

$$w_i : KO(X) \to H^i(X; \mathbb{Z}_2).$$

The total characteristic classes

$$c : K(X) \to \bar{H}^*(X)$$

and

$$w : KO(X) \to \bar{H}^*(X; \mathbb{Z}_2)$$

are exponential in the sense that

$$c(\alpha + \beta) = c(\alpha)c(\beta) \quad and \quad w(\alpha + \beta) = w(\alpha)w(\beta).$$

Here $\bar{H}^*(X)$ is the direct product $\bar{H}^*(X) = \prod_q H^q(X)$.

As an immediate application of these product formulas, we can deduce a "splitting principle" for characteristic classes. We now explain this principle.

Recall that an n - dimensional bundle ζ over X splits as a sum of n line bundles if and only if its associated principal bundle has an $O(1) \times \cdots \times O(1)$ - structure. That is, the classifying map $f_\zeta : X \to BO(n)$ lifts to the n -fold product, $BO(1)^n$. The analogous observation also holds for complex vector bundles. If we have such a lifting, then in cohomology, $f_\zeta^* : H^*(BO(n); \mathbb{Z}_2) \to H^*(X; \mathbb{Z}_2)$ factors through $\otimes_n H^*(BO(1); \mathbb{Z}_2)$.

The "splitting principle" for characteristic classes says that this cohomological property always happens.

To state this more carefully, recall that $H^*(BO(1);\mathbb{Z}_2) = \mathbb{Z}_2[w_1]$. Thus

$$H^*(BO(1)^n;\mathbb{Z}_2) \cong \mathbb{Z}_2[x_1, \cdots, x_n]$$

where $x_j \in H^1$ is the generator of the cohomology of the j^{th} factor in this product. Similarly,

$$H^*(BU(1)^n) \cong \mathbb{Z}[y_1, \cdots, y_n]$$

where $y_j \in H^2$ is the generator of the cohomology of the j^{th} factor in this product.

Notice that the symmetric group Σ_n acts on these polynomial algebras by permuting the generators. The subalgebra consisting of polynomials fixed under this symmetric group action is called the algebra of symmetric polynomials, $Sym[x_1, \cdots, x_n]$ or $Sym[y_1, \cdots, y_n]$.

Theorem 5.20. *(***Splitting Principle.)** The maps

$$\mu : BU(1)^n \to BU(n) \quad \text{and} \quad \mu : BO(1)^n \to BO(n)$$

induce injections in cohomology

$$\mu^* : H^*(BU(n)) \to H^*(BU(1)^n) \quad \text{and} \quad \mu^* : H^*(BO(n);\mathbb{Z}_2) \to H^*(BO(1)^n;\mathbb{Z}_2).$$

Furthermore the images of these monomorphisms are the symmetric polynomials

$$H^*(BU(n)) \cong Sym[y_1, \cdots, y_n] \quad \text{and} \quad H^*(BO(n);\mathbb{Z}_2) \cong Sym[x_1, \cdots, x_n].$$

Proof. By the Whitney sum formula,

$$\mu^*(w_j) = \sum_{j_1+\cdots+j_n=j} w_{j_1} \otimes \cdots \otimes w_{j_n} \in H^*(BO(1);\mathbb{Z}_2) \otimes \cdots \otimes H^*(BO(1);\mathbb{Z}_2).$$

But $w_i(\gamma_1) = 0$ unless $i = 0, 1$. So

$$\mu^*(w_j) = \sum_{1 \le i_1 < \cdots < i_j \le n} x_{i_1} \cdots x_{i_j} \in \mathbb{Z}_2[x_1, \cdots, x_n].$$

This is the j^{th} - elementary symmetric polynomial, $\sigma_j(x_1, \cdots, x_n)$. Thus the image of $\mathbb{Z}_2[w_1, \cdots, w_n] = H^*(BO(n);\mathbb{Z}_2)$ is the subalgebra of $\mathbb{Z}_2[x_1, \cdots, x_n]$ generated by the elementary symmetric polynomials, $\mathbb{Z}_2[\sigma_1, \cdots, \sigma_n]$. But it is well known that the elementary symmetric polynomials generate $Sym[x_1, \cdots, x_n]$ (see [54]). The complex case is proved similarly. □

This result gives another way of producing characteristic classes which is particularly useful in index theory.

Let $p(x)$ be a power series in one variable, which is assumed to have a grading equal to one. Say

$$p(x) = \sum_i a_i x^i.$$

Consider the corresponding symmetric power series in n -variables,

$$p(x_1, \cdots, x_n) = p(x_1) \cdots p(x_n).$$

Let $p_j(x_1, \cdots, x_n)$ be the homogeneous component of $p(x_1, \cdots, x_n)$ of grading j. So

$$p_j(x_1, \cdots, x_n) = \sum_{i_1 + \cdots + i_n = j} a_{i_1} \cdots a_{i_n} x_1^{i_1} \cdots x_n^{i_n}.$$

Since p_j is symmetric, by the splitting principle we can think of

$$p_j \in H^j(BO(n); \mathbb{Z}_2)$$

and hence determines a characteristic class (i.e a polynomial in the Stiefel - Whitney classes).

Similarly if we give x grading 2, we can think of $p_j \in H^{2j}(BU(n))$ and so determines a polynomial in the Chern classes.

In particular, given a real valued smooth function $y = f(x)$, its Taylor series $p_f(x) = \sum_k \frac{f^{(k)}(0)}{k!} x^k$ determines characteristic classes $f_i \in H^i(BO(n); \mathbb{Z}_2)$ or $f_i \in H^{2i}(BU(n); \mathbb{Z}_2)$.

Exercise. Consider the examples $f(x) = e^x$, and $f(x) = tanh(x)$. Write the low dimensional characteristic classes f_i in $H^*(BU(n))$ for $i = 1, 2, 3$, as explicit polynomials in the Chern classes.

5.4 Applications

In this section all cohomology will be taken with $\mathbb{Z}_2$ - coefficients, even if not explicitly written.

5.4.1 Characteristic classes of manifolds

We have seen that the characteristic classes of trivial bundles are trivial. However the converse is not true, as we will now see, by examining the characteristic classes of manifolds.

Definition 5.6. *The characteristic classes of a manifold M, $w_j(M)$, $c_i(M)$, are defined to be the characteristic classes of the tangent bundle, TM.*

Theorem 5.21. *$w_j(S^n) = 0$ for all $j, n > 0$.*

Proof. As we saw in chapter 1, the normal bundle of the standard embedding $S^n \hookrightarrow \mathbb{R}^{n+1}$ is a trivial line bundle. Thus

$$TS^n \oplus \epsilon_1 \cong \epsilon_{n+1}$$

and so TS^{n+1} is stably trivial. The theorem follows. □

Of course we know TS^2 is nontrivial since it has no nowhere zero cross sections. Thus the Stiefel- Whitney classes do not form a complete invariant of the bundle. However they do constitute a very important class of invariants, as we will see below.

Write $a \in H^1(\mathbb{RP}^n; \mathbb{Z}_2) \cong \mathbb{Z}_2$ as the generator. Then the total Stiefel - Whitney class of the canonical line bundle γ_1 is

$$w(\gamma_1) = 1 + a \in H^*(\mathbb{RP}^n).$$

This allows us to compute the Stiefel - Whitney classes of $\mathbb{RP}^n$ (i.e of the tangent bundle $T(\mathbb{RP}^n)$).

Theorem 5.22. *$w(\mathbb{RP}^n) = (1 + a)^{n+1} \in H^*(\mathbb{RP}^n; \mathbb{Z}_2)$. So $w_j(\mathbb{RP}^n) = \binom{n+1}{j} a^j \in H^j(\mathbb{RP}^n)$.*

Note: Even though the polynomial $(1 + a)^{n+1}$ has highest degree term a^{n+1}, this class is zero in $H^*(\mathbb{RP}^n)$ since $H^{n+1}(\mathbb{RP}^n) = 0$.

Proof. As seen in Chapter 3,

$$T\mathbb{RP}^n \oplus \epsilon_1 \cong \oplus_{n+1}\gamma_1.$$

Thus

$$\begin{aligned} w(T\mathbb{RP}^n) &= w(T(\mathbb{RP}^n) \oplus \epsilon_1) \\ &= w(\oplus_{n+1}\gamma_1) \\ &= w(\gamma_1)^{n+1}, \quad \text{by the Whitney sum formula} \\ &= (1 + a)^{n+1}. \end{aligned}$$

□

Observation. The same argument shows that the total Chern class of $\mathbb{CP}^n$ is

$$c(\mathbb{CP}^n) = (1 + b)^{n+1} \tag{5.10}$$

where $b \in H^2(\mathbb{CP}^n \mathbb{Z})$ is the generator.

This calculation of the Stiefel - Whitney classes of $\mathbb{RP}^n$ allows us to rule out the possibility that many of these projective spaces are parallelizable.

Corollary 5.23. *If $\mathbb{RP}^n$ is parallelizable, then n is of the form $n = 2^k - 1$ for some k.*

Proof. We show that if $n \neq 2^k - 1$ then there is some $j > 0$ such that $w_j(\mathbb{RP}^n) \neq 0$. But $w_j(\mathbb{RP}^n) = \binom{n+1}{j} a^j$, so we are reduced to verifying that if m is not a power of 2, then there is a $j \in \{1, \cdots, m-1\}$ such that $\binom{m}{j} \equiv 1 \,\mathrm{mod} 2$. This follows immediately from the following combinatorial lemma, whose proof we leave to the reader.

Lemma 5.24. *Let $j \in \{1, \cdots, m-1\}$. Write j and m in their binary representations,*

$$m = \sum_{i=0}^{k} a_i 2^i$$

$$j = \sum_{i=0}^{k} b_i 2^i$$

where the a_i's and b_i's are either 0 or 1. Then

$$\binom{m}{j} \equiv \prod_{i=0}^{k} \binom{a_i}{b_i} \, mod\, 2.$$

Note. Here we are adopting the usual conventions that $\binom{0}{0} = 1$, $\binom{1}{0} = 1$, and $\binom{0}{1} = 0$.

□

Since we know that Lie groups are parallelizable, this result says that $\mathbb{RP}^n$ can only have a Lie group structure if n is of the form $2^k - 1$. However a famous theorem of Adams [2] says that the only $\mathbb{RP}^n$'s that are parallelizable are $\mathbb{RP}^1$, $\mathbb{RP}^3$, and $\mathbb{RP}^7$.

Now as seen in chapter 2 an n - dimensional vector bundle ζ^n has k - linearly independent cross sections if and only if

$$\zeta^n \cong \rho^{n-k} \oplus \epsilon_k$$

for some $n-k$ dimensional bundle ρ. Moreover, having this structure is equivalent to the classifying map

$$f_\zeta : X \to BO(n)$$

having a lift (up to homotopy) to a map $f_\rho : X \to BO(n-k)$.

Now the Stiefel - Whitney classes give natural obstructions to the existence of such a lift because the map $\iota : BO(n-k) \to BO(n)$ induces the map of rings

$$\iota^* : \mathbb{Z}_2[w_1, \cdots, w_n] \to \mathbb{Z}_2[w_1, \cdots w_{n-k}]$$

that maps w_j to w_j for $j \leq n-k$, and w_j to 0 for $n \geq j > n-k$. We therefore have the following result.

Theorem 5.25. *Let ζ be an n -dimensional bundle over X. Suppose $w_k(\zeta)$ is nonzero in $H^k(X; \mathbb{Z}_2)$. Then ζ has no more than $n-k$ linearly independent cross sections. In particular, if $w_n(\zeta) \neq 0$, then ζ does not have a nowhere zero cross section.*

This result has applications to the existence of linearly independent vector fields on a manifold. The following is an example.

Theorem 5.26. *If m is even, $\mathbb{RP}^m$ does not have a nowhere zero vector field.*

Proof. By 5.22

$$\begin{aligned} w_m(\mathbb{RP}^m) &= \binom{m+1}{m} a^m \\ &= (m+1)a^m \in H^m(\mathbb{RP}^m; \mathbb{Z}_2). \end{aligned}$$

For m even this is nonzero. Hence $w_m(\mathbb{RP}^m) \neq 0$. □

5.4.2 Normal bundles and immersions

Theorem 5.25 has important applications to the existence of immersions of a manifold M in Euclidean space, which we now discuss.

Let $e : M^n \looparrowright \mathbb{R}^{n+k}$ be an immersion. Recall that this means that the derivative at each point,

$$De(x) : T_x M^n \to T_{e(x)} \mathbb{R}^{n+k} = \mathbb{R}^{n+k}$$

is injective. Recall also that the Inverse Function Theorem implies that an immersion is a local embedding.

The immersion e defines a k - dimensional normal bundle ν_e^k whose fiber at $x \in M$ is the orthogonal complement of the image of $T_x M^n$ in $\mathbb{R}^{n+k}$ under $De(x)$. In particular we have

$$TM^n \oplus \nu_e^k \cong e^*(T\mathbb{R}^{n_k}) \cong \epsilon_{n+k}.$$

Thus we have the Whitney sum relation among the Stiefel - Whitney classes

$$w(M^n) \cdot w(\nu_e^k) = 1. \tag{5.11}$$

So we can compute the Stiefel - Whitney clases of the normal bundle formally as the power series

$$w(\nu_e^k) = 1/w(M) \in \bar{H}^*(M; \mathbb{Z}_2).$$

This proves the following:

Proposition 5.27. *The Stiefel - Whitney classes of the normal bundle to an immersion $e : M^n \looparrowright \mathbb{R}^{n+k}$ are independent of the immersion. They are called the normal Stiefel - Whitney classes, and are written $\bar{w}_i(M)$. These classes are determined by the formula*

$$w(M) \cdot \bar{w}(M) = 1.$$

Example. $\bar{w}(\mathbb{RP}^n) = 1/(1+a)^{n+1} \in \bar{H}^*(\mathbb{RP}^n; \mathbb{Z}_2)$.

So for example, when $n = 2^k$, $k > 0$, $w(\mathbb{RP}^{2^k}) = 1 + a + a^{2^k}$. This is true since by 5.24 $\binom{2^k+1}{r} \equiv 1 \bmod 2$ if and only if $r = 0, 1, 2^k$. Thus the total normal Stiefel - Whitney class is given by

$$\bar{w}(\mathbb{RP}^{2^k}) = 1/(1 + a + a^{2^k}) = 1 + a + a^2 + \cdots + a^{2^k - 1}.$$

Note. The reason this series is truncated a a^{2^k-1} is because

$$(1 + a + a^{2^k})(1 + a + a^2 + \cdots + a^{2^k-1}) = 1 \in H^*(\mathbb{RP}^n; \mathbb{Z}_2)$$

since $H^q(\mathbb{RP}^n) = 0$ for $q > n$.

Corollary 5.28. *There is no immersion of $\mathbb{RP}^{2^k}$ in $\mathbb{R}^N$ for $N \leq 2^{k+1} - 2$.*

Proof. The above calculation shows that $\bar{w}_{2^k-1}(\mathbb{RP}^{2^k}) \neq 0$. Thus it cannot have a normal bundle of dimension less than $2^k - 1$. The result follows. □

5.5 Pontrjagin Classes

In this section we define and study Pontrjagin classes. These are integral characteristic classes for real vector bundles and are defined in terms of the Chern classes of the complexification of the bundle. We will then show that polynomials in Pontrjagin classes and the Euler class define all possible characteristic classes for oriented, real vector bundles when the values of the characteristic classes is cohomology with coefficients in an integral domain R which contains $1/2$. By the classification theorem, to deduce this we must compute $H^*(BSO(n); R)$. For this calculation we follow the treatment given in Milnor and Stasheff [74].

5.5.1 Orientations and Complex Conjugates

We begin with a reexamination of certain basic properties of complex vector bundles.

Let V be an n - dimensional $\mathbb{C}$ - vector space with basis $\{v_1, \cdots, v_n\}$. By multiplication of these basis vectors by the complex number i, we get a collection of $2n$ - vectors $\{v_1, iv_1, v_2, iv_2, \cdots, v_n, iv_n\}$ which forms a basis for V as a real $2n$ - dimensional vector space. This basis then determines an orientation of the underlying real vector space V.

Exercise. Show that the orientation of V that the basis $\{v_1, iv_1, v_2, iv_2, \cdots, v_n, iv_n\}$ determines is independent of the choice of the original basis $\{v_1, \cdots, v_n\}$

Thus every complex vector space V has a canonical orientation. By choosing this orientation for every fiber of a complex vector bundle ζ, we see that every complex vector bundle has a canonical orientation. By the results of section 2 this means that every n - dimensional complex vector bundle ζ over a space X has a canonical choice of Thom class $u \in H^{2n}(T(\zeta))$ and hence Euler class

$$\chi(\zeta) = c_n(\zeta) \in H^{2n}(X).$$

Now given a complex bundle ζ there exists a *conjugate bundle* $\bar{\zeta}$ which is equal to ζ as a real, $2n$ - dimensional bundle, but whose complex structure is conjugate. More specifically, recall that a complex structure on a $2n$ - dimensional real bundle ζ determines and is determined by a linear transformation

$$J_\zeta : \zeta \to \zeta$$

with the property that $J_\zeta^2 = J_\zeta \circ J_\zeta = -id$. If ζ has a complex structure then

J_ζ is just scalar multiplication by the complex number i on each fiber. If we replace J_ζ by $-J_\zeta$ we define a new complex structure on ζ referred to as the *conjugate* complex structure. We write $\bar{\zeta}$ to denote ζ with this structure. That is,

$$J_{\bar{\zeta}} = -J_\zeta.$$

Notice that the identity map

$$id : \zeta \to \bar{\zeta}$$

is anti-complex linear (or conjugate complex linear) in the sense that

$$id(J_\zeta \cdot v) = -J_{\bar{\zeta}} \cdot id(v).$$

We note that the conjugate bundle $\bar{\zeta}$ is often not isomorphic to ζ as complex vector bundles. For example, consider the two dimensional sphere as complex projective space

$$S^2 = \mathbb{CP}^1 = \mathbb{C} \cup \infty.$$

The tangent bundle $T(\mathbb{CP}^1)$ has the induced structure as a complex line bundle.

Proposition 5.29. *The complex line bundles $T(S^2)$ and $\bar{T}(S^2)$ are not isomorphic.*

Proof. Suppose $\phi : T(S^2) \to \bar{T}(S^2)$ is a isomorphism as complex vector bundles. Then at every tangent space

$$\phi_x : T_xS^2 \to T_xS^2$$

is a an isomorphism that conjugates the complex structure. Any such isomorphism is given by reflection through a line ℓ_x in the tangent plane T_xS^2. Therefore for every x we have picked a line $\ell_x \subset T_xS^2$. This defines a (real) one dimensional subbundle ℓ of $T(S^2)$, which, by the classification theorem is given by an element of

$$[S^2, BO(1)] \cong H^1(S^2, \mathbb{Z}_2) = 0.$$

Thus ℓ is a trivial subbundle of $T(S^2)$. Hence we can find a nowhere vanishing vector field on S^2, which gives us a contradiction. □

Exercise. Let $\bar{\gamma}_n$ be the conjugate of the universal bundle γ_n over $BU(n)$. By the classification theorem, $\bar{\gamma}_n$ is classified by a map

$$q : BU(n) \to BU(n)$$

having the property that $q^*(\gamma_n) = \bar{\gamma}_n$. Using the Grassmannian model of $BU(n)$, find an explicit description of a map $q : BU(n) \to BU(n)$ with this property.

The following describes the effect of conjugating a vector bundle on its Chern classes.

Theorem 5.30. $c_k(\bar{\zeta}) = (-1)^k c_k(\zeta)$

Proof. Suppose ζ is an n - dimensional bundle. By the classification theorem and the functorial property of Chern classes it suffices to prove this theorem when ζ is the universal bundle γ_n over $BU(n)$. Now in our calculations of the cohomology of these classifying spaces, we proved that the inclusion $\iota : BU(k) \to BU(n)$ induces an isomorphism in cohomology in dimension k,

$$\iota^* : H^{2k}(BU(n)) \xrightarrow{\cong} H^{2k}(BU(k)).$$

Hence it suffices to prove this theorem for the universal k - dimensional bundle γ_k over $BU(k)$.

Now $c_k(\gamma_k) = \chi(\gamma_k)$ and similarly, $c_k(\bar{\gamma}_k) = \chi(\bar{\gamma}_k)$. So it suffices to prove that

$$\chi(\gamma_k) = (-1)^{-k}\chi(\bar{\gamma}_k).$$

But by the observations above, this is equivalent to showing that the canonical orientation of the underlying real $2k$ - dimensional bundle from the complex structures of γ_k and $\bar{\gamma}_k$ are the same if k is even, and opposite if k is odd. To do this we only need to compare the orientations at a single point. Let $x \in BU(k)$ be given by $\mathbb{C}^k \subset \mathbb{C}^\infty$ as the first k - coordinates. If $\{e_1, \cdots, e_k\}$ forms the standard basis for $\mathbb{C}^k$, then the orientations of $\gamma_k(x)$ determined by the complex structures of γ_k and $\bar{\gamma}_k$ are respectively represented by the real bases

$$\{e_1, ie_1, \cdots, e_k, ie_k\} \quad \text{and} \quad \{e_1, -ie_1, \cdots, e_k, -ie_k\}.$$

The change of basis matrix between these two basis has determinant $(-1)^k$. The theorem follows. □

Now suppose η is a real n - dimensional vector bundle over a space X, we then let $\eta_{\mathbb{C}}$ be its complexification

$$\eta_{\mathbb{C}} = \eta \otimes_{\mathbb{R}} \mathbb{C}.$$

$\eta_{\mathbb{C}}$ has the obvious structure as an n - dimensional complex vector bundle.

Proposition 5.31. *There is an isomorphism*

$$\phi : \eta_{\mathbb{C}} \xrightarrow{\cong} \bar{\eta}_{\mathbb{C}}.$$

Proof. Define

$$\phi : \eta_{\mathbb{C}} \to \bar{\eta}_{\mathbb{C}}$$
$$\eta \otimes \mathbb{C} \to \eta \otimes \bar{\mathbb{C}}$$
$$v \otimes z \to v \otimes \bar{z}$$

for $v \in \eta$ and $z \in \mathbb{C}$. Clearly ϕ is an isomorphism of complex vector bundles. □

Corollary 5.32. *For a real n - dimensional bundle η, then for k odd,*

$$2c_k(\eta_{\mathbb{C}}) = 0.$$

Proof. By 5.30 and 5.31

$$c_k(\eta_{\mathbb{C}}) = (-1)^k c_k(\eta_{\mathbb{C}}).$$

Hence for k odd $c_k(\eta_{\mathbb{C}})$ has order 2. □

5.5.2 Pontrjagin classes

We now use these results to define Pontrjagin classes for real vector bundles.

Definition 5.7. *Let η be an n - dimensional real vector bundle over a space X. Then define the i^{th} - Pontrjagin class*

$$p_i(\eta) \in H^{4i}(X; \mathbb{Z})$$

by the formula

$$p_i(\eta) = (-1)^i c_{2i}(\eta_{\mathbb{C}}).$$

Remark. The signs used in this definition are done to make calculations in the next section come out easily.

As we've done with Stiefel - Whitney and Chern classes, define the total Pontrjagin class

$$p(\eta) = 1 + p_1(\eta) + \cdots + p_i(\eta) + \cdots \in \bar{H}^*(X, \mathbb{Z}).$$

The following is the Whitney sum formula for Pontrjagin classes, and follows immediately for the Whitney sum formula for Chern classes and 5.32.

Theorem 5.33. *For real bundles η and ξ over X, we have*

$$2(p(\eta \oplus \xi) - p(\eta)p(\xi)) = 0 \in H^*(X;\mathbb{Z}).$$

In particular if R is a commutative integral domain containing $1/2$, then viewed as characteristic classes with values in $H^*(X;R)$, we have

$$p(\eta \oplus \xi) = p(\eta)p(\xi) \in \bar{H}^*(X : R).$$

Remark. Most often Pontryagin classes are viewed as having values in rational cohomology, and so the formula $p(\eta \oplus \xi) = p(\eta)p(\xi)$ applies.

We now study the Pontrjagin classes of a complex vector bundle. Let ζ be a complex n - dimensional bundle over a space X, and let $\zeta_{\mathbb{C}} = \zeta \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of its underlying real $2n$ - dimensional bundle. So $\zeta_{\mathbb{C}}$ is a complex $2n$ - dimensional bundle. We leave the proof of the following to the reader.

Proposition 5.34. *As complex $2n$ - dimensional bundles,*

$$\zeta_{\mathbb{C}} \cong \zeta \oplus \bar{\zeta}.$$

This result, together with 5.30 and the definition of Pontrjagin classes imply the following.

Corollary 5.35. *Let ζ be a complex n - dimensional bundle. Then its Pontryagin classes are determined by its Chern classes according to the formula*

$$\begin{aligned} 1 - p_1 + p_2 - \cdots \pm p_n &= (1 - c_1 + c_2 - \cdots \pm c_n)(1 + c_1 + c_2 + \cdots + c_n) \\ &\in H^*(X, \mathbb{Z}). \end{aligned}$$

Example. We will compute the Pontrjagin classes of the tangent bundle of projective space, $T(\mathbb{CP}^n)$. Recall that the total Chern class is given by

$$c(T(\mathbb{CP}^n)) = (1+a)^{n+1}$$

where $a \in H^2(\mathbb{CP}^n) \cong \mathbb{Z}$ is the generator. Notice that this implies that for the conjugate, $\bar{T}(\mathbb{CP}^n)$ we have

$$c(\bar{T}(\mathbb{CP}^n)) = (1-a)^{n+1}$$

Thus by the above formula we have

$$\begin{aligned} 1 - p_1 + p_2 - \cdots \pm p_n &= (1+a)^{n+1}(1-a)^{n+1} \\ &= (1-a^2)^{n+1}. \end{aligned}$$

We therefore have the formula

$$p_k(\mathbb{CP}^n) = \binom{n+1}{k} a^{2k} \in H^{4k}(\mathbb{CP}^n).$$

Now let η be an oriented real n- dimensional vector bundle. Then the complexification $\eta_{\mathbb{C}} = \eta \otimes \mathbb{C} = \eta \oplus i\eta$ which is simply $\eta \oplus \eta$ as real vector bundles.

Lemma 5.36. *The above isomorphism*

$$\eta_{\mathbb{C}} \cong \eta \oplus \eta$$

of real vector bundles takes the canonical orientation of $\eta_{\mathbb{C}}$ to $(-1)^{\frac{n(n-1)}{2}}$ times the orientation of $\eta \oplus \eta$ induced from the given orientation of η.

Proof. Pick a particular fiber, η_x. Let $\{v_1, \cdots, v_n\}$ be a $\mathbb{C}$ - basis for V. Then the basis $\{v_1, iv_1, \cdots, v_n iv_n\}$ determines the orientation for $\eta_x \otimes \mathbb{C}$. However the basis $\{v_1, \cdots, v_n, iv_1, \cdots iv_n\}$ gives the natural basis for $(\eta \oplus i\eta)_x$. The change of basis matrix has determinant $(-1)^{\frac{n(n-1)}{2}}$. □

Corollary 5.37. *If η is an oriented $2k$ - dimensional real vector bundle, then*

$$p_k(\eta) = \chi(\eta)^2 \in H^{4k}(X).$$

Proof.

$$\begin{aligned} p_k(\eta) &= (-1)^k c_{2k}(\eta \times \mathbb{C}) \\ &= (-1)^k \chi(\eta \otimes \mathbb{C}) \\ &= (-1)^k (-1)^{k(2k-1)} \chi(\eta \oplus \eta) \\ &= \chi(\eta \oplus \eta) \\ &= \chi(\eta)^2. \end{aligned}$$

□

5.5.3 Oriented characteristic classes

We now use the results above to show that Pontrjagin classes and the Euler class yield all possible characteristic classes for oriented vector bundles, if the coefficient ring contains 1/2. More specifically we prove the following.

Theorem 5.38. *Let R be an integral domain containing $1/2$. Then*

$$\begin{aligned} H^*(BSO(2n+1); R) &= R[p_1, \cdots, p_n] \\ H^*(BSO(2n); R) &= R[p_1, \cdots, p_{n-1}, \chi(\gamma_{2n})] \end{aligned}$$

Remark. This theorem can be restated by saying that $H^(BSO(n);R)$ is generated by $\{p_1,\cdots,p_{[n/2]}\}$ and χ, subject only to the relations

$$\chi = 0 \quad \text{if } n \text{ is odd}$$
$$\chi^2 = p_{[n/2]} \quad \text{if } n \text{ is even.}$$

Proof. In this proof all cohomology will be taken with R coefficients. We first observe that since $SO(1)$ is the trivial group, $BSO(1)$ is contractible, and so $H^*(BSO(1)) = 0$. This will be the first step in an inductive proof. So we assume the theorem has been proved for $BSO(n-1)$, and we now compute $H^*(BSO(n))$ using the Gysin sequence:

$$\cdots \to H^{q-1}(BSO(n-1)) \xrightarrow{\delta} H^{q-n}(BSO(n)) \xrightarrow{\cup\chi} H^q(BSO(n)) \xrightarrow{\iota^*} H^q(BSO(n-1)) \xrightarrow{\delta} H^{q-n+1}(BSO(n)) \xrightarrow{\cup\chi} H^{q+1}(BSO(n)) \to \cdots \tag{5.12}$$

Case 1. n is even.

Since the first $n/2-1$ Pontrjagin classes are defined in $H^*(BSO(n))$ as well as in $H^*(BSO(n-1))$, the inductive assumption implies that ι^* : $H^*(BSO(n)) \to H^*(BSO(n-1))$ is surjective. Thus the Gysin sequence reduces to short exact sequences

$$0 \to H^q(BSO(n)) \xrightarrow{\cup\chi} H^{q+n}(BSO(n)) \xrightarrow{\iota^*} H^{q+n}(BSO(n-1)) \to 0.$$

The inductive step then follows.

Case 2. n is odd, say $n = 2m+1$.

By 5.13 in this case the Euler class χ has order two in integral cohomology. Thus since R contains $1/2$, in cohomology with R coefficients, the Euler class is zero. Thus the Gysin sequence reduces to short exact sequences:

$$0 \to H^j(BSO(2m+1)) \xrightarrow{\iota^*} H^*(BSO(2m)) \to H^{j-2m}(BSO(2m+1)) \to 0.$$

Thus the map ι^* makes $H^*(BSO(2m+1))$ a subalgebra of $H^*(BSO(2m))$. This subalgeabra contains the Pontrjagin classes and hence it contains the graded algebra $A^* = R[p_1,\cdots p_m]$. By computing ranks we will now show that this is the entire image of ι^*. This will complete the inductive step in this case.

So inductively assume that the rank of A^{j-1} is equal to the rank of

$H^j(BSO(2m+1))$. Now we know that every element of $H^j(BSO(2m))$ can be written uniquely as a sum $a + \chi b$ where $a \in A^j$ and $b \in A^{j-2m}$. Thus

$$H^j(BSO(2m)) \cong A^j \oplus A^{j-2m}$$

which implies that

$$rk(H^j(BSO(2m)) = rk(A^j) + rk(A^{j-2m}).$$

But by the exactness of the above sequence,

$$rk(H^j(BSO(2m)) = rk(H^j(BSO(2m+1)) + rk(H^{j-2m}(BSO(2m+1))).$$

Comparing these two equations, and using our inductive assumption, we conclude that

$$rk(H^j(BSO(2m+1)) = rk(A^j).$$

Thus $A^j = \iota^*(H^j(BSO(2m+1)))$, which completes the inductive argument.
□

5.6 Connections, Curvature, and Characteristic Classes

In this section we describe how Chern and Pontrjagin classes can be defined using connections (i.e covariant derivatives) on vector bundles. What we will describe is an introduction to the theory of Chern and Weil that describe the cohomology of a classifying space of a compact Lie group in terms of invariant polynomials on its Lie algebra. The treatment we will follow is from Milnor and Stasheff [74].

Definition 5.8. *Let $M_n(\mathbb{C})$ be the ring of $n \times n$ matrices over $\mathbb{C}$. Then an invariant polynomial* on $M_n(\mathbb{C})$ is a function

$$P : M_n(\mathbb{C}) \to \mathbb{C}$$

which can be expressed as a complex polynomial in the entries of the matrix, and satisfies,

$$P(ABA^{-1}) = P(B)$$

for every $B \in M_n(\mathbb{C})$ and $A \in GL(n, \mathbb{C})$.

Examples. The trace function $(a_{i,j}) \to \sum_{j=1}^{n} a_{j,j}$ and the determinant function are examples of invariant polynomials on $M_n(\mathbb{C})$.

Now let $D_A : \Omega^0(M;\zeta) \to \Omega^1(M;\zeta)$ be a connection (or covariant derivative) on a complex n - dimensional vector bundle ζ. Its curvature is a a two- form with values in the endomorphism bundle

$$F_A \in \Omega^2(M; End(\zeta))$$

The endomorphism bundle can be described alternatively as follows. Let E_ζ be the principal $GL(n,\mathbb{C})$ bundle associated to ζ. Then of course $\zeta = E_\zeta \otimes_{GL(n,\mathbb{C})} \mathbb{C}^n$. The endomorphism bundle can then be described as follows. The proof is an easy exercise that we leave to the reader.

Proposition 5.39.

$$End(\zeta) \cong ad(\zeta) = E_\zeta \times_{GL(n,\mathbb{C})} M_n(\mathbb{C})$$

where $GL(n,\mathbb{C})$ acts on $M_n(bc)$ by conjugation,

$$A \cdot B = ABA^{-1}.$$

Let ω be a differential p - form on M with values in $End(\zeta)$,

$$\omega \in \Omega^p(M; End(\zeta)) \cong \Omega^p(M; ad(\zeta)) = \Omega^p(M; E_\zeta \times_{GL(n,\mathbb{C})} M_n(\mathbb{C})).$$

Then on a coordinate chart $U \subset M$ with local trivialization $\psi : \zeta_{|_U} \cong U \times \mathbb{C}^n$ for ζ, and hence the induced coordinate chart and local trivialization for $ad(\zeta)$, ω can be viewed as an $n \times n$ matrix of p -forms on M. We write

$$\omega = (\omega_{i,j}).$$

Of course this description depends on the coordinate chart and local trivialization chosen, but at any $x \in U$, then by the above proposition, two trivializations yield conjugate matrices. That is, if $(\omega_{i,j}(x))$ and $(\omega'_{i,j}(x))$ are two matrix descriptions of $\omega(x)$ defined by two different local trivializations of $\zeta_{|_U}$, then there exists an $A \in GL(n,\mathbb{C})$ with

$$A(\omega_{i,j}(x))A^{-1} = (\omega'_{i,j}(x)).$$

Now let P be an invariant polynomial on $M_n(\mathbb{C})$ of degree d. Then using the wedge bracket we can apply P to a matrix of p forms, and produce a differential form of top dimension pd on $U \subset M$: $P(\omega_{i,j}) \in \Omega^{pd}(U)$. Now since the polynomial P is invariant under conjugation the form $P(\omega_{i,j})$ is independent of the local trivialization of $\zeta_{|_U}$. These forms therefore fit together to give a well defined global form

$$P(\omega) \in \Omega^*(M). \tag{5.13}$$

If $P$is homogeneous of degree d, then

$$P(\omega) \in \Omega^{pd}(M) \tag{5.14}$$

An important example is when $\omega = F_A \in \Omega^2(M; End(\zeta))$ is the curvature form of a connection D_A on ζ. We have the following fundamental lemma, that will allow us to define characteristic classes in terms of these forms and invariant polynomials.

Lemma 5.40. *For any connection D_A and invariant polynomial (or invariant power series) P, the differential form $P(F_A)$ is closed. That is,*

$$dP(F_A) = 0.$$

Proof. (following Milnor and Stasheff [74]) Let P be an invariant polynomial or power series. We write $P(A) = P(a_{i,j})$ where the $a_{i,j}$'s are the entries of the matrix. We can then consider the matrix of partial derivatives $(\partial P/\partial(x_{i,j}))$ where the $x_{i,j}$'s are indeterminates. Let $F_A = (\omega_{i,j})$ be the curvature matrix of two - forms on an open set U with a given trivialization. Then the exterior derivative has the following local expression

$$dP(F_A) = \sum (\partial P/\partial \omega_{i,j}) d\omega_{i,j}. \tag{5.15}$$

In matrix notation this can be written as

$$dP(F_A) = trace(P'(F_A) dF_A)$$

Now as seen in chapter 1, on a trivial bundle, and hence on this local coordinate patch, a connection D_A can be viewed as a matrix valued one form,

$$D_A = (\alpha_{i,j})$$

and with respect to which the curvature F_A has the formula

$$\omega_{i,j} = d\alpha_{i,j} - \sum_k \omega_{i,k} \wedge \omega_{k,j}.$$

In matrix notation we write

$$F_A = d\alpha - \alpha \wedge \alpha.$$

Differentiating yields the following form of the Bianchi identity

$$dF_A = \alpha \wedge F_A - F_A \wedge \alpha. \tag{5.16}$$

We need the following observation.

Claim. The transpose of the matrix of first derivatives of an invariant polynomial (or power series) $P'(A)$ commutes with A.

Proof. Let $E_{j,i}$ be the matrix with entry 1 in the (j,i)-th place and zeros in all other coordinates. Now differentiate the equation

$$P((I + tE_{j,i})A) = P(A(I + tE_{j,i}))$$

with respect to t and then setting $t = 0$ yields

$$\sum_k A_{i,k}(\partial P/\partial A_{j,k}) = \sum_k (\partial P/\partial A_{k,i})A_{k,i}.$$

Thus the matrix A commutes with the transpose of $(\partial P/\partial Ai, j)$ as claimed. □

We now complete the proof of the lemma. Substituting F_A for the matrix of indeterminates in the above claim means we have

$$F_A \wedge P'(F_A) = P'(F_A) \wedge F_A. \tag{5.17}$$

Now for notational convenience let $X = P'(F_A) \wedge \alpha$. Then substituting the Bianchi identity 5.16 into 5.15 and using 5.17 we obtain

$$\begin{aligned} dP(F_A) &= \mathit{trace}\,(X \wedge F_A - F_A \wedge X) \\ &= \sum (X_{i,j} \wedge \omega_{j,i} - \omega_{j,i} \wedge X_{i,j}). \end{aligned}$$

Since each $X_{i,j}$ commutes with the 2 - form $\omega_{j,i}$, this sum is zero, which proves the lemma. □

Thus for any connection D_A on the complex vector bundle ζ over M, and invariant polynomial P, the form $P(F_A)$ represents a deRham cohomology class with complex coefficients. That is,

$$[P(F_A)] \in H^*(M : \mathbb{C}).$$

Theorem 5.41. *The cohomology class* $[P(F_A)] \in H^*(X, \mathbb{C})$ *is independent of the connection* D_A.

Proof. Let D_{A_0} and D_{A_1} be two connections on ζ. Pull back the bundle ζ over $M \times \mathbb{R}$ via the projection map $M \times \mathbb{R} \to M$. Call this pull - back bundle $\bar{\zeta}$ over $M \times \mathbb{R}$. We get the induced pull back connections $\bar{D}_{A_i}$, $i = 0, 1$ as well. We can then form the linear combination of connections

$$D_A = t\bar{D}_{A_1} + (1 - t)D_{A_0}.$$

Then $P(F_A)$ is a deRham cocycle on $M \times \mathbb{R}$. Now let $i = 0$ or 1 and consider the inclusions $j_i : M = M \times \{i\} \hookrightarrow M \times \mathbb{R}$. The induced connection $j_i^*(D_A) = D_{A_i}$ on ζ. But since there is an obvious homotopy between j_0 and j_1 and hence the cohomology classes

$$[j_0^*(P(F_A)) = P(F_{A_0})] = [j_1^*(P(F_A)) = P(F_{A_1})].$$

This proves the theorem. □

Thus the invariant polynomial P determines a cohomology class given any bundle ζ over a smooth manifold. It is immediate that these classes are preserved under pull - back, and are hence characteristic classes for $U(n)$ bundles, and hence are given by elements of

$$H^*(BU(n);\mathbb{C}) \cong \mathbb{C}[c_1, \cdots, c_n].$$

In order to see how an invariant polynomial corresponds to a polynomial in the Chern classes we need the following bit of algebra.

Recall the elementary symmetric polynomials $\sigma_1, \cdots \sigma_n$ in n -variables, discussed in section 3. If we view the n - variables as the eigenvalues of an $n \times n$ matrix, we can write

$$det(I + tA) = 1 + t\sigma_1(A) + \cdots + t^n\sigma_n(A). \tag{5.18}$$

Lemma 5.42. *Any invariant polynomial on $M_n(\mathbb{C})$ can be expressed as a polynomial of $\sigma_1, \cdots, \sigma_n$.*

Proof. Given $A \in M_n(\mathbb{C})$, chose a B such that BAB^{-1} is in Jordan canonical form. Replacing B with $diag(\epsilon, \epsilon^2, \cdots, \epsilon^n)B$, we can make the off diagonal entries arbitrarily close to zero. By continuity it follows that $P(A)$ depends only on the diagonal entries of BAB^{-1}, i,e the eigenvalues of A. Since $P(A)$ is invariant, it must be a symmetric polynomial of these eigenvalues. Hence it is a polynomial in the elementary symmetric polynomials. □

So we now consider the elementary symmetric polynomials, viewed as invariant polynomials in $M_n(\mathbb{C})$. Hence by the above constructions they determine characteristic classes $[\sigma_r(F_A)] \in H^{2r}(M;\mathbb{C})$ where F_A is a connection on a vector bundle ζ over M.

Now we've seen the elementary symmetric functions before in the context of characteristic classes. Namely we've seen that $H^*(BU(n))$ can be viewed as the subalgebra of symmetric polynomials in $\mathbb{Z}[x_1, \cdots x_n] = H^*(BU(1) \times \cdots \times BU(1))$, with the Chern class C_r corresponding to the elementary symmetric polynomial σ_r. This was the phenomenon of the *splitting principle*.

We will now use a splitting principle argument to prove the following.

Theorem 5.43. *Let ζ be a complex n - dimensional vector bundle with connection D_A. Then the cohomology class $[\sigma_r(F_A)] \in H^{2r}(X;\mathbb{C})$ is equal to $(2\pi i)^r c_r(\zeta)$, for $r = 1, \cdots, n$.*

Proof. We first prove this theorem for complex line bundles. That is, $n = 1$. In this case $\sigma_1(F_A) = F_A$ which is a closed form in $\Omega^2(M; ad(\zeta)) = \Omega^2(M; \mathbb{C})$ because the adjoint action of $GL(1, \mathbb{C})$ is trivial since it is an abelian group. In particular F_A is closed in this case by 5.40. Thus F_A represents a cohomology class in $H^2(M; \mathbb{C})$. Moreover as seen above, this cohomology class $[F_A]$ is a characteristic class for line bundles and hence is an element of $H^2(BU(1); \mathbb{C}) \cong \mathbb{C}$ generated by the first Chern class $c_1 \in H^2(BU(1))$. So for this case we need to prove the following generalization of the Gauss - Bonnet theorem.

Lemma 5.44. *Let ζ be a complex line bundle over a manifold M with connection D_A. Then the curvature form F_A is a closed two - form representing the cohomology class*

$$[F_A] = 2\pi i c_1(\zeta) = 2\pi i \chi(\zeta).$$

Before we prove this lemma we show how this lemma can in fact be interpreted as a generalization of the classical Gauss - Bonnet theorem. So let D_A be a unitary connection on ζ. (That is, D_A is induced by a connection on an associated principal $U(1)$ - bundle.) If we view ζ as a two dimensional, oriented vector bundle which, to keep notation straight we refer to as $\zeta_{\mathbb{R}}$, then D_A induces (and is induced by) a connection $D_{A_{\mathbb{R}}}$ on the real bundle $\zeta_{\mathbb{R}}$. Notice that since $SO(2) \cong U(1)$ then orthogonal connections on oriented real two dimensional bundes are equivalent to unitary connections on complex line bundles.

Since $SO(2)$ is abelian, the real adjoint bundle

$$ad(\zeta_{\mathbb{R}}) = E_{\zeta_{\mathbb{R}}} \times_{SO(2)} M_2(\mathbb{R})$$

is trivial. Hence the curvature $F_{A_{\mathbb{R}}}$ is then a 2×2 matrix valued two - form.

$$F_{A_{\mathbb{R}}} \in \Omega^2(M; M_2(\mathbb{R})).$$

Moreover, since the Lie algebra of $SO(2)$ consists of skew symmetric 2×2 real matrices, then it is straightforward to check the following relation between the original complex valued connection $F_A \in \Omega^2(M; \mathbb{C})$ and the real curvature form $F_{A_{\mathbb{R}}} \in \Omega^1(M; M_2(\mathbb{R}))$.

Claim. If $F_{A_{\mathbb{R}}}$ is written as the skew symmetric matrix of $2 - forms$

$$F_{A_{\mathbb{R}}} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \in \Omega^2(M; M_2(\mathbb{R}))$$

then

$$F_A = i\omega \in \Omega^2(M; \mathbb{C}).$$

When the original connection $D_{A_{\mathbb{R}}}$ is the *Levi - Civita* connection associated to a Riemannian metric on the tangent bundle of a Riemann surface, the curvature form

$$\omega \in \Omega^2(M, \mathbb{R})$$

is referred to as the "Gauss - Bonnet"" connection. If dA denotes the area form with respect to the metric, then we can write

$$\omega = \kappa \, dA$$

then κ is a scalar valued function called the "Gaussian curvature" of the Riemann surface M. In this case, by the claim we have $[F_A] = 2\pi i \chi(T(M))$, and since

$$\langle \chi(T(M)), [M] \rangle = \chi_M,$$

Where χ_M the Euler characteristic of M, we have

$$\langle [F_A], [M] \rangle = \int_M F_A = i \int_M \omega = i \int_M \kappa \, dA.$$

Thus the above lemma applied to this case, which states that

$$\langle [F_A], [M] \rangle = 2\pi i \chi_M$$

is equivalent to the classical Gauss - Bonnet theorem which states that

$$\int_M \kappa \, dA = 2\pi \chi_M = 2\pi(2 - 2g) \tag{5.19}$$

where g is the genus of the Riemann surface M.

We now prove the above lemma.

Proof. As mentioned above, since $[F_A]$ is a characteristic class for line bundles, and so it is some multiple of the first Chern class, say $[F_A] = qc_1(\zeta)$. By the naturality, the coefficient q is independent of the bundle. So to evaluate q it is enough to compute it on a specific bundle. We choose the tangent bundle of the unit sphere $T(S^2)$, equipped with the Levi - Civita connection D_A corresponding to the usual round metric (or equivalently the metric coming from the complex strucure $S^2 = \mathbb{CP}^1$). In this case the Gaussian curvature is constant at one,

$$\kappa = 1.$$

Moreover since $T(S^2) \oplus \epsilon_1 \cong \gamma_1 \oplus \gamma_1$, the Whitney sum formula yields

$$\langle c_1(S^2), [S^2] \rangle = 2 \langle c_1(\gamma_1), [S^2] \rangle = 2.$$

Thus we have

$$\begin{aligned} \langle [F_A], [S^2] \rangle &= q \langle c_1(S^2), [S^2] \rangle \\ &= 2q. \end{aligned}$$

Putting these facts together yields that

$$\begin{aligned} 2q &= \langle [F_A], [S^2] \rangle \\ &= \int_{S^2} F_A \\ &= i \int_{S^2} \kappa \, dA \\ &= i \int_{S^2} dA \qquad = i \cdot \text{surface area of } S^2 \\ &= i \cdot 4\pi. \end{aligned}$$

Hence $q = 2\pi i$, as claimed. □

We now proceed with the proof of theorem 5.43 in the case when the bundle is a sum of line bundles. By the splitting principal we will then be able to conclude the theorem is true for all bundles.

So let $\zeta = L_1 \oplus \cdots \oplus L_n$ where $L_1, \cdots, L_n$ are complex line bundles over M. Let $D_1, \cdots, D_n$ be connections on $L_!, \cdots, L_n$ respectively. Now let D_A be the connection on ζ given by the sum of these connections

$$D_A = D_1 \oplus \cdots \oplus D_n.$$

Notice that with respect to any local trivialization, the curvature matrix F_A is the diagonal $n \times n$ matrix with diagonal entries, the curvatures $F_1, \cdots, F_n$ of the connections $D_1, \cdots D_n$ respectively. Thus the invariant polynomial applied to the curvature form $\sigma_r(F_A)$ is given by the symmetric polynomial in the diagonal entries,

$$\sigma_r(F_A) = \sigma_r(F_1, \cdots, F_r).$$

Now since the curvatures F_i are closed 2 - forms on M, we have an equation of cohomology classes

$$[\sigma_r(F_A)] = \sigma_r([F_1], \cdots, [F_r]).$$

By the above lemma we therefore have

$$\begin{aligned} [\sigma_r(F_A)] &= \sigma_r([F_1], \cdots, [F_n]) \\ &= \sigma_r((2\pi i)c_1(L_1), \cdots, (2\pi i)c_1(L_n)) \\ &= (2\pi i)^r \sigma_r(c_1(L), \cdots, c_1(L_n)) \quad \text{since } \sigma_r \text{ is symmetric} \\ &= (2\pi i)^r c_r(L_1 \oplus \cdots \oplus L_n) \quad \text{by the splitting principal } 5.20 \\ &= (2\pi i)^r c_r(\zeta) \end{aligned}$$

as claimed.

This proves the theorem when ζ is a sum of line bundles. As observed above, the splitting principal implies that the theorem then must be true for all bundles. □

We end this section by describing two corollaries of this important theorem.

Corollary 5.45. *For any real vector bundle η, the deRham cocycle $\sigma_{2k}(F_A)$ represent the cohomology class $(2\pi)^{2k} p_k(\eta) \in H^{4k}(M;\mathbb{R})$, while $[\sigma_{2k+1}(F_A)]$ is zero in $H^{4k+2}(M;\mathbb{R})$.*

Proof. This just follows from the definition of the Pontrjagin classes in terms of the even Chern classes of the complexification, and the fact that the odd Chern classes of the complexification have order two and therefore represent the zero class in $H^*(M;\mathbb{R})$. □

Recall that a flat connection is one whose curvature is zero. The following is immediate form the above theorem.

Corollary 5.46. *If a real (or complex) vector bundle has a flat connection, then all its Pontrjagin (or Chern) classes with rational coefficients are zero.*

We recall that a bundle has a flat connection if and only if its structure group can be reduced to a discrete group. Thus a complex vector bundle with a discrete structure group has zero Chern classes with rational coefficients. This can be interpreted as saying that if $\iota : G \subset GL(n,\mathbb{C})$ is the inclusion of a discrete subgroup, then the map in cohomology,

$$\mathbb{Q}[c_1, \cdots, c_n] = H^*(BU(n);\mathbb{Q}) = H^*(BGL_n(\mathbb{C});\mathbb{Q}) \xrightarrow{\iota^*} H^*(BG;\mathbb{Q})$$

is zero.

6

Embeddings and Immersions in Euclidean Space

6.1 The existence of embeddings: The Whitney Embedding Theorem

The following result is often known as the "Easy Whitney Embedding Theorem". It tells us that we may view any manifold as a submanifold of Euclidean space.

Theorem 6.1. *Let M^n be a C^r manifold of dimension n. Then there is a C^r-embedding $e : M^n \hookrightarrow \mathbb{R}^L$ for L sufficiently large.*

Proof. We prove this theorem in the case when M^n is closed. We refer the reader to [44] for the general case. Since M^n is compact we can find a *finite* atlas $\{\phi_i, U_i\}_{i=1}^{M}$ with the following properties:

1. For all $i = 1, \cdots, m, B_2(0) \subset \phi_i(U_i) \subset \mathbb{R}^n$, and
2. $M^n = \bigcup_{i=1}^{m} Int\, \phi_i^{-1}(B_1(0))$.

Here $B_r(0) \subset \mathbb{R}^n$ is the open ball around the origin of radius r.

Let $\lambda : \mathbb{R}^n \to [0, 1]$ be a C^∞ "bump function" such that

$$\lambda(x) = \begin{cases} 1 & \text{on} \quad B_1(0) \\ 0 & \text{on} \quad \mathbb{R}^n - B_2(0) \end{cases}$$

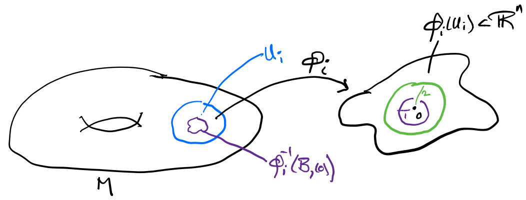
Define $\lambda_i : M^n \to [0, 1]$ by

$$\lambda_i = \begin{cases} \lambda \circ \phi_i & \text{on} \quad U_i \\ 0 & \text{on} \quad M^n - U_i. \end{cases}$$

These are "local bump functions". Notice that the sets $S_i = \lambda_i^{-1}(1) \subset U_i$, $i = 1, \cdots, m$ cover M^n.

Now define $f_i : M^n \to \mathbb{R}^n$ by

$$f_i(x) = \begin{cases} \lambda_i(x)\phi_i(x) & \text{if} \quad x \in U_i \\ 0 & \text{if} \quad x \in M - U_i \end{cases}$$



Notice that f_i is C^r. Define $g_i(x) = (f_i(x), \lambda_i(x)) \in \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$, and

$$g = (g_1, \cdots, g_m) : M^n \to \mathbb{R}^{n+1} \times \cdots \times \mathbb{R}^{n+1} = \mathbb{R}^{m(n+1)}.$$

g is a C^r map. We claim it is an embedding.

If $x \in S_i$, g_i is immersive at x, so therefore g is immersive at x. Since the S_i's cover M^n, g_i is an immersion. We observe that g is one-to-one.

Suppose $x \neq y$ and $y \in S_i$. If x also lies in S_i, then since

$$f_{i_{|S_i}} = \phi_{i_{|S_i}}$$

then $f_i(x) \neq f_i(y)$ since ϕ_i is injective. If x *does not* lie in S_i, then

$$\lambda_i(y) = 1 \neq \lambda_i(x).$$

So $g(x) \neq g(y)$.

So $g : M^n \to \mathbb{R}^{n(m+1)}$ is an injective immersion. Since M^n is compact, g is an embedding. □

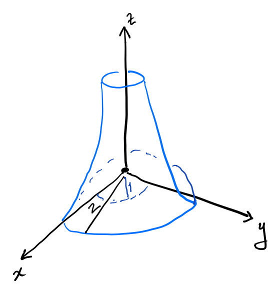


FIGURE 6.1
A graph of λ when $n = 2$.

Remark. Notice that this theorem implies that a compact n-manifold M^n can be embedding in *any* manifold N^m if the dimension of N^m is sufficiently large. This is because N^m looks locally like Euclidean space, and so by the above theorem N^m can be embedding in an open set inside M^n.

6.1.1 Obstructions to the existence of embeddings and immersions, and the immersion conjecture

A stronger version of Theorem 6.1 was proved by H. Whitney in a seminal paper published in 1944 [106].

Theorem 6.2. *[106] A. (Whitney Embedding Theorem) Let M^n be a compact*

C^r manifold of dimension n, with $r \geq 1$. Then there is a C^r-embedding $e : M^n \hookrightarrow \mathbb{R}^{2n}$. Furthermore there is a C^r-immersion $j : M^n \looparrowright \mathbb{R}^{2n-1}$.

An extension of Whitney's theorem to the setting of manifolds with boundary is the following:

Theorem 6.3. *Let M^n be a C^r-n-dimensional compact manifold with boundary, with $r \geq 1$. Then there is a neat C^r embedding of M^n into $\mathbb{H}^{2n}$.*

It is natural to ask if Whitney's theorem is the best possible. More specfically, one can ask the following question. From now on all manifolds we consider are closed and C^∞, unless specifically stated otherwise.

Question 1. What is the smallest positive integer $\phi(n)$ so that every compact n-dimensional manifold can be embedded in $\mathbb{R}^{n+\phi(n)}$? Notice that Whitney's theorem says that $\phi(n) \leq n$.

Question 2. What is the smallest positive integer $\psi(n)$ so that every compact n-dimensional manifold can be immersed in $\mathbb{R}^{n+\psi(n)}$? Whitney's theorem says that $\psi(n) \leq n-1$.

Question 1 poses a problem that as of this date is unsolved. There are many results of the best possible embedding dimension for particular n-manifolds, but general the answer to Question 1 is unknown. However in the case when n is a power of 2 one can prove that Whitney's result is best possible. That is, if $n = 2^k$, then $\phi(2^k) = 2^k$. We give a sketch of a proof of this fact by proving the following.

Proposition 6.4. *The projective space $\mathbb{RP}^{2^k}$ embeds in $\mathbb{R}^{2^{k+1}}$ by Whitney's theorem, but it does not embedd in $\mathbb{R}^{2^{k+1}-1}$.*

Proof. We give a sketch of an argument that uses a theory of Haefliger developed in [42]. For X any space, consider the configuration space of k ordered, distinct points in X:

$$F(X,k) = \{(x_1, \cdots, x_k) \in X^k \,:\, x_i \neq x_j \quad \text{if} \quad i \neq j\}.$$

Notice that the symmetric group Σ_k acts freely on $F(X,k)$ by permuting the order of the elements.

Notice that if $e : M^n \hookrightarrow \mathbb{R}^L$ is an embedding of a manifold into Euclidean space, there is an induced map of configuration spaces

$$F(e) : F(M^n, 2)/\Sigma_2 \to F(\mathbb{R}^L, 2)/\Sigma_2.$$

We claim that $F(\mathbb{R}^L, 2)/\Sigma_2$ has the homotopy type of the projective space $\mathbb{RP}^{L-1}$. To see this, notice that $F(\mathbb{R}^L, 2)$ is diffeomorphic to $\mathbb{R}^L \times (\mathbb{R}^L - \{0\})$ via the map that sends (x_1, x_2) to $(x_1 + x_2, x_1 - x_2)$. This is a Σ_2-equivariant diffeomorphism, where the action on $\mathbb{R}^L \times (\mathbb{R}^L - \{0\})$ is given by

$(u, v) \to (u, -v)$. But clearly with respect to this action $\mathbb{R}^L \times (\mathbb{R}^L - \{0\})$ is Σ_2-equivariantly homotopy equivalent to the sphere S^{L-1} with the antipodal Σ_2-action. Then claim then follows.

Now since any compact n- manifold M^n embeds in $\mathbb{R}^L$ for L sufficiently large, and since any two embeddings into sufficiently large dimensional Euclidean space are isotopic (to be discussed below), then one always comes equipped with a map, well defined up to homotopy,

$$\omega : F(M^n, 2)/\Sigma_2 \to F(\mathbb{R}^\infty, 2)/\Sigma_2 \simeq \mathbb{RP}^\infty.$$

Furthermore, by the above claim, if M^n embeds in $\mathbb{R}^L$, this map factors, up to homotopy, through a map $\omega_L : F(M^n, 2)/\Sigma_2 \to \mathbb{RP}^{L-1}$. By Whitney's theorem, one can always find such a ω_L for $L = n$. However in the case of $M^n = \mathbb{RP}^{2^k}$, Haefliger showed using obstruction theory that there is no map $\omega_{2^k-1} : F(\mathbb{RP}^{2^k}, 2)/\Sigma_2) \to \mathbb{RP}^{2^k-2}$ that factors $\omega : F(\mathbb{RP}^{2^k}, 2)/\Sigma_2) \to \mathbb{RP}^\infty$. This means that $\mathbb{RP}^{2^k}$ cannot be embedded in $\mathbb{R}^{2^{k+1}-1}$. □

Notice that this proposition says that in the case $n = 2^k$ the answer to Question 1 above is $\phi(2^k) = 2^k$. But as was mentioned above, in general Question 1 is unresolved. However, as we have observed, Haefligger's theory supplies a homotopy theoretic obstruction to embedding manifolds in Euclidean space. We remark that in recent years Haefligger's theory has been generalized to a theory of "Embedding Calculus", as developed by T. Goodwillie, M. Weiss, and others [38], [39], [104] [105]. This is a beautiful and effective theory for studying spaces of embeddings of one manifold into an other, using sophisticated homotopy theoretic techniques. We encourage the reader to learn more about this theory.

The situation with immersions instead of embeddings is considerably easier, due to the following famous result of Hirsch and Smale [45]. This is an early example of the h-principle (where "h" stands for homotopy) as defined by Gromov [37] and developed further by Eliashberg and Mishachev [30]. We now describe the Hirsch-Smale result.

Suppose $f : M^n \looparrowright P^{n+k}$ is an immersion between smooth (C^∞) manifolds. Then one has the induced map of tangent bundles yielding the commutative diagram

$$\begin{array}{ccc} TM^n & \xrightarrow{Df} & TP^{n+k} \\ \downarrow & & \downarrow \\ M^n & \xrightarrow[f]{} & P^{n+k} \end{array}$$

This is an example of a *bundle monomorphism,* meaning a map of vector bundles

$$\begin{array}{ccc} \zeta & \xrightarrow{\bar{\gamma}} & \xi \\ \downarrow & & \downarrow \\ X & \xrightarrow[\gamma]{} & Y \end{array}$$

so that $\gamma_x : \zeta_x \to \xi_{\gamma(x)}$ is a linear monomorphism of vector spaces for each $x \in X$. We denote the space of such bundle monomorphisms by $Mono(\zeta, \xi)$. Let $Imm(M^n, P^{n+k})$ be the space of immersions, topologized in the space of all maps given the compact-open topology. Then differentiation induces a map

$$D : Imm(M^n, P^{n+k}) \to Mono(TM^n, TP^{n+k}).$$

Theorem 6.5. *(Hirsch and Smale [45]). Let M^n be a compact, smooth manifold of dimension n, and P^{n+k} be a smooth manifold of dimension $n+k$, with $k \geq 1$. Then the map*

$$D : Imm(M^n, P^{n+k}) \to Mono(TM^n, TP^{n+k}).$$

is a weak homotopy equivalence.

Notice that in particular, if $Mono(TM^n, T\mathbb{R}^{n+k})$ is nonempty, then there exists an immersion $M^n \looparrowright \mathbb{R}^{n+k}$, for $k \geq 1$.

Notice furthermore that a bundle monomorphism $\gamma : TM^n \to T\mathbb{R}^{n+k}$ determines a k-dimensional normal bundle,

$$\pi : \nu^k_\gamma \to M^n$$

where $\pi^{-1}(x) = \{v \in \mathbb{R}^{n+k} \text{ such that } v \perp \gamma(T_x M^n)\}$. That is ν^k_γ is the *orthogonal complement* to TM^n, inside $T\mathbb{R}^{n+k}$. In otherwords,

$$TM^n \oplus \nu^k_\gamma \cong M \times \mathbb{R}^{n+k}.$$

The following is is a direct consequence of the Hirsch-Smale theorem.

Corollary 6.6. *A compact n-manifold M^n immerses in $\mathbb{R}^{n+k}$ if and only if there is a k-dimensional bundle $\nu^k \to M^n$ such that*

$$TM^n \oplus \nu^k \cong M^n \times \mathbb{R}^{n+k}.$$

We now give an interpretation of these results in terms of classifying spaces. We use [23] as a reference. This allows one to recast the question of immersing manifolds into Euclidean space into a homotopy theoretic problem.

As above, let $BO(k)$ denote the classifying space of k-dimensional vector bundles, and let $BO = \lim_{k\to\infty} BO(k)$. Since every manifold immerses, and indeed embeds in sufficiently high dimensional Euclidean space means there is a map

$$\nu : M^n \to BO$$

representing this high dimensional (or "stable") normal bundle. This map is well-defined up to homotopy for the following reason. Given any compact space X with basepoint, the homotopy classes of basepoint preserving maps $[X, BO]$ represents the set of stable vector bundles $SVect(S)$, which is isomorphic to the reduced K-theory, $\tilde{KO}(X)$, and is therefore an abelian group. (We refer the reader to Theorem 3.17 for a discussion of this fact.) In particular the addition in this abelian group corresponds to the Whitney sum of vector bundles. In this abelian group structure, the stable normal bundle is the inverse of the stable tangent bundle represented by the composite

$$\tau M : M^n \to BO(n) \to BO.$$

Thus the stable normal bundle map is well-defined, up to homotopy. We may therefore restate Corollary 6.6 as follows.

Theorem 6.7. *Let M^n be a compact n-manifold and $\nu : M^n \to BO$ represent its stable normal bundle. Then M^n immerses in $\mathbb{R}^{n+k}$ if and only if there is a map $\nu^k : M^n \to BO(k)$ so that the composite*

$$M^n \xrightarrow{\nu^k} BO(k) \to BO$$

is homotopic to the stable normal bundle map $\nu : M^n \to BO$.

Using this theorem, the work of Brown and Peterson [17] [18] [19], and the author [21], combined to give a resolution of Question 2 above. We now outline how this was achieved.

In [60] Massey showed that for every closed n-manifold M^n, the homomorphism induced by the stable normal bundle map

$$\nu^* : H^*(BO; \mathbb{Z}/2) \to H^*(M^n; \mathbb{Z}/2)$$

factors through $H^*(BO(n - \alpha(n))$, where $\alpha(n)$ is the number of ones in the dyadic (base 2) expansion of n. That is to say, there is a homomorphism $\tilde{\nu}^* : H^*(BO(2n - \alpha(n); \mathbb{Z}/2) \to H^*(M^n; \mathbb{Z}/2)$ so that the composition

$$H^*(BO; \mathbb{Z}/2) \xrightarrow{\iota^*} H^*(BO(n - \alpha(n)); \mathbb{Z}/2) \xrightarrow{\tilde{\nu}^*} H^*(M^n; \mathbb{Z}/2)$$

is equal to ν^*. Here $\iota : (BO(n - \alpha(n)) \to BO$ is the usual inclusion. Now recall from Theorem 5.15 that $H^*(BO; \mathbb{Z}/2) \cong \mathbb{Z}/2[\omega_1, \cdots, \omega_k, \cdots]$ and that $H^*(BO(m); \mathbb{Z}/2) \cong \mathbb{Z}/2[\omega_1, \cdots, \omega_m]$ for every m. So Massey's result can be restated as the following.

Theorem 6.8. *(Massey [60]) Let M^n be a closed n-dimensional manifold, and let $\nu_M : M^n \to BO$ classify its stable normal bundle. Then*

$$\omega_i(\nu_M) = 0$$

for all $i > n - \alpha(n)$.

For a closed n-manifold M^n, let $I_{M^n} \subset H^*(BO; \mathbb{Z}/2) \cong \mathbb{Z}/2[w_1, \cdots, w_i, \cdots]$ be the kernel of the stable normal bundle homomorphism, $\nu^* : H^*(BO; \mathbb{Z}/2) \to H^*(M^n; \mathbb{Z}/2)$. Let I_n be the intersection

$$I_n = \bigcap_{M^m} I_{M^n}.$$

Here the intersection is taken over all closed n-manifolds. I_n is an ideal in $\mathbb{Z}/2[w_1, \cdots, w_i, \cdots]$, and by Massey's result we know that $w_i \in I_n$ for all $i > n - \alpha(n)$. In [17] [18] Brown and Peterson computed I_n explicitly, thus refining Massey's theorem. In [19] they went further and constructed a "universal space" for normal bundles of n-manifolds, and proved the following theorem.

Theorem 6.9. *(Brown and Peterson [19]). For every n there is a space BO/I_n equipped with a map $\rho_n : BO/I_n \to BO$ satisfying the following properties.*

1. *In cohomology $\rho_n^* : H^*(BO; \mathbb{Z}/2) \to H^*BO/I_n; \mathbb{Z}/2)$ is surjective, with kernel I_n. That is, ρ_n^* induces an isomorphism*

$$H^*(BO/I_n; \mathbb{Z}/2) \cong H^*(BO; \mathbb{Z}/2)/I_n.$$

2. *Every closed n-manifold M^n admits a map $\tilde{\nu}_{M^n} : M^n \to BO/I_n$ such that the composition*

$$M^n \xrightarrow{\tilde{\nu}_{M^n}} BO/I_n \xrightarrow{\rho_n} BO$$

is homotopic to the stable normal bundle map $\nu_{M^n} : M^n \to BO$.

Notice that by combining the work of Massey and Brown-Peterson, we have the following commutative diagram for every closed n-manifold M^n:

$$\begin{array}{ccc} H^*(BO;\mathbb{Z}/2) & \xrightarrow{\iota^*_{n-\alpha(n)}} & H^*(BO(n-\alpha(n);\mathbb{Z}/2) \\ \nu^*_{M^n}\downarrow & & \downarrow \rho^*_n \\ H^*(M^n;\mathbb{Z}/2) & \xleftarrow[\tilde{\nu}^*_{M^n}]{} & H^*(BO;\mathbb{Z}/2))/I_n \end{array}$$

Brown and Peterson's work [19] can be viewed as realizing a part of this cohomology diagram as coming from a diagram of spaces:

$$\begin{array}{ccc} BO & \xleftarrow{\iota_{n-\alpha(n)}} & BO(n-\alpha(n)) \\ {\scriptstyle \nu_{M^n}}\uparrow & & \\ M^n & \xrightarrow[\tilde{\nu}_{M^n}]{} & BO/I_n \end{array}$$

In [21] the topological realization of this cohomology diagram was made complete when the author proved the following.

Theorem 6.10. *([21]) For every n there is a map $\tilde{\rho}_n : BO/I_n \to BO(n - \alpha(n))$ such that the composition $BO/I_n \xrightarrow{\tilde{\rho}_n} BO(n-\alpha(n)) \xrightarrow{\iota_{n-\alpha(n)}} BO$ is homotopic to $\rho_n : BO/I_n \to BO$ as in Theorem 6.9.*

Now let M^n be an n-manifold, and let $\tilde{\nu}_{M^n} : M^n \to BO/I_n$ be as in Theorem 6.9. Combining Theorem 6.9 with Theorem 6.10 implies that the composition

$$\tilde{\nu}_{M^n} : M^n \xrightarrow{\tilde{\nu}_{M^n}} BO/I_n \xrightarrow{\tilde{\rho}_n} BO(n-\alpha(n))$$

factors (up to homotopy) the stable normal bundle map $\nu_{M^n} : M^n \to BO$. Then by Theorem 6.7 we can conclude the following theorem.

Theorem 6.11. *([21]) Every closed n-manifold M^n admits an immersion*

$$j_{M^n} : M^n \looparrowright \mathbb{R}^{2n-\alpha(n)}.$$

We end this section by describing why this is the best possible result. That is, the answer to Question 2 above, which asks what is the smallest integer $\psi(n)$ such that every closed n-manifold immerses in $\mathbb{R}^{n+\psi(n)}$ is $\psi(n) = n - \alpha(n)$.

We will actually describe a closed manifold M^n whose normal Stiefel-Whitney class, $w_{n-\alpha(n)}(\nu_{M^n})$ is nonzero. This would then supply an obstruction to immersing M^n into $\mathbb{R}^{2n-\alpha(n)-1}$.

The manifold M^n can be described as follows. Write n as a sum of distinct powers of 2:

$$n = 2^{i_1} + 2^{i_2} + \cdots + 2^{i_r}.$$

Note that r, the number of distinct powers of 2 in this description, is equal to $\alpha(n)$. We then define

$$M^n = \mathbb{RP}^{2^{i_1}} \times \mathbb{RP}^{2^{i_2}} \times \cdots \times \mathbb{RP}^{2^{i_r}}.$$

We then need to prove the following.

Proposition 6.12. *The normal Stiefel-Whitney class*

$$w_{n-\alpha(n)}(\nu_{M^n}) \in H^{n-\alpha(n)}(M^n; \mathbb{Z}/2)$$

is nonzero.

Proof. . We first observe that the case when n is a power of 2 was proved in Corollary 5.28. This used the fact that

$$\bar{w}(\mathbb{RP}^m) = w(\nu_{\mathbb{RP}^m}) = \frac{1}{w(\mathbb{RP}^m)} = \frac{1}{(1+a)^{m+1}} \in \prod_k H^k(\mathbb{RP}^m;\mathbb{Z}/2),$$

where $a \in H^1(\mathbb{RP}^m;\mathbb{Z}/2) \cong \mathbb{Z}/2$ is the generator. This was proved in the example following Proposition 5.27 above. In particular,

$$\bar{w}(\mathbb{RP}^{2^j}) = w(\nu_{\mathbb{RP}^{2^j}}) = 1 + a + a^2 \cdots + a^{2^j-1}.$$

We now turn to the general case.

Write $n = 2^{i_1} + 2^{i_2} + \cdots + 2^{i_r}$ as above, and let $M^n = \mathbb{RP}^{2^{i_1}} \times \cdots \times \mathbb{RP}^{2^{i_r}}$. Then the total normal Stiefel-Whitney class is given by

$$\bar{w}(M^n) = w(\nu_{M^n}) = \otimes_{j=1}^r \bar{w}(\mathbb{RP}^{2^{i_j}}) = \otimes_{j=1}^r (1 + a_j + \cdots a_j^{2^{i_j}-1}) \in \otimes_{j=1}^r H^*(\mathbb{RP}^{2^{i_j}};\mathbb{Z}/2) \cong H^*(M^n;\mathbb{Z}/2).$$

Notice that the highest dimensional nonzero monomial in this expression is

$$a^{2^{i_1}-1} \otimes \cdots \otimes a^{2^{i_r}-1}$$

which lies in dimension $\sum_{j=1}^r (2^{i_j} - 1) = n - r = n - \alpha(n)$. Thus

$$\bar{w}_{n-\alpha(n)}(\nu_{M^n}) = a^{2^{i_1}-1} \otimes \cdots \otimes a^{2^{i_r}-1} \in H^{n-\alpha(n)}(\prod_{j=1}^r \mathbb{RP}^{2^{i_j}};\mathbb{Z}/2) = H^{n-\alpha(n)}(M^n;\mathbb{Z}/2),$$

and this class is clearly nonzero. □

To summarize, this proposition says that for M^n defined as the product of projective spaces as above, then $w_{n-\alpha(n)}(\nu_{M^n}) \in H^{n-\alpha(n)}(M^n;\mathbb{Z}/2)$ is nonzero. Thus, even though M^n admits an immersion into $\mathbb{R}^{2n-\alpha(n)}$, there is no immersion of M^n into $\mathbb{R}^{2n-\alpha(n)-1}$. In particular this says that the answer to Question 2 above is $\psi(n) = n - \alpha(n)$.

6.2 "Turning a sphere inside-out".

In the last subsection we used the Smale-Hirsch theorem (Theorem 6.5) to discuss the existence or nonexistence of immersions of manifolds into Euclidean spaces of varying dimensions. In this subsection we discuss the first application of this theorem, which was to show that any two immersions of S^2 into $\mathbb{R}^3$ are *isotopic* (sometimes referred to as "regularly homotopic").

Specifically we will give Smale's proof of his famous theorem saying that the identity embedding $\iota : S^2 \hookrightarrow \mathbb{R}^3$ defined by $\iota(x,y,z) = (x,y,z)$, is isotopic as immersions to its opposite, $-\iota(x,y,z) = -(x,y,z)$. That is, there exists a one-parameter family of immersions connecting ι to $-\iota$. Such a one parameter family is called an "eversion" of the sphere. The fact that such an eversion exists is perhaps counter-intuitive. It is sometimes described as "turning the sphere inside out", and indeed there are now videos showing such eversions. However Smale's original proof was a nonconstructive one, which relied on (an early version of) Theorem 6.5.

Notice that the statement that two immersions $f, g : M \looparrowright N$ are isotopic (or "regularly homotopic") is equivalent to the statement that f and g lie in the same path component of $Imm(M,N)$. To prove that the immersions ι and j of S^2 into $\mathbb{R}^3$ are isotopic, Smale proved the following:

Theorem 6.13. *(Smale [86]) The space $Imm(S^2, \mathbb{R}^3)$ is path connected.*

Note. The proof of this theorem uses some of the basics of the homotopy theory of fibrations developed in the next chapter. The student may want to delay the proof of this theorem until these homotopy theoretic techniques have been learned. We place the proof of this theorem in this chapter though, because of its historical impact on the development of immersion theory.

Proof. By Theorem 6.5 one has a weak homotopy equivalence

$$D : Imm(S^2, \mathbb{R}^3) \xrightarrow{\simeq} Mono(TS^2, T\mathbb{R}^3).$$

We can think about the space $Mono(TS^2, T\mathbb{R}^3)$ in the following way. Consider the fiber bundle

$$Mono(\mathbb{R}^2, \mathbb{R}^3) \to I(TS^2, \mathbb{R}^3) \xrightarrow{p} S^2 \tag{6.1}$$

where $I(TS^2, \mathbb{R}^3)$ is defined to be the space

$$I(TS^2, \mathbb{R}^3) = \{(x, \psi) \,:\, x \in S^2, \quad \text{and } \psi : T_xS^2 \to \mathbb{R}^3 \text{ is a linear monomorphism.}\}$$

Then $p(x, \psi) = x \in S^2$. So each fiber of p is equivalent to the Stiefel manifold $V_{2,3} = Mono(\mathbb{R}^2, \mathbb{R}^3)$. Notice that $V_{2,3}$ has the homotopy type of $O(3)/O(1) \cong SO(3)$. This is true by the following reasoning. Using the Gram-Schmidt process, one sees that $Mono(\mathbb{R}^2, \mathbb{R}^3)$ is homotopy equivalent to the space of inner-product preserving monomorphisms, $Mono^{<,>}(\mathbb{R}^2, \mathbb{R}^3)$. Now this space has a transitive action of the orthogonal group $O(3)$, and the isotropy subgroup of the inclusion of $\mathbb{R}^2$ in $\mathbb{R}^3$ given by $(x,y) \to (0,x,y)$ is $O(1) < O(3)$.

Notice there is a natural homeomorphism

$$Mono(TS^2, T\mathbb{R}^3) \xrightarrow{\cong} \Gamma_{S^2}(I(TS^2, \mathbb{R}^3))$$

where $\Gamma_{S^2}(I(TS^2,\mathbb{R}^3))$ is the space of (differentiable) sections of the bundle (6.1). To prove the theorem it then suffices to prove the following.

Lemma 6.14. *The space of sections $\Gamma_{S^2}(I(TS^2,\mathbb{R}^3))$ is path connected.*

Proof. For ease of notation let γ represent the space $\Gamma_{S^2}(I(TS^2,\mathbb{R}^3)) \cong Mono(TS^2, T\mathbb{R}^3) \simeq Imm(S^2,\mathbb{R}^3)$. Let α, and $\beta \in \gamma$ be any two sections. We will show that they live in the same path component of γ. Write $S^2 = \mathbb{R}^2 \cup \infty$, and fix an identification of $T_\infty S^2$ with $\mathbb{R}^2$. Without loss of generality we may assume that

$$\alpha(\infty) = \beta(\infty) = (\infty, \iota) \in I(TS^2,\mathbb{R}^3)$$

where $\iota : T_\infty S^2 \cong \mathbb{R}^2 \hookrightarrow \mathbb{R}^3$ is the natural inclusion $(u,v) \to (0,u,v)$. This is because the group $SO(3)$ acts transitively on $V_{2,3}$, and so one may rotate α and β if necessary so that they satisfy this basepoint relation. Since $SO(3)$ is connected such rotations preserve the path components of α and β.

So we may assume that α and β lie in $\gamma_b \subset \gamma$ which we define to be the space of sections satisfying this basepoint condition. Notice that γ_b can be viewed as a subspace of the space of all maps S^2 to $I(TS^2,\mathbb{R}^3)$ that take ∞ to (∞,ι). This is the two-fold based loop space $\Omega^2 I(TS^2,\mathbb{R}^3)$. Indeed γ_b is exactly that subspace of $\Omega^2 I(TS^2,\mathbb{R}^3)$ which maps to the identity element in $\Omega^2 S^2$ under the map $\Omega^2 p : \Omega^2 I(TS^2,\mathbb{R}^3) \to \Omega^2 S^2$. This map, being the two-fold loop map of the fibration (6.1), defines a fibration

$$\Omega^2 V_{2,3} \to \Omega^2 I(TS^2,\mathbb{R}^3) \xrightarrow{\Omega^2 p} \Omega^2 S^2. \tag{6.2}$$

We make a couple of observations about this fibration. First recall that the homotopy group $\pi_2(V_{2,3}) = \pi_2(SO(3)) = 0$. This is because $SO(3) \cong \mathbb{RP}^3$ and the universal cover of $\mathbb{RP}^3$ is S^3, whose second homotopy group vanishes. This implies that $\Omega^2\mathbb{RP}^3 = \Omega^2 SO(3) \cong \Omega^2 V_{2,3}$ is path connected. By considering this fibration sequence one then deduces that there is a bijection between the path components of $\Omega^2 I(TS^2,\mathbb{R}^3)$ and $\Omega^2 S^2$. In fact this bijection is an isomorphism between abelian groups. This is because the path components of two-fold loop spaces are abelian groups and $\Omega^2 p$ is a map that preserves this two-fold loop structure. Thus we may conclude that

$$\pi_0(\Omega^2 I(TS^2,\mathbb{R}^3)) \cong \pi_0(\Omega^2 S^2) \cong \pi_2(S^2) \cong \mathbb{Z}.$$

Furthermore, observe that the path components of a two-fold loop space are all homotopy equivalent. This is seen as follows. Let $\Omega^2 Y$ be a two-fold loop space. Let g and h represent elements of this space and $\Omega^2_g Y$ and $\Omega^2 hY$ be the path components of this space containing g and h respectively. "Multiplying by $g^{-1}h$ defines a map $\times g^{-1}h : \Omega^2_g Y \to \Omega^2_h Y$ which has homotopy inverse $\times h^{-1}g : \Omega^2_h Y \to \Omega^2_g Y$.

We conclude that we can restrict two-fold loop fibration (6.2) to any path component of $\Omega^2 S^2$ to obtain a homotopy fibration sequence

$$\Omega^2 V_{2,3} \to \Omega^2_{[n]} I(TS^2,\mathbb{R}^3) \xrightarrow{\Omega^2 p} \Omega^2_{[n]} S^2.$$

(By "homotopy fibration sequence" we mean that the fibers are homotopy equivalent to $\Omega^2 V_{2,3}$). Here $\Omega^2_{[n]} S^2$ is the component of $\Omega^2 S^2$ containing maps of degree n. But notice that when $n = 1$, then by the definition of what a section means, γ_b is the fiber of $\Omega^2 p$ over the identity map of S^2, $id \in \Omega^2_{[1]} S^2$. We may then conclude that $\gamma_b \simeq \Omega^2 V_{2,3}$, which as just observed, is path connected. In particular our original sections α and β in γ live in the same path component. □

□

Exercises

Let M^n be a closed differentiable manifold, and let $e_0 : M^n \looparrowright \mathbb{R}^N$ and $e_1 \looparrowright \mathbb{R}^N$ be two immersions of M^n. We say that e_0 and e_1 are *isotopic* if there is a one-parameter family of immersions connecting e_0 and e_1. That is, e_0 and e_1 are isotopic if there is a continuous map $H : M^n \times [0,1] \to \mathbb{R}^N$ so that

- $H(x,0) = e_0(x)$ and $H(x,1) = e_1(x)$ for all $x \in M^n$
- The map $H_t : M^n \to \mathbb{R}^N$ defined by $H_t(x) = H(x,t)$ is a differentiable immersion for every $t \in [0,1]$.

Smale's theorem about "turning a sphere inside out" says that any two immersions $S^2 \looparrowright \mathbb{R}^3$ are isotopic.

1. Show, however, that there are infinitely many distinct isotopy classes of immersions $S^1 \looparrowright \mathbb{R}^2$. You may use Smale's theorem saying that the space of immersions $M \looparrowright \mathbb{R}^N$ is weakly homotopy equivalent to the space of bundle monomorphiisms $TM \to T\mathbb{R}^N$.
2. Describe a representative of each isotopy class you find.

Final Remark. In discussing eversions of spheres, we proved (ala Smale) that all immersion of S^2 in $\mathbb{R}^3$ are regularly homotopic (isotopic). Ultimately, using Hirsch-Smale theory, this was because $\pi_2(V_{2,3}) = 0$. However, somewhat surprisingly, there are infinitely many isotopy classes of immersions of S^2 into $\mathbb{R}^4$. This is because $\pi_2(V_{2,4}) \cong \mathbb{Z}$. We leave it to the reader to fill in the details of this striking result.

7

Homotopy Theory of Fibrations

In this chapter we study the basic algebraic topological properties of fiber bundles, and their generalizations, "Serre fibrations". We begin with a discussion of homotopy groups and their basic properties. We then show that fibrations yield long exact sequences in homotopy groups and use it to show that the loop space of the classifying space of a group is homotopy equivalent to the group. We then develop basic obstruction theory for liftings in fibrations, use it to interpret characteristic classes as obstructions, and apply them in several geometric contexts, including vector fields, Spin structures, and classification of $SU(2)$ - bundles over four dimensional manifolds. We also use obstruction theory to prove the existence of Eilenberg - MacLane spaces, and to prove their basic property of classifying cohomology. We then develop the theory of spectral sequences and then discuss the famous Leray - Serre spectral sequence of a fibration. We use it in several applications, including a proof of the theorem relating homotopy groups and homology groups, a calculation of the homology of the loop space ΩS^n, and a calculation of the homology of the Lie groups $U(n)$ and $O(n)$.

7.1 Homotopy Groups

We begin by adopting some conventions and notation. In this chapter, unless otherwise specified, we will assume that all spaces are connected and come equipped with a basepoint. When we write $[X, Y]$ we mean homotopy classes of basepoint preserving maps $X \to Y$. Suppose $x_0 \in X$ and $y_0 \in Y$ are the basepoints. Then a basepoint preserving homotopy between basepoint preserving maps f_0 and $f_1 : X \to Y$ is a map

$$F : X \times I \to Y$$

such that each $F_t : X \times \{t\} \to Y$ is a basepoint preserving map and $F_0 = f_0$ and $F_1 = f_1$. If $A \subset X$ and $B \subset Y$, are subspaces that contain the basepoints, ($x_0 \in A$, and $y_0 \in B$), we write $[X, A; Y, B]$ to mean homotopy classes of maps $f : X \to Y$ so that the restriction $f_{|_A}$ maps A to B. Moreover homotopies are assumed to preserve these subsets as well. That is, a homotopy defining this equivalence relation is a map $F : X \times I \to Y$ that restricts to a basepoint preserving homotopy $F : A \times I \to B$. We can now give a careful strict definition of homotopy groups.

Definition 7.1. *The n^{th} homotopy group of a space X with basepoint $x_0 \in X$ is defined to be the set*

$$\pi_n(X) = \pi_n(X, x_0) = [S^n, X].$$

Equivalently, this is the set

$$\pi_n(X) = [D^n, S^{n-1}; X, x_0]$$

where $S^{n-1} = \partial D^n$ is the boundary sphere.

Exercise. Prove that these two definitions are in fact equivalent.

Remarks. 1. It will often helpful to us to use as our model of the disk D^n the n - cube $I^n = [0,1]^n$. Notice that in this model the boundary ∂I^n consists of n - tuples $(t_1, \cdots, t_n)$ with $t_i \in [0,1]$ where at least one of the coordinates is either 0 or 1.
2. Notice that for $n = 1$, this definition of the first homotopy group is the usual definition of the fundamental group.

So far the homotopy "groups" have only been defined as sets. We now examine the group structure. To do this, we will define our homotopy groups via the cube I^n, which we give the basepoint $(0, \cdots, 0)$. Let

$$f \quad \text{and} \quad g : (I^n, \partial I^n) \longrightarrow (X, x_0)$$

be two maps representing elements $[f]$ and $[g] \in \pi_n(X, x_0)$. Define

$$f \cdot g : I^n \longrightarrow X$$

by

$$f \cdot g(t_1, t_2, \cdots, t_n) = \begin{cases} f(2t_1, t_2, \cdots, t_n) & \text{for } t_1 \in [0, 1/2] \\ g(2t-1, t_2, \cdots, t_n) & \text{for } t_1 \in [1/2, 1] \end{cases}$$

The map $f \cdot g : (I^n, \partial I^n) \to (X, x_0)$ represents the product of the classes

$$[f \cdot g] = [f] \cdot [g] \in \pi_n(X, x_0).$$

Notice that in the case $n = 1$ this is precisely the definition of the product structure on the fundamental group $\pi_1(X, x_0)$. The same proof that this product structure is well defined and gives the fundamental group the structure of an associative group extends to prove that all of the homotopy groups are in fact groups under this product structure. We leave the details of checking this to the reader. We refer the reader to any introductory textbook on algebraic topology for the details.

As we know the fundamental group of a space can be quite complicated. Indeed any group can be the fundamental group of a space. In particular fundamental groups can be very much noncommutative. However we recall the relation of the fundamental group to the first homology group, for which we again refer the reader to any introductory textbook:

Theorem 7.1. *Let X be a connected space. Then the abelianization of the fundamental group is isomorphic to the first homology group,*

$$\pi_1(X)/[\pi_1, \pi_1] \cong H_1(X)$$

where $[\pi_1, \pi_1]$ is the commutator subgroup of $\pi_1(X)$.

We also have the following basic result about higher homotopy groups.

Proposition 7.2. *For $n \geq 2$, the homotopy group $\pi_n(X)$ is abelian.*

Proof. Let $[f]$ and $[g]$ be elements of $\pi_n(X)$ represented by basepoint preserving maps $f : (I^n, \partial I^n) \to (X, x_0)$ and $g : (I^n, \partial I^n) \to (X, x_0)$, respectively. We need to find a homotopy between the product maps $f \cdot g$ and $g \cdot f$ defined above. The following schematic diagram suggests such a homotopy. We leave it to the reader to make this into a well defined homotopy.

□

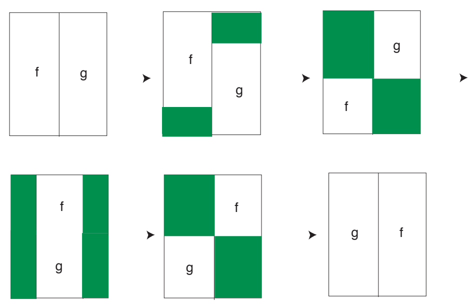
Now assume $A \subset X$ is a subspace containing the basepoint $x_0 \in A$.

Definition 7.2. *For $n \geq 1$ we define the relative homotopy group $\pi_n(X, A) = \pi_n(X, A, x_0)$ to be homotopy classes of maps of pairs*

$$\pi_n(X, A) = [(D^n, \partial D^n, t_0); (X, A, x_0)].$$

where $t_0 \in \partial D^n = S^{n-1}$ and $x_0 \in A$ are the basepoints.

Exercise. Show that for $n > 1$ the relative homotopy group $\pi_n(X, A)$ is in fact a group. Notice here that the zero element is represented by any basepoint preserving map of pairs$f : (D^n, \partial I^n) \to (X, A)$ that is homotopic (through maps of pairs) to one whose image lies entirely in $A \subset X$.



Again, let $A \in X$ be a subset containing the basepoint $x_0 \in A$, and let $i : A \hookrightarrow X$ be the inclusion. This induces a homomorphism of homotopy groups

$$i_* : \pi_n(A, x_0) \to \pi_n(X, x_0).$$

Also, by ignoring the subsets, a basepoint preserving map $f : (D^n, \partial D^n) \to (X, x_0)$ defines a map of pairs $f : (D^n, \partial D^n, t_0) \to (X, A, x_0)$ which defines a homomorphism

$$j_* : \pi_n(X, x_0) \to \pi_n(X, A, x_0).$$

Notice furthermore, that by construction, the composition

$$j_* \circ i_* : \pi_n(A) \to \pi_n(X) \to \pi_n(X, A)$$

is zero. Finally, if given a map of pairs $g : (D^n, S^{n-1}, t_0) \to (X, A, x_0)$, then we can restrict g to the boundary sphere S^{n-1} to produce a basepoint preserving map

$$\partial g : (S^{n-1}, t_0) \to (A, x_0).$$

This defines a homomorphism

$$\partial_* : \pi_n(X, A, x_0) \to \pi_{n-1}(A, x_0).$$

Notice here that the composition

$$\partial_* \circ j_* : \pi_n(X) \to \pi_n(X, A) \to \pi_{n-1}(A)$$

is also zero, since the application of this composition to any representing map $f : (D^n, S^{n-1}) \to (X, x_0)$ yields the constant map $S^{n-1} \to x_0 \in A$. We now have the following fundamental property of homotopy groups. Compare with the analogous theorem in homology.

Theorem 7.3. *Let $A \subset X$ be a subspace containing the basepoint $x_0 \in A$. Then we have a long exact sequence in homotopy groups*

$$\cdots \xrightarrow{\partial_*} \pi_n(A) \xrightarrow{i_*} \pi_n(X) \xrightarrow{j_*} \pi_n(X, A) \xrightarrow{\partial_*} \pi_{n-1}(A) \to \cdots$$
$$\xrightarrow{\partial_*} \pi_1(A) \xrightarrow{i_*} \pi_1(X) \to$$

Proof. We've already observed that $j_* \circ i_*$ and $\partial_* \circ j_*$ are zero. Similarly, $i_* \circ \partial_*$ is zero because an element in the image of ∂_* is represented by a basepoint preserving map $S^{n-1} \to A$ that extends to a map $D^n \to X$. Thus the image under i_*, namely the composition $S^{n-1} \to A \hookrightarrow X$ has an extension to D^n and is therefore null homotopic. We therefore have

$$image(\partial_*) \subset kernel(i_*)$$
$$image(i_*) \subset kernel(j_*)$$
$$image(j_*) \subset kernel(\partial_*).$$

To finish the proof we need to show that all of these inclusions are actually equalities. Consider the kernel of (i_*). An element $[f] \in \pi_n(A)$ is in $ker(i_*)$ if and only if the basepoint preserving composition $f : S^n \to A \subset X$ is null homotopic. Such a null - homotopy gives an extension of this map to the disk $F : D^{n+1} \to X$. The induced map of pairs $F : (D^{n+1}, S^n) \to (X, A)$ represents an element in $\pi_{n+1}(X, A)$ whose image under ∂_* is $[f]$. This proves that $image(\partial_*) = kernel(i_*)$. The other equalities are proved similarly, and we leave their verification to the reader. □

Remark. Even though this theorem is analogous to the existence of exact sequences for pairs in homology, notice that its proof is much easier.

Notice that $\pi_0(X)$ is the set of path components of X. So a space is (path) - connected if and only if $\pi_0(X) = 0$ (i.e the set with one element). We generalize this notion as follows.

Definition 7.3. *A space X is said to be m - connected if $\pi_q(X) = 0$ for $0 \leq q \leq m$.*

We now do our first calculation.

Proposition 7.4. *An n - sphere is $n-1$ connected.*

Proof. We need to show that any map $S^k \to S^n$, where $k < n$ is null homotopic. Now since spheres can be given the structure of simplicial complexes, the simplicial approximation theorem says that any map $f : S^k \to S^n$ is homotopic to a simplicial map (after suitable subdivisions). So we assume without loss of generality that f is simplicial. But since $k < n$, the image of of f lies in the k - skeleton of the n - dimensional simplicial complex S^n. In particular this means that $f : S^k \to S^n$ is not surjective. Let $y_0 \in S^n$ be a point that is not in the image of f. Then f has image in $S^n - y_0$ which is homeomorphic to the open disk D^n, and is therefore contractible. This implies that f is null homotopic. □

7.2 Fibrations

Recall that in chapter 2 we proved that locally trivial fiber bundles satisfy the Covering Homotopy Theorem 4.2. A generalization of the notion of a fiber bundle, due to Serre, is simply a map that satisfies this type of lifting property.

Definition 7.4. *A Serre fibration is a surjective, continuous map $p : E \to B$ that satisfies the* Homotopy Lifting Property for CW - complexes. That is, if X is any CW - complex and $F : X \times I \to B$ is any continuous homotopy so that $F_0 : X \times \{0\} \to B$ factors through a map $f_0 : X \to E$, then there exists a lifting $\bar{F} : X \times I \to E$ that extends f_0 on $X \times \{0\}$, and makes the following diagram commute:

$$\begin{array}{ccc} X \times I & \xrightarrow{\bar{F}} & E \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle p} \\ X \times I & \xrightarrow[F]{} & B. \end{array}$$

A Hurewicz fibration is a surjective, continuous map $p : E \to B$ that satisfies the homotopy lifting property for all spaces.

Remarks. 1. Obviously every Hurewicz fibration is a Serre fibration. The converse is false. In these notes, unless otherwise stated, we will deal with Serre fibrations, which we will simply refer to as fibrations.
2. The Covering Homotopy Theorem implies that a fiber bundle is a fibration in this sense.

The following is an important example of a fibration.

Proposition 7.5. *Let X be any connected space with basepoint $x_0 \in X$. Let PX denote the space of based paths in X. That is,*

$$PX = \{\alpha : I \to X : \alpha(0) = x_0\}.$$

The path space PX is topologized using the compact - open function space topology. Define

$$p : PX \to X$$

by $p(\alpha) = \alpha(1)$. Then PX is a contractible space, and the map $p : PX \to X$ is a fibration, whose fiber at x_0, $p^{-1}(x_0)$ is the loop space ΩX.

Proof. The fact that PX is contractible is straightforward. For a null homotopy of the identity map one can take the map $H : PX \times I \to PX$, defined by $H(\alpha, s)(t) = \alpha((1-s)t)$.

To prove that $p : PX \to X$ is a fibration, we need to show it satisfies the Homotopy Lifting Property. So let $F : Y \times I \to X$ and $f_0 : X \to PX$ be maps making the following diagram commute:

$$\begin{array}{ccc} Y \times \{0\} & \xrightarrow{f_0} & PX \\ \cap\downarrow & & \downarrow p \\ Y \times I & \xrightarrow[F]{} & X \end{array}$$

Then we can define a homotopy lifting, $\bar{F} : Y \times I \to PX$ by defining for $(y, s) \in Y \times I$, the path

$$\bar{F}(y,s) : I \to X$$
$$\bar{F}(y,s)(t) = \begin{cases} f_0(y)(\frac{2t}{2-s}) & \text{for } t \in [0, \frac{2-s}{2}] \\ F(y, 2t-2+s) & \text{for } t \in [\frac{2-s}{2}, 1] \end{cases}$$

One needs to check that this definition makes $\bar{F}(y,s)(t)$ a well defined continous map and satisfies the boundary conditions

$$\bar{F}(y,0)(t) = f_0(y,t)$$
$$\bar{F}(y,s)(0) = x_0$$
$$\bar{F}(y,s)(1) = F(y,s)$$

These verifications are all straightforward. □

The following is just the observation that one can pull back the Homotopy Lifting Property.

Proposition 7.6. *Let $p : E \to B$ be a fibration, and $f : X \to B$ a continuous map. Then the pull back, $p_f : f^*(E) \to X$ is a fibration, where*

$$f^*(E) = \{(x, e) \in X \times E \, such \; that \, f(x) = p(e)\}$$

and $p_f(x, e) = x$.

The following shows that in the setting of homotopy theory, every map can be viewed as a fibration in this sense.

Theorem 7.7. *Every continuous map $f : X \to Y$ is homotopic to a fibration in the sense that there exists a fibration*

$$\tilde{f} : \tilde{X} \to Y$$

and a homotopy equivalence

$$h : X \xrightarrow{\simeq} \tilde{X}$$

making the following diagram commute:

$$\begin{array}{ccc} X & \xrightarrow[\simeq]{h} & \tilde{X} \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \tilde{f}} \\ Y & = & Y. \end{array}$$

Proof. Define $\tilde{X}$ to be the space

$$\tilde{X} = \{(x, \alpha) \in X \times Y^I \; \text{such that} \, \alpha(0) = x.\}$$

where here Y^I denotes the space of continuous maps $\alpha : [0, 1] \to Y$ given the compact open topology. The map $\tilde{f} : \tilde{X} \to Y$ is defined by $\tilde{f}(x, \alpha) = \alpha(1)$. The fact that $\tilde{f} : \tilde{X} \to Y$ is a fibration is proved in the same manner as theorem 7.5, and so we leave it to the reader.

Define the map $h : X \to \tilde{X}$ by $h(x) = (x, \epsilon_x) \in \tilde{X}$, where $\epsilon_x(t) = x$ is the constant path at $x \in X$. Clearly $\tilde{f} \circ h = f$ so the diagram in the statement of the theorem commutes. Now define $g : \tilde{X} \to X$ by $g(x, \alpha) = x$. Clearly $g \circ h$ is the identity map on X. To see that $h \circ g$ is homotopic to the identity on $\tilde{X}$, consider the homotopy $F : \tilde{X} \times I \to \tilde{X}$, defined by $F((x, \alpha), s) = (x, \alpha_s)$, where $\alpha_s : I \to X$ is the path $\alpha_s(t) = \alpha(st)$. So in particular $\alpha_0 = \epsilon_x$ and $\alpha_1 = \alpha$. Thus F is a homotopy between $h \circ g$ and the identity map on $\tilde{X}$. Thus h is a homotopy equivalence, which completes the proof of the theorem. □

The homotopy fiber of a map $f : X \to Y$, F_f, is defined to be the fiber of the fibration $\tilde{f} : \tilde{X} \to Y$ defined in the proof of this theorem. That is,

Definition 7.5. *The homotopy fiber F_f of a basepoint preserving map $f : X \to Y$ is defined to be*

$$F_f = \{(x, \alpha) \in X \times Y^I \text{ such that } \alpha(0) = f(x) \text{ and } \alpha(1) = y_0.\}$$

where $y_0 \in Y$ is the basepoint.

So for example, the homotopy fiber of the inclusion of the basepoint $y_0 \hookrightarrow Y$ is the loop space ΩY. The homotopy fiber of the identity map $id : Y \to Y$ is the path space PY. The homotopy fibers are important invariants of the map $f : X \to Y$.

The following is the basic homotopy theoretic property of fibrations.

Theorem 7.8. *Let $p : E \to B$ be a fibration over a connected space B with fiber F. So we are assuming the basepoint of E, is contained in F, $e_0 \in F$, and that $p(e_0) = b_0$ is the basepoint in B. Let $i : F \hookrightarrow E$ be the inclusion of the fiber. Then there is a long exact sequence of homotopy groups:*

$$\cdots \xrightarrow{\partial_*} \pi_n(F) \xrightarrow{i_*} \pi_n(E) \xrightarrow{p_*} \pi_n(B) \xrightarrow{\partial_*} \pi_{n-1}(F) \to$$
$$\cdots \to \pi_1(F) \xrightarrow{i_*} \pi_1(E) \xrightarrow{p_*} \pi_1(B).$$

Proof. Notice that the projection map $p : E \to B$ induces a map of pairs

$$p : (E, F) \to (B, b_0).$$

By the exact sequence for the homotopy groups of the pair (E, F), 7.3 it is sufficient to prove that the induced map in homotopy groups

$$p_* : \pi_n(E, F) \to \pi_n(B, b_0)$$

is an isomorphism for all $n \geq 1$. We first show that p_* is surjective. So let $f : (I^n, \partial I^n) \to (B, b_0)$ represent an element of $\pi_n(B)$. We can think of a map from a cube as a homotopy of maps of cubes of one lower dimension. Therefore by induction on n, the homotopy lifting property says that that $f : I^n \to B$ has a basepoint preserving lifting $\bar{f} : I^n \to E$. Since $p \circ \bar{f} = f$, and since the restriction of f to the boundary ∂I^n is constant at b_0, then the image of the restriction of $\bar{f}$ to the boundary ∂I^n has image in the fiber F. That is, $\bar{f}$ induces a map of pairs

$$\bar{f} : (I^n, \partial I^n) \to (E, F)$$

which in turn represents an element $[\bar{f}] \in \pi_n(E, F)$ whose image under p_* is $[f] \in \pi_n(B, b_0)$. This proves that p_* is surjective.

We now prove that $p_* : \pi_n(E, F) \to \pi_n(B, b_0)$ is injective. So let f :

$(D^n, \partial D^n) \to (E, F)$ be a map of pairs that represents an element in the kernel of p_*. That means $p \circ f : (D^n, \partial D^n) \to (B, b_0)$ is null homotopic. Let $F : (D^n, \partial D^n) \times I \to (B, b_0)$ be a null homotopy between $F_0 = f$ and the constant map $\epsilon : D^n \to b_0$. By the Homotopy Lifting Property there exists a basepoint preserving lifting

$$\bar{F} : D^n \times I \to E$$

having the properties that $p \circ \bar{F} = F$ and $\bar{F} : D^n \times \{0\} \to E$ is equal to $f : (D^n, \partial D^n) \to (E, F)$. Since $p \circ \bar{F} = F$ maps $\partial D^n \times I$ to the basepoint b_0, we must have that $\bar{F}$ maps $\partial D^n \times I$ to $p^{-1}(b_0) = F$. Thus $\bar{F}$ determines a homotopy of pairs,

$$\bar{F} : (D^n, \partial D^n) \times I \to (E, F)$$

with $\bar{F}_0 = f$. Now consider $\bar{F}_1 : (D^n, \partial D^n) \times \{1\} \to E$. Now $p \circ \bar{F}_1 = F_1 = \epsilon : D^n \to b_0$. Thus the image of $\bar{F}_1$ lies in $p^{-1}(b_0) = F$. Thus $\bar{F}$ gives a homotopy of the map of pairs $f : (D^n, \partial D^n) \to (E, F)$ to a map of pairs whose image lies entirely in F. Such a map represents the zero element of $\pi_n(E, F)$. This completes the proof that p_* is injective, and hence is an isomorphism. As observed earlier, this is what was needed to prove the theorem. □

We now use this theorem to make several important calculations of homotopy groups. In particular, we prove the following seminal result of Hopf.

Theorem 7.9.

$\pi_2(S^2) \cong \pi_3(S^3) \cong \mathbb{Z}$.
$\pi_k(S^3) \cong \pi_k(S^2)$ *for all* $k \geq 3$. *In particular,*
$\pi_3(S^2) \cong \mathbb{Z}$, *generated by the Hopf map* $\eta : S^3 \to S^2$.

Proof. Consider the Hopf fibration $\eta : S^3 \to S^2 = \mathbb{CP}^1$ with fiber S^1. Recall that S^1 is an Eilenberg - MacLane space $K(\mathbb{Z}, 1)$ since it is the classifying space of $\mathbb{Z}$. Thus

$$\pi_q(S^1) = \begin{cases} \mathbb{Z} & \text{for q} = 1 \\ 0 & \text{for all other } q. \end{cases}$$

(**Remark.** The fact that the classifying space $B\pi$ of a *discrete* group π is an Eilenberg - MacLane space $K(\pi, 1)$ can now be given a simpler proof, using the exact sequence in homotopy groups of the universal bundle $E\pi \to B\pi$.)

Using this fact in the exact sequence in homotopy groups for the Hopf fibration $\eta : S^3 \to S^2$, together with the fact that $\pi_q(S^3) = 0 \quad \text{for } q \leq 2$, one is led to the facts that $\pi_2(S^2) \cong \pi_1(S^1) = \mathbb{Z}$, and that $\eta_* : \pi_k(S^3) \to \pi_k(S^2)$ is an isomorphism for $k \geq 3$. To examine the case $k = 3$, consider the homomorphism (called the *Hurewicz homomorphism*)

$$h : \pi_3(S^3) \to H_3(S^3) = \mathbb{Z}$$

defined by sending a class represented by a self map $f : S^3 \to S^3$, to the image of the fundamental class in homology, $f_*([S^3]) \in H^3(S^3) \cong \mathbb{Z}$. Clearly this is a homomorphism (check this!). Moreover it is surjective since the image of the identity map is the fundamental class, and thus generates, $H_3(S^3)$, $H([id]) = [S^3] \in H_3(S^3)$. Thus $\pi_3(S^3)$ contains an integral summand generated by the identity. In particular, since $\eta_* : \pi_3(S^3) \to \pi_3(S^2)$ is an isomorphism, this implies that $\pi_3(S^2)$ contains an integral summand generated by the Hopf map $[\eta] \in \pi_3(S^2)$. The fact that these integral summands generate the entire groups $\pi_3(S^3) \cong \pi_3(S^2)$ will follow once we know that the Hurewicz homomorphism is an isomorphism in this case. Later in this chapter we will prove the more general "Hurewicz theorem" that says that for any $k > 1$, and any $(k-1)$ - connected space X, the Hurewicz homomorphism is an isomorphism in dimension k: $h : \pi_k(X) \cong H_k(X)$. □

Remark. As we remarked earlier in these notes. these were the first nontrivial elements found in the higher homotopy groups of spheres, $\pi_{n+k}(S^n)$, and Hopf's proof of their nontriviality is commonly viewed as the beginning of modern Homotopy Theory [102]

Before we continue to apply the notion of fibrations to homotopy theory, we point out that there is a dual notion of a *cofibration* that is also very important. Instead of satisfying a homotopy lifting property, cofibratons satisfy a *homotopy extension property,*

Definition 7.6. *A map $\iota : A \to X$ of topological spaces is called a cofibration if for any map $f : X \to Y$ and any homotopy*

$$H : A \times [0,1] \to Y$$

with $H(a,0) = f(\iota(a))$, then there is an extension of the homotopy H to $X \times I$,

$$\bar{H} : X \times [0,1] \to Y.$$

so that $\bar{H}(x,0) = f(x)$ for all $x \in X$, and $\bar{H}(\iota(a),t) = H(a,t)$ for all $a \in A$ and $t \in [0,1]$.

Exercise. Show that if X is a CW-complex and $A \subset X$ is a subcomplex then the inclusion map $\iota : A \hookrightarrow X$ is a cofibration.

Notice we have the following analogue of Theorem 7.7:

Theorem 7.10. *Every map $g : A \to X$ is homotopic to a cofibration in the sense that there is a space $\bar{X}$ equipped with a deformation retraction $j : X \xrightarrow{\simeq} \bar{X}$ and a cofibration*

$$\bar{g} : A \to \bar{X}$$

that is homotopic to $j \circ g : A \to X \xrightarrow{\simeq} \bar{X}$.

Proof. Define $\bar{X}$ to be the mapping cylinder

$$\bar{X} = X \cup (A \times I)/\sim$$

where $(a, 0) \in A \times I$ is identified with $g(a) \in X$. Define $\bar{g} : A \to \bar{X}$ to be the inclusion as $A \times \{0\}$. We leave it to the reader to verify that the pair $(\bar{X}, \bar{g} : A \to \bar{X})$ satisfies the required properties. □

Definition 7.7. *Let $\iota : A \to X$ be a cofibration. The cofiber of ι is the quotient space X/A defined to be*

$$X/A = X/\sim$$

where the equivalence relation is given by $\iota(a) \sim \iota(b)$ for any two points $a, b \in A$. Notice that in the case where ι is the inclusion of a subcomplex $\iota : A \subset X$ of a CW complex, the cofiber is the quotient complex, X/A.

Exercises.

1. Show that if $\iota : A \to X$ is a cofibration, it's cofiber X/A is homotopy equivalent to the mapping cone

$$X \cup_\iota c(A)$$

where $c(A) = A \times [0, 1]/A \times \{1\}$, and the notation $X \cup_\iota c(A)$ refers to the disjoint union of X with $c(A)$, modulo the identification $(a, 0) \in c(A)$ is identified with $\iota(a) \in X$ for all $a \in A$.

2. Show that if $\iota : A \to X$ is a cofibration, then there is an isomorphism of homology groups,

$$H_*(X, A) \cong \tilde{H}_*(X/A).$$

Remark. . Since any map $f : X \to Y$ is homotopic to a cofibration with cofiber the mapping cone $Y \cup_f c(X)$, the mapping cone is sometimes referred to as the *"homotopy cofiber"* of f. Notice furthermore that the inclusion of Y into the mapping cone,

$$Y \subset Y \cup_f c(X)$$

is a cofibration with cofiber the suspension $\Sigma X = c(X)/X \times \{0\}$.

We end this section with an application to the "homotopy stability" of the orthogonal and unitary groups, as well as their classifying spaces.

Theorem 7.11. *The inclusion maps*

$$\iota : O(n) \hookrightarrow O(n+1) \quad \textit{and}$$
$$U(n) \hookrightarrow U(n+1)$$

induce isomorphisms in homotopy groups through dimensions $n-2$ and $2n-1$ respectively. Also, the induced maps on classifying spaces,

$$B\iota : BO(n) \to BO(n+1) \quad \textit{and}$$
$$BU(n) \to BU(n+1)$$

induce isomorphisms in homotopy groups through dimensions $n-1$ and $2n$ respectively.

Proof. The first two statements follow from the existence of fiber bundles

$$O(n) \hookrightarrow O(n+1) \to S^n$$

and

$$U(n) \hookrightarrow U(n+1) \to S^{2n+1},$$

the connectivity of spheres 7.4, and by applying the exact sequence in homotopy groups to these fiber bundles. The second statement follows from the same considerations, after recalling from 4.28 the sphere bundles

$$S^n \to BO(n) \to BO(n+1)$$

and

$$S^{2n+1} \to BU(n) \to BU(n+1).$$

□

7.3 Obstruction Theory

In this section we discuss the obstructions to obtaining a lifting to the total space of a fibration of a map to the base space. As an application we prove the important "Whitehead theorem" in homotopy theory, and we prove general results about the existence of cross sections of principal $O(n)$ or $U(n)$ - bundles. We do not develop a formal theory here - we just develop what we will need for our applications to fibrations. For a full development of obstruction theory we refer the reader to [101].

Let X be a CW - complex. Recall that its cellular k - chains, $C_k(X)$ is the free abelian group generated by the k - dimensional cells in X. The co-chains with coefficients in a group G are defined by

$$C^k(X, G) = Hom(C_k(X), G).$$

Theorem 7.12. *Let $p : E \to B$ be a fibration with fiber F. Let $f : X \to B$ be a continuous map, where X is a CW - complex. Suppose there is a lifting of the $(k-1)$ - skeleton $\tilde{f}_{k-1} : X^{(k-1)} \to E$. That is, the following diagram commutes:*

$$\begin{array}{ccc} X^{(k-1)} & \xrightarrow{\tilde{f}_{k-1}} & E \\ \cap\downarrow & & \downarrow p \\ X & \xrightarrow[f]{} & B. \end{array}$$

Then the obstruction to the existence of a lifting to the k -skeleton, $\tilde{f}_k : X^{(k)} \to E$ that extends $\tilde{f}_{k-1}$, is a cochain $\gamma \in C^k(X; \pi_{k-1}(F))$. That is, $\gamma = 0$ if and only if such a lifting $\tilde{f}_k$ exists.

Proof. We will first consider the special case where $X^{(k)}$ is obtained from $X^{(k-1)}$ by adjoining a single k -dimensional cell. So assume

$$X^{(k)} = X^{(k-1)} \cup_\alpha D^k$$

where $\alpha : \partial D^k = S^{(k-1)} \to X^{k-1)}$ is the attaching map. We therefore have the following commutative diagram:

$$\begin{array}{ccccc} S^{k-1} & \xrightarrow{\alpha} & X^{(k-1)} & \xrightarrow{\tilde{f}_{k-1}} & E \\ \cap\downarrow & & \cap\downarrow & & \downarrow p \\ D^k & \xrightarrow[\subset]{} & X^{(k-1)} \cup_\alpha D^k & \xrightarrow[f]{} & B \end{array}$$

Notice that $\bar{f}_{k-1}$ has an extension to $X^{(k-1)} \cup_\alpha D^k = X^{(k)}$ that lifts f, if and only if the composition $D^k \subset X^{(k-1)} \cup_\alpha D^k \xrightarrow{f} B$ lifts to E in such a way that it extends $\bar{f}_{k-1} \circ \alpha$.

Now view the composition $D^k \subset X^{(k-1)} \cup_\alpha D^k \xrightarrow{f} B$ as a a map from the cone on S^{k-1} to B, or in other words, as a null homotopy$F : S^{k-1} \times I \to B$ from $F_0 = p \circ \bar{f}_{k-1} \circ \alpha : S^{k-1} \to X^{(k-1)} \to E \to B$ to the constant map $F_1 = \epsilon : S^{(k-1)} \to b_0 \in B$. By the Homotopy Lifting Property, F lifts to a homotopy

$$\bar{F} : S^{(k-1)} \times I \to E$$

with $\bar{F}_0 = \bar{f}_{k-1} \circ \alpha$. Thus the extension f_k exists on $X^{(k-1)} \cup_\alpha D^k$ if and only if this lifting $\bar{F}$ can be chosen to be a null homotopy of $\bar{f}_{k-1} \circ \alpha$. But we know $\bar{F}_1 : S^{k-1} \times \{1\} \to E$ lifts F_1 which is the constant map $\epsilon : S^{k-1} \to b_0 \in B$. Thus the image of $\bar{F}_1$ lies in the fiber F, and therefore determines an element $\gamma \in \pi_{k-1}(F)$. The homotopy $\bar{F}_1$ can be chosen to be a null homotopy if and only if $\bar{F}_1 : S^{k-1} \to F$ is null homotopic. (Because combining $\bar{F}$ with a null homotopy of $\bar{F}_1$, i.e an extension of $\bar{F}_1$ to a map $D^k \to F$, is still a lifting of

F, since the extension lives in a fiber over a point.) But this is only true if the homotopy class $\gamma = 0 \in \pi_{k-1}(F)$.

This proves the theorem in the case when $X^{(k)} = X^{(k-1)} \cup_\alpha D^k$. In the general case, suppose that $X^{(k)}$ is obtained from $X^{(k-1)}$ by attaching a collection of k - dimensional disks, indexed on a set, say J. That is,

$$X^{(k)} = X^{(k-1)} \bigcup_{j \in J} \cup_{\alpha_j} D^k.$$

The above procedure assigns to every $j \in J$ an "obstruction" $\gamma_j \in \pi_{k-1}(F)$. An extension $\bar{f}_k$ exists if and only if all these obstructions are zero. This assignment from the indexing set of the k - cells to the homotopy group can be extended linearly to give a homomorphism γ from the free abelian group generated by the k - cells to the homotopy group $\pi_{k-1}(F)$, which is zero if and only if the extension $\bar{f}_k$ exists. Such a homomorphism γ is a cochain, $\gamma \in C^k(X; \pi_{k-1}(F))$. This completes the proof of the theorem. □

We now discuss several applications of this obstruction theory.

Corollary 7.13. *Any fibration $p : E \to B$ over a CW - complex with an aspherical fiber F admits a cross section.*

Proof. Since $\pi_q(F) = 0$ for all q, by the theorem, there are no obstructions to constructing a cross section inductively on the skeleta of B. □

Proposition 7.14. *Let X be an n - dimensional CW - complex, and let ζ be an m - dimensional vector bundle over X, with $m \geq n$. Then ζ has $m - n$ linearly independent cross sections. If ξ is a d - dimensional complex bundle over X, then ξ admits $d - [n/2]$ linearly independent cross sections, where $[n/2]$ is the integral part of $n/2$.*

.

Proof. Let ζ be classified by a map $f_m : X \to BO(m)$. To prove the theorem we need to prove that f_m lifts (up to homotopy) to a map $f_n X \to BO(n)$. We would then have that

$$\zeta \cong f_m^*(\gamma_m) \cong f_n^*(\gamma_n) \oplus \epsilon_{m-n}$$

where γ_k is the universal k - dimensional vector bundle over $BO(k)$, and ϵ_j represents the j - dimensional trivial bundle. These isomorphisms would then produce the $m-n$ linearly independent cross sections of ζ. over X. Now recall there is a fibration

$$O(m)/O(n) \to BO(n) \to BO(m).$$

That is, the fiber of $p : BO(n) \to BO(m)$ is the quotient space $O(m)/O(n)$. Now by a simple induction argument using 7.11 shows that the fiber $O(m)/O(n)$ is $n-1$ connected. That is, $\pi_q(O(m)/O(n)) = 0$ for $q \leq n-1$. By 7.12 all obstructions vanish for lifting the n - skeleton of X to the total space $BO(n)$. Since we are assuming X is n - dimensional, this completes the proof. The complex case is proved similarly. □

Corollary 7.15. *Let X be a compact, n - dimensional CW complex. Then every element of the reduced real K - theory, $\tilde{K}O(X)$ can be represented by a n - dimensional vector bundle. Every element of the complex K - theory, $\tilde{K}(X)$ can be represented by an $[n/2]$ - dimensional complex vector bundle.*

Proof. By 4.32 we know

$$\tilde{K}O(X) \cong [X, BO] \quad \text{and}$$
$$\tilde{K}(X) \cong [X, BU].$$

But by the above proposition, any element $\alpha \in [X, BO]$ lifts to an element $\alpha_n \in [X, BO(n)]$ which in turn classifies an n - dimensional real vector bundle representing the $\tilde{K}O$ - class α.

Similarly, any element $\beta \in [X, BU]$ lifts to an element $\alpha_n \in [X, BU([n/2])]$ which in turn classifies an $[n/2]$ - dimensional complex vector bundle representing the $\tilde{K}$ - class β. □

We now use this obstruction theory to prove the well known "Whitehead Theorem", one of the most important foundational theorems in homotopy theory.

Theorem 7.16. *Suppose X and Y are CW - complexes and $f : X \to Y$ a continuous map that induces an isomorphism in homotopy groups,*

$$f_* : \pi_k(X) \xrightarrow{\cong} \pi_k(Y) \quad \textit{for all } k \geq 0$$

Then $f : X \to Y$ is a homotopy equivalence.

Proof. By 7.7 we can replace $f : X \to Y$ by a homotopy equivalent fibration

$$\tilde{f} : \tilde{X} \to Y.$$

That is, there is a homotopy equivalence $h : X \to \tilde{X}$ so that $\tilde{f} \circ h = f$. Since f induces an isomorphism in homotopy groups, so does $\tilde{f}$. By the exact sequence in homotopy groups for this fibration, this means that the fiber of the fibration $\tilde{f} : \tilde{X} \to Y$, i.e the homotopy fiber of f, is aspherical. thus by 7.12 there are no obstructions to finding a lifting $\tilde{g} : Y \to \tilde{X}$ of the identity map of Y. Thus $\tilde{g}$ is a section of the fibration, so that $\tilde{f} \circ \tilde{g} = id : Y \to Y$. Now let $h^{-1} : \tilde{X} \to X$ denote a homotopy inverse to the homotopy equivalence h. Then if we define

$$g = h^{-1} \circ \tilde{g} : Y \to X$$

we then have $f \circ g : Y \to Y$ is given by

$$\begin{aligned} f \circ g &= f \circ h^{-1} \circ \tilde{g} \\ &= \tilde{f} \circ h \circ h^{-1} \circ \tilde{g} \\ &\sim \tilde{f} \circ \tilde{g} \\ &= id : Y \to Y. \end{aligned}$$

Thus $f \circ g$ is homotopic to the identity of Y. To show that $g \circ f$ is homotopic to the identity of X, we need to construct a homotopy $X \times I \to X$ that lifts a homotopy $X \times I \to Y$ from $f \circ g \circ f$ to f. This homotopy is constructed inductively on the skeleta of X, and like in the argument proving 7.12, one finds that there are no obstructions in doing so because the homotopy fiber of f is aspherical. We leave the details of this obstruction theory argument to the reader. Thus f and g are homotopy inverse to each other, which proves the theorem. □

The following is an immediate corollary.

Corollary 7.17. *An aspherical CW - complex is contractible.*

Proof. If X is an aspherical CW - complex, then the constant map to a point, $\epsilon : X \to pt$ induces an isomorphism on homotopy groups, and is therefore, by the above theorem, a homotopy equivalence. □

The Whitehead theorem will now allow us to prove the following important relationship between the homotopy type of a topological group and its classifying space.

Theorem 7.18. *Let G be a topological group with the homotopy type of a CW complex., and BG its classiftying space. Then there is a homotopy equivalence between G and the loop space,*

$$G \simeq \Omega BG.$$

Proof. It was shown in chapter 4 that there is a model for a universal G - bundle, $p : EG \to BG$ with EG a G - equivariant CW - complex. In particular, EG is aspherical, and hence by the Whitehead theorem, it is contractible. Let

$$H : EG \times I \to EG$$

be a contraction. That is, $H_0 : EG \times \{0\} \to EG$ is the constant map at the basepoint $e_0 \in EG$, , and $H_1 : EG \times \{1\} \to EG$ is the identity. Composing with the projection map,

$$\Phi = p \circ H : EG \times I \to BG$$

is a homotopy between the constant map to the basepoint $\Phi_0 : EG \times \{0\} \to b_0 \in BG$ and the projection map $\Phi_1 = p : EG \times \{1\} \to BG$. Consider the adjoint of Φ,

$$\bar{\Phi} : EG \to P(BG) = \{\alpha : I \to BG \text{ such that } \alpha(0) = b_0.\}$$

defined by $\bar{\Phi}(e)(t) = \Phi(e,t) \in BG$. Then by definition, the following diagram commutes:

$$\begin{array}{ccc} EG & \xrightarrow{\bar{\Phi}} & (BG) \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ BG & = & BG \end{array}$$

where $q(\alpha) = \alpha(1)$, for $\alpha \in P(BG)$. Thus Φ is a map of fibrations that induces a map on fibers

$$\phi : G \to \Omega BG.$$

Comparing the exact sequences in homotopy groups of these two fibrations, we see that ϕ induces an isomorphism in homotopy groups. A result of Milnor [71] that we will not prove says that if X is a CW complex, then the loop space ΩX has the homotopy type of a CW - complex. Then the Whitehead theorem implies that $\phi : G \to \Omega BG$ is a homotopy equivalence. $\square$

7.4 Eilenberg - MacLane Spaces

In this section we prove a classification theorem for cohomology. Recall that in chapter 4 we proved that there are spaces BG that classify principal G - bundles over a space X, in the sense that homotopy classes of basepoint preserving maps, $[X, BG]$ are in bijective correspondence with isomorphism classes of principal G - bundles. Similarly $BO(n)$ and $BU(n)$ classify real and complex n - dimensional vector vector bundles in this same sense, and BO and BU classify K -theory. In this section we show that there are classifying spaces $K(G,n)$ that classify n - dimensional cohomology with coefficients in G in this same sense. These are Eilenberg - MacLane spaces. We have discussed these spaces earlier in these notes, but in this section we prove their existence and their classification properties.

7.4.1 Obstruction theory and the existence of Eilenberg - MacLane spaces

In chapter 4 we proved that for any topological group G there is a space BG classifying G bundles. For G discrete, we saw that $BG = K(G, 1)$, an Eilenberg - MacLane space whose fundamental group is G, and whose higher homotopy groups are all zero. In this section we generalize this existence theorem as follows.

Theorem 7.19. *Let G be any abelian group and n an integer with $n \geq 2$. Then there exists a space $K(G, n)$ with*

$$\pi_k(K(G,n)) = \begin{cases} G, & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

This theorem will basically be proven using obstruction theory. For this we will assume the following famous theorem of Hurewicz, which we will prove later in this chapter. We first recall the Hurewicz homomorphism from homotopy to homology.

Let $f : (D^n, S^{n-1}) \to (X, A)$ represent an element $[f] \in \pi_n(X, A)$. Let $\sigma_n \in H_n(D^n, S^{n-1}) \cong \mathbb{Z}$ be a preferred, fixed generator. Define $h([f]) = f_*(\sigma_n) \in H_n(X, A)$. The following is straightforward, and we leave its verification to the reader.

Lemma 7.20. *The above construction gives a well defined homomorphism*

$$h_* : \pi_n(X, A) \to H_n(X, A)$$

called the "Hurewicz homomorphism".

The following is the "Hurewicz theorem".

Theorem 7.21. *Let X be simply connected, and let $A \subset X$ be a simply connected subspace. Suppose that the pair (X, A) is $(n-1)$ - connected, for $n > 2$. That is,*

$$\pi_k(X, A) = 0 \quad \text{if} \quad k \leq n - 1.$$

Then the Hurewicz homomorphism $h_* : \pi_n(X, A) \to H_n(X, A)$ is an isomorphism.

We now prove the following basic building block type result concerning how the homotopy groups change as we build a CW - complex cell by cell.

Theorem 7.22. *Let X be a simply connected, CW - complex and let*

$$f : S^k \to X$$

be a map. Let X' be the mapping cone of f. That is,

$$X' = X \cup_f D^{n+1}$$

which denotes the union of X with a disk D^{n+1} glued along the boundary sphere $S^k = \partial D^{k+1}$ via f. That is we identify $t \in S^k$ with $f(t) \in X$. Let

$$\iota : X \hookrightarrow X'$$

be the inclusion. Then

$$\iota_* : \pi_k(X) \to \pi_k(X')$$

is surjective, with kernel equal to the cyclic subgroup generated by $[f] \in \pi_k(X)$.

Proof. Let $g : S^q \to X'$ represent an element in $\pi_q(X')$ with $q \leq k$. By the cellular approximation theorem, g is homotopic to a cellular map, and therefore one whose image lies in the q - skeleton of X'. But for $q \leq k$, the q - skeleton of X' is the q - skeleton of X. This implies that

$$\iota_* : \pi_q(X) \to \pi_q(X')$$

is surjective for $q \leq k$. Now assume $q \leq k-1$, then if $g : S^q \to X \subset X'$ is null homotopic, any null homotopy, i.e extension to the disk $G : D^{q+1} \to X'$ can be assumed to be cellular, and hence has image in X. This implies that for $q \leq k-1$, $\iota_* : \pi_q(X) \to \pi_q(X')$ is an isomorphism. By the exact sequence in homotopy groups of the pair (X', X), this implies that the pair (X', X) is k - connected. By the Hurewicz theorem that says that

$$\pi_{k+1}(X', X) \cong H_{k+1}(X', X) = H_{k+1}(X \cup_f D^{k+1}, X)$$

which, by analyzing the cellular chain complex for computing $H_*(X')$ is $\mathbb{Z}$ if and only if $f : S^k \to X$ is zero in homology, and zero otherwise. In particular, the generator $\gamma \in \pi_{k+1}(X', X)$ is represented by the map of pairs given by the inclusion

$$\gamma : (D^{k+1}, S^k) \hookrightarrow (X \cup_f D^{k+1}, X)$$

and hence in the long exact sequence in homotopy groups of the pair (X', X),

$$\cdots \to \pi_{k+1}(X', X) \xrightarrow{\partial_*} \pi_k(X) \xrightarrow{\iota_*} \pi_k(X') \to \cdots$$

we have $\partial_*(\gamma) = [f] \in \pi_k(X)$. Thus $\iota_* : \pi_k(X) \to \pi_k(X')$ is surjective with kernel generated by $[f]$. This proves the theorem. □

We will now use this basic homotopy theory result to establish the existence of Eilenberg - MacLane spaces.

Proof. of 7.19 Fix the group G and the integer $n \geq 2$. Let $\{\gamma_\alpha : \alpha \in \mathcal{A}\}$ be a set of generators of G, where $\mathcal{A}$ denotes the indexing set for these generators. Let $\{\theta_\beta : \beta \in \mathcal{B}\}$ be a corresponding set of relations. In other words G is isomorphic to the free abelian group $F_\mathcal{A}$ generated by $\mathcal{A}$, modulo the subgroup $R_\mathcal{B}$ generated by $\{\theta_\beta : \beta \in \mathcal{B}\}$.

Consider the wedge of spheres $\bigvee_\mathcal{A} S^n$ indexed on the set $\mathcal{A}$. Then by the Hurewicz theorem,

$$\pi_n(\bigvee_\mathcal{A} S^n) \cong H_n(\bigvee_\mathcal{A} S^n) \cong F_\mathcal{A}.$$

Now the group $R_\mathcal{B}$ is a subgroup of a free abelian group, and hence is itself free abelian. Let $\bigvee_\mathcal{B} S^n$ be a wedge of spheres whose n^{th} - homotopy group (which by the Hurewicz theorem is isomorphic to its homology, which is free abelian) is $R_\mathcal{B}$. Moreover there is a natural map

$$j : \bigvee_\mathcal{B} S^n \to \bigvee_\mathcal{A} S^n$$

which, on the level of the homotopy group π_n is the inclusion $R_\mathcal{B} \subset F_\mathcal{A}$. Let X_{n+1} be the mapping cone of j:

$$X_{n+1} = \bigvee_\mathcal{A} S^n \cup_j \bigcup_\mathcal{B} D^{n+1}$$

where the disk D^{n+1} corresponding to a generator in $R_\mathcal{B}$ is attached via the map $S^n \to \bigvee_\mathcal{A} S^n$ giving the corresponding element in $\pi_n(\bigvee_\mathcal{A} S^n) = F_\mathcal{A}$. Then by using 7.22 one cell at a time, we see that X_{n+1} is an $n-1$ - connected space and $\pi_n(X_n)$ is generated by $F_\mathcal{A}$ modulo the subgroup $R_\mathcal{B}$. In other words,

$$\pi_n(X_{n+1}) \cong G.$$

Now inductively assume we have constructed an space X_{n+k} with

$$\pi_q(X_{n+k}) = \begin{cases} 0 & \text{if } q < n, \\ G & \text{if } q = n \quad \text{and} \\ 0 & \text{if } n < q \leq n+k-1 \end{cases}$$

Notice that we have begun the inductive argument with $k = 1$, by the construction of the space X_{n+1} above. So again, assume we have constructed X_{n+k}, and we need to show how to construct X_{n+k+1} with these properties. Once we have done this, by induction we let $k \to \infty$, and clearly X_∞ will be a model for $K(G, n)$.

Now suppose $\pi = \pi_{n+k}(X_{n+k})$ is has a generating set $\{\gamma_u : u \in \mathcal{C}\}$, where $\mathcal{C}$

is the indexing set. Let $F_{\mathcal{C}}$ be the free abelian group generated by the elements in this generating set. Let $\bigvee_{u\in\mathcal{C}} S^{n+k}_u$ denote a wedge of spheres indexed by this indexing set. Then, like above, by applying the Hurewicz theorem we see that

$$\pi_{n+k}(\bigvee_{u\in\mathcal{C}} S^{n+k}_u) \cong H_{n+k}(\bigvee_{\mathcal{C}} S^{n+k}) \cong F_{\mathcal{C}}.$$

Let

$$f : \bigvee_{\mathcal{C}} S^{n+k} \to X_{n+k}$$

be a map which, when restricted to the sphere S^{n+k}_u represents the generator $\gamma_u \in \pi = \pi_{n+k}(X_{n+k})$. We define X_{n+k+1} to be the mapping cone of f:

$$X_{n+k+1} = X_{n+k} \cup_f \bigcup_{u\in\mathcal{C}} D^{n+k+1}.$$

Then by 7.22 we have that $\pi_q(X_{n+k}) \to \pi_q(X_{n+k+1})$ is an isomorphism for $q < n+k$, and

$$\pi_{n+k}(X_{n+k}) \to \pi_{n+k}(X_{n+k+1})$$

is surjective, with kernel the subgroup generated by $\{\gamma_u : u \in \mathcal{C}\}$. But since this subgroup generates $\pi = \pi_{n+k}(X_{n+k})$ we see that this homomorphism is zero. Since it is surjective, that implies $\pi_{n+k}(X_{n+k+1}) = 0$. Hence X_{n+k+1} has the required properties on its homotopy groups, and so we have completed our inductive argument. □

7.4.2 The Hopf - Whitney theorem and the classification theorem for Eilenberg - MacLane spaces

We now know that the Eilenberg - MacLane spaces $K(G,n)$ exist for every n and every abelian group G, and when $n = 1$ for every group G. Furthermore, by their construction in the proof of 7.19 they can be chosen to be CW - complexes. In this section we prove their main property, i.e they classify cohomology.

In order to state the classification theorem properly, we need to recall the universal coefficient theorem, which says the following.

Theorem 7.23. *(Universal Coefficient Theorem) Let G be an abelian group. Then there is a split short exact sequence*

$$0 \to Ext(H_{n-1}(X); G) \to H^n(X;G) \to Hom(H^n(X), G) \to 0.$$

Corollary 7.24. *If Y is $(n-1)$ - connected, and $\pi = \pi_n(Y)$, then*

$$H^n(Y;\pi) \cong Hom(\pi,\pi).$$

Proof. Since Y is $(n-1)$ connected, $H_{n-1}(Y) = 0$, so the universal coefficient theorem says that $H^n(Y;\pi) \cong Hom(H_n(Y), \pi)$. But the Hurewicz theorem says that the Hurewicz homomorphism $h_* : \pi = \pi_n(Y) \to H_n(Y)$ is an isomorphism. The corollary follows by combining these two isomorphisms. □

For an $(n-1)$ - connected space Y as above, let $\iota \in H^n(Y;\pi)$ be the class corresponding to the identity map $id \in Hom(\pi, \pi)$ under the isomorphism in this corollary. This is called the fundamental class. Given any other space X, we therefore have a set map

$$\phi : [X, Y] \to H^n(X, \pi)$$

defined by $\phi([f]) = f^*(\iota) \in H^n(X;\pi)$. The classification theorem for Eilenberg - MacLane spaces is the following.

Theorem 7.25. *For $n \geq 2$ and π any abelian group, let $K(\pi, n)$ denote an Eilenberg - MacLane space with $\pi_n(K(\pi, n)) = \pi$, and all other homotopy groups zero. Let $\iota \in H^n(K(\pi, n);\pi)$ be the fundamental class. Then for any CW - complex X, the map*

$$\begin{aligned} \phi : [X, K(\pi, n)] &\to H^n(X;\pi) \\ [f] &\to f^*(\iota) \end{aligned}$$

is a bijective correspondence.

We have the following immediate corollary, giving a uniqueness theorem regarding Eilenberg - MacLane spaces.

Corollary 7.26. *Let $K(\pi, n)_1$ and $K(\pi, n)_2$ be CW - complexes that are both Eilenberg - MacLane spaces with the same homotopy groups. Then there is a natural homotopy equivalence between $K(\pi, n)_1$ and $K(\pi, n)_2$.*

Proof. Let $f : K(\pi, n)_1 \to K(\pi, n)_2$ be a map whose homotopy class is the inverse image of the fundamental class under the bijection

$$\phi : [K(\pi, n)_1, K(\pi, n)_2] \xrightarrow{\cong} H^n(K(\pi, n)_1;\pi) \cong Hom(\pi, \pi).$$

This means that $f : K(\pi, n)_1 \to K(\pi, n)_2$ induces the identity map in $Hom(\pi, \pi)$, and in particular induces an isomorphism on π_n. Since all other homotopy groups are zero in both of these complexes, f induces an isomorphism in homotopy groups in all dimensions. Therefore by the Whitehead theorem 7.16, f is a homotopy equivalence. □

We begin our proof of this classification theorem by proving a special case, known as the Hopf - Whitney theorem. This predates knowledge of the existence of Eilenberg - MacLane spaces.

Theorem 7.27. *(Hopf-Whitney theorem) Let Y be any $(n-1)$ - connected space with $\pi = \pi_n(Y)$. Let X be any n - dimensional CW complex. Then the map*

$$\phi : [X, Y] \to H^n(X; \pi)$$
$$[f] \to f^*(\iota)$$

is a bijective correspondence.

Remark. This theorem is most often used in the context of manifolds, where it implies that if M^n is any closed, orientable manifold the correspondence

$$[M^n, S^n] \to H^n(M^n; \mathbb{Z}) \cong \mathbb{Z}$$

is a bijection.

Exercise. Show that this correspondence can alternatively be described as assigning to a smooth map $f : M^n \to S^n$ its degree, $deg(f) \in \mathbb{Z}$.

Proof. (Hopf - Whitney theorem) We first set some notation. Let Y be $(n-1)$ - connected, and have basepoint $y_0 \in Y$. Let $X^{(m)}$ denote the m - skeleton of the n - dimensional complex X. Let $C_k(X) = H_k(X^{(k)}, X^{(k-1)})$ be the cellular k - chains in X. Alternatively, $C_k(X)$ can be thought of as the free abelian group on the k - dimensional cells in the CW - decomposition of X. Let $Z^k(X)$ and $B^k(X)$ denote the subgroups of cocycles and coboundaries respectively. Let J_k be the indexing set for the set of k - cells in this CW - structure. So that there are attaching maps

$$\alpha_k : \bigvee_{j \in J_k} S^k_j \to X^{(k)}$$

so that the $(k+1)$ - skeleton $X^{(k+1)}$ is the mapping cone

$$X^{(k+1)} = X^{(k)} \cup_{\alpha_k} \bigcup_{j \in J_k} D^{k+1}_j.$$

We prove this theorem in several steps, each translating between cellular cochain complexes or cohomology on the one hand, and homotopy classes of maps on the other hand. The following is the first step.

Step 1. There is a bijective correspondence between the following set of homotopy classes of maps of pairs, and the cochain complex with values in π:

$$\phi : [(X^{(n)}, X^{(n-1)}), (Y, y_0)] \to C^n(X; \pi).$$

Proof. A map of pairs $f : (X^{(n)}, X^{(n-1)}) \to (Y, y_0)$ is the same thing as a basepoint preserving map from the quotient,

$$f : X^{(n)}/X^{(n-1)} = \bigvee_{j \in J_n} S^n_j \to Y.$$

So the homotopy class of f defines and is defined by an assignment to every $j \in J_n$, an element $[f_j] \in \pi_n(Y) = \pi$. But by extending linearly, this is the same as a homomorphism from the free abelian group generated by J_n, i.e the chain group $C_n(X)$, to π. That is, this is the same thing as a cochain $[f] \in C^n(X;\pi)$. □

Step 2. The map $\phi : [X, Y] \to H^n(X;\pi)$ is surjective.

Proof. . Notice that since X is an n - dimensional CW - complex, all n - dimensional cochains are cocycles, $C^n(X;\pi) = Z^n(X;\pi)$. So in particular there is a surjective homomorphism $\mu : C^n(X;\pi) = Z^n(X;\pi) \to Z^n(X;\pi)/B^n(X;\pi) = H^n(X;\pi)$. A check of the definitions of the maps defined so far yields that the following diagram commutes:

$$\begin{array}{ccc} [(X^{(n)}, X^{(n-1)}), (Y, y_0)] & \xrightarrow[\cong]{\phi} & C^n(X;\pi) \\ \rho\downarrow & & \downarrow\mu \\ [X, Y] & \xrightarrow[\phi]{} & H^n(X;\pi) \end{array}$$

where ρ is the obvious restriction map. By the commutativity of this diagram, since μ is surjective and $\phi : [(X^{(n)}, X^{(n-1)}), (Y, y_0)] \to C^n(X;\pi)$ is bijective, then we must have that $\phi : [X, Y] \to H^n(X;\pi)$ is surjective, as claimed. □

In order to show that ϕ is injective, we will need to examine the coboundary map

$$\delta : C^{n-1}(X;\pi) \to C^n(X;\pi)$$

from a homotopy point of view. To do this, recall that the boundary map on the chain level, $\partial : C_k(X) \to C_{k-1}(X)$ is given by the connecting homomorphism $H_n(X^{(k)}, X^{(k-1)}) \to H_{k-1}(X^{(k-1)}, X^{(k-2)})$ from the long exact sequence in homology of the triple, $(X^{(k)}, X^{(k-1)}, X^{(k-2)})$. This boundary map can be realized homotopically as follows. Let $c(X^{(k-1)})$ be the cone on the subcomplex $X^{(k-1)}$,

$$c(X^{(k-1)}) = X^{(k-1)} \times I/(X^{(k-1)} \times \{1\} \cup \{x_0\} \times I),$$

which is obviously a contractible space. Consider the mapping cone of the inclusion $X^{(k-1)} \hookrightarrow X^{(k)}$, $X^{(k)} \cup c(X^{(k-1)}$. By projecting the cone to a point, there is a projection map

$$p_k : X^{(k)} \cup c(X^{(k-1)}) \to X^{(k)}/X^{(k-1)} = \bigvee_{j \in J_k} S^k_j$$

which is a homotopy equivalence. (**Note.** The fact that this map induces an isomorphism in homology is straight forward by computing the homology exact sequence of the pair $(X^{(k)} \cup c(X^{(k-1)}), X^{(k)})$. The fact that this map is a homotopy equivalence is a basic point set topological property of CW - complexes coming from the so - called "Homotopy Extension Property". However it can be proved directly, by hand, in this case. We leave its verification to the reader.) Let

$$u_k : X^{(k)} \to \bigvee_{j \in J_k} S^k_j$$

be the composition

$$X^{(k)} \hookrightarrow X^{(k)} \cup c(X^{(k-1)}) \xrightarrow{p_k} X^{(k)}/X^{(k-1)} = \bigvee_{j \in J_k} S^k_j.$$

Then the composition of u_k with the attaching map

$$\alpha_{k+1} : \bigvee_{j \in J_{k+1}} S^k_j \to X^{(k)}$$

(whose mapping cone defines the $(k+1)$ - skeleton $X^{(k+1)}$), is a map between wedges of k - spheres,

$$d_{k+1} : \bigvee_{j \in J_{k+1}} S^k_j \xrightarrow{\alpha_{k+1}} X^{(k)} \xrightarrow{u_k} \bigvee_{j \in J_k} S^k_j.$$

The following is immediate from the definitions.

Step 3. The induced map in homology,

$$(d_{k+1})_* : H_k(\bigvee_{j \in J_{k+1}} S^k_j) \to H_k(\bigvee_{j \in J_k} S^k_j)$$
$$C_{k+1}(X) \to C_k(X)$$

is the boundary homomorphism in the chain complex $\partial_{k+1} : C_{k+1}(X) \to C_k(X)$.

Now consider the map

$$[(X^{(n)}, X^{(n-1)}), (Y, y_0)] \xrightarrow[\cong]{\phi} C^n(X; \pi) = Z^n(X; \pi) \xrightarrow{\mu} H^n(X; \pi).$$

We then have the following corollary.

Step 4. A map $f : X^{(n)}/X^{(n-1)} = \bigvee_{j \in J_n} S^n_j \to Y$ has the property that

$$\mu \circ \phi([f]) = 0 \in H^n(X; \pi)$$

if and only if there is a map

$$f_{n-1} : \bigvee_{j \in J_{(n-1)}} S_j^n \to Y$$

so that f is homotopic to the composition

$$\bigvee_{j \in J_n} S_j^n \xrightarrow{d_n} \bigvee_{j \in J_{n-1}} S_j^n \xrightarrow{f_{n-1}} Y.$$

Proof. Since $\phi : [(X^{(n)}, X^{(n-1)}), (Y, y_0)] \to C^n(X;\pi) = Z^n(X;\pi)$ is a bijection, $\mu \circ \phi([f]) = 0$ if and only if $\phi([f])$ is in the image of the coboundary map. The result then follows from step 3. □

Step 5. The composition

$$X^{(n)} \xrightarrow{u_n} \bigvee_{j \in J_n} S_j^n \xrightarrow{d_n} \bigvee_{j \in J_{n-1}} S_j^n$$

is null homotopic.

Proof. The map u_n was defined by the composition

$$X^{(n)} \hookrightarrow X^{(n)} \cup c(X^{(n-1)}) \xrightarrow[\simeq]{p_n} \bigvee_{j \in J_n} S_j^n.$$

But notice that if we take the quotient $X^{(n)} \cup c(X^{(n-1)}/X^{(n)}$ we get the suspension

$$X^{(n)} \cup c(X^{(n-1)})/X^{(n)} = \Sigma X^{(n-1)}.$$

Furthermore, the map between the wedges of the spheres, $d_n : \bigvee_{j \in J_n} S_j^n \to \bigvee_{j \in J_{n-1}} S_j^n$ is directly seen to be the composition

$$d_n : \bigvee_{j \in J_n} S_j^n \simeq X^{(n)} \cup c(X^{(n-1)}) \xrightarrow{proj.} X^{(n)} \cup c(X^{(n-1)})/X^{(n)} = \Sigma X^{(n-1)} \xrightarrow{\Sigma u_{n-1}} \bigvee_{j \in J_{n-1}} S_j^n.$$

Thus the composition $d_n \circ u_n : X^{(n)} \to \bigvee_{j \in J_n} S_j^n \to \bigvee_{j \in J_{n-1}} S_j^n$ factors as the composition

$$X^{(n)} \hookrightarrow X^{(n)} \cup c(X^{(n-1)}) \xrightarrow{proj.} X^{(n)} \cup c(X^{(n-1)})/X^{(n)} = \Sigma X^{(n-1)} \xrightarrow{\Sigma u_{n-1}} \bigvee_{j \in J_{n-1}} S_j^n.$$

But the composite of the first two terms in this composition,

$$X^{(n)} \hookrightarrow X^{(n)} \cup c(X^{(n-1)}) \xrightarrow{proj.} X^{(n)} \cup c(X^{(n-1)})/X^{(n)}$$

is clearly null homotopic, and hence so is $d_n \circ u_n$. □

We now complete the proof of the theorem by doing the following step.

Step. 6. The correspondence $\phi : [X, Y] \to H^n(X; \pi)$ is injective.

Proof. Let f, $g : X \to Y$ be maps with $\phi([f]) = \phi([g]) \in H^n(X; \pi)$. Since Y - is $(n-1)$ - connected, given any map $h : X \to Y$, the restriction to its $(n-1)$ - skeleton is null homotopic. (**Exercise.** Check this!) Null homotopies define maps

$$\tilde{f},\, \tilde{g} : X \cup c(X^{(n-1)}) \to Y$$

given by f and g respectively on X, and by their respective null homotopies on the cones, $c(X^{(n-1)})$. Using the homotopy equivalence $p_n : X^{(n)} \cup c(X^{(n-1)}) \simeq X^{(n)}/X^{(n-1)} = \bigvee_{j \in J_n} S^n_j$, we then have maps

$$\bar{f},\, \bar{g} : X^{(n)}/X^{(n-1)} \to Y$$

which, when composed with the projection $X = X^{(n)} \to X^{(n)}/X^{(n-1)}$ are homotopic to f and g respectively. Now by the commutativity of the diagram in step 2, since $\phi([f]) = \phi([g])$, then $\mu \circ \phi([\bar{f}]) = \mu \circ \phi([\bar{g}])$. Or equivalently,

$$\mu \circ \phi([\bar{f}] - [\bar{g}]) = 0$$

where we are using the fact that

$$[(X^{(n)}, X^{(n-1)}), Y] = [\bigvee_{j \in J_n} S^n_j, Y] = \oplus_{j \in J_n} \pi_n(Y)$$

is a group, and maps to $C^n(X; \pi)$ is a group isomorphism.

Let $\psi : X^{(n)}/X^{(n-1)} \to Y$ represent $[\bar{f}] - [\bar{g}] \in [\bigvee_{j \in J_n} S^n_j, Y]$. Then $\mu \circ \phi(\psi) = 0$. Then by step 4, there is a map $\psi_{n-1} : \bigvee_{j \in J_{n-1}} S^n_j \to Y$ so that $\psi_{n-1} \circ d_n$ is homotopic to ψ. Thus the composition

$$X \xrightarrow{proj.} X/X^{(n-1)} \xrightarrow{\psi} Y$$

is homotopic to the composition

$$X \to X^{(n)}/X^{n-1} = \bigvee_{j \in J_n} S^n_j \xrightarrow{d_n} \bigvee_{j \in J_{n-1}} S^n_j \xrightarrow{\psi_{n-1}} Y.$$

But by step 5, this compostion is null homotopic. Now since ψ represents $[\bar{f}] - [\bar{g}]$, a null homotopy of the composition

$$X \xrightarrow{proj.} X/X^{(n-1)} \xrightarrow{\psi} Y$$

defines a homotopy between the compositions

$$X \xrightarrow{proj.} X/X^{(n-1)} \xrightarrow{\bar{f}} Y \quad \text{and} \quad X \xrightarrow{proj.} X/X^{(n-1)} \xrightarrow{\bar{g}} Y.$$

The first of these maps is homotopic to $f : X \to Y$, and the second is homotopic to $g : X \to Y$. Hence $f \simeq g$, which proves that ϕ is injective. $\square$

We now know that the correspondence $\phi : [X, Y] \to H^n(X; \pi)$ is surjective (step 2) and injective (step 6). This completes the proof of this theorem. □

We now proceed with the proof of the main classification theorem for cohomology, using Eilenberg - MacLane spaces (7.25).

Proof. The Hopf Whitney theorem proves this theorem when X is an n - dimensional CW - complex. We split the proof for general CW - complexes into two cases.

Case 1. X is $n+1$ - dimensional.

Consider the following commutative diagram

$$\begin{array}{ccc} [X, K(\pi, n)] & \xrightarrow{\phi} & H^n(X; \pi) \\ \rho\downarrow & & \downarrow\rho \\ [X^{(n)}, K(\pi, n)] & \xrightarrow[\cong]{\phi_n} & H^n(X^{(n)}; \pi) \end{array} \tag{7.1}$$

where the vertical maps ρ denote the obvious restriction maps, and ϕ_n denotes the restriction of the correspondence ϕ to the n - skeleton, which is an isomorphism by the Hopf - Whitney theorem.

Now by considering the exact sequence for cohomology of the pair $(X, X^{(n)}) = (X^{(n+1)}, X^{(n)})$, one sees that the restriction map $\rho : H^n(X, \pi) \to H^n(X^{(n)}, \pi)$ is injective. Using this together with the fact that ϕ_n is an isomorphism and the commutativity of this diagram, one sees that to show that $\phi : [X, K(\pi, n)] \to H^n(X; \pi)$ is surjective, it suffices to show that for $\gamma \in H^n(X, \pi)$ with $\rho(\gamma) = \phi_n([f_n])$, where $f_n : X^{(n)} \to K(\pi, n)$, then f_n can be extended to a map $f : X \to K(\pi, n)$.

Using the same notation as was used in the proof of the Hopf - Whitney theorem, since $X = X^{(n+1)}$, we can write

$$X = X^{(n)} \cup_{\alpha_{n+1}} \bigcup_{j \in J_{n+1}} D^{(n+1)}$$

where $\alpha_{n+1} : \bigvee_{j \in J_{n+1}} S^n_j \to X^{(n)}$ is the attaching map. Thus the obstruction to finding an extension $f : X \to K(\pi, n)$ of the map $f_n : X^{(n)} \to K(\pi, n)$, is the compostion

$$\bigvee_{j \in J_{n+1}} S^n_j \xrightarrow{\alpha_{n+1}} X^{(n)} \xrightarrow{f_n} K(\pi, n).$$

Now since $\bigvee_{j \in J_{n+1}} S^n_j$ is n - dimensional, the Hopf - Whitney theorem says that this map is determined by its image under ϕ,

$$\phi([f_n \circ \alpha_{n+1}]) \in H^n(\bigvee_{j \in J_{n+1}} S^n_j; \pi).$$

But this class is $\alpha_{n+1}^*(\phi([f_n]))$, which by assumption is $\alpha_{n+1}^*(\rho(\gamma))$. But the composition

$$H^n(X;\pi) \xrightarrow{\rho} H^n(X^{(n)},\pi) \xrightarrow{\alpha_{n+1}^*} H^n(\bigvee_{j\in J_{n+1}} S_j^n;\pi)$$

are two successive terms in the long exact sequence in cohomology of the pair $(X^{(n+1)}, X^{(n)})$ and is therefore zero. Thus the obstruction to finding the extension $f : X \to K(\pi, n)$ is zero. As observed above this proves that $\phi : [X, K(\pi, n)] \to H^n(X;\pi)$ is surjective.

We now show that ϕ is injective. So suppose $\phi([f]) = \phi([g])$ for f, $g : X \to K(\pi, n)$. To prove that ϕ is injective we need to show that this implies that f is homotopic to g. Let f_n and g_n be the restrictions of f and g to $X^{(n)}$. That is,

$$f_n = \rho([f]) : X^{(n)} \to K(\pi, n) \quad \text{and} \quad g_n = \rho([g]) : X^{(n)} \to K(\pi, n)$$

Now by the commutativity of diagram 7.1 and the fact that ϕ_n is an isomorphism, we have that f_n and g_n are homotopic maps. Let

$$F_n : X^{(n)} \times I \to K(\pi, n)$$

be a homotopy between them. That is, $F_0 = f_n : X^{(n)} \times \{0\} \to K(\pi, n)$ and $F_1 = g_n : X^{(n)} \times \{1\} \to K(\pi, n)$. This homotopy defines a map on the $(n+1)$ -subcomplex of $X \times I$ defined to be

$$\tilde{F} : (X \times \{0\}) \cup (X \times \{1\}) \cup X^{(n)} \times I \to K(\pi, n)$$

where $\tilde{F}$ is defined to be f and g on $X \times \{0\}$ and $X \times \{1\}$ respectively, and F on $X^{(n)} \times I$. But since X is $(n+1)$ - dimensional, $X \times I$ is $(n+2)$ - dimensional, and this subcomplex is its $(n+1)$ - skeleton. So $X \times I$ is the union of this complex with $(n+2)$ - dimensional disks, attached via maps from a wedge of $(n+1)$ - dimensional spheres. Hence the obstruction to extending $\tilde{F}$ to a map $F : X \times I \to K(\pi, n)$ is a cochain in $C^{n+2}(X \times I; \pi_{n+1}(K(\pi, n))$. But this group is zero since $\pi_{n+1}(K(\pi, n)) = 0$. Thus there is no obstruction to extending $\tilde{F}$ to a map $F : X \times I \to K(\pi, n)$, which is a homotopy between f and g. As observed before this proves that ϕ is injective. This completes the proof of the theorem in this case.

General Case. Since, by case 1, we know the theorem for $(n+1)$ - dimensional CW - complexes, we assume that the dimension of X is $\geq n+2$. Now consider the following commutative diagram:

$$\begin{array}{ccc} [X, K(\pi,n)] & \xrightarrow{\phi} & H^n(X;\pi) \\ \rho\downarrow & & \downarrow\rho \\ [X^{(n+1)}, K(\pi,n)] & \xrightarrow[\cong]{\phi_{n+1}} & H^n(X^{(n+1)};\pi) \end{array}$$

where, as earlier, the maps ρ denote the obvious restriction maps, and ϕ_{n+1} denotes the restriction of ϕ to the $(n+1)$ skeleton, which we know is an isomorphism, by the result of case 1.

Now in this case the exact sequence for the cohomology of the pair $(X, X^{(n+1)})$ yields that the restriction map $\rho : H^n(X;\pi) \to H^n(X^{(n+1)}, \pi)$ is an isomorphism. Therefore by the commutativity of this diagram, to prove that $\phi : [X, K(\pi, n)] \to H^n(X;\pi)$ is an isomorphism, it suffices to show that the restriction map

$$\rho : [X, K(\pi, n)] \to [X^{(n+1)}, K(\pi, n)]$$

is a bijection. This is done by induction on the skeleta $X^{(K)}$ of X, with $K \geq n+1$. To complete the inductive step, one needs to analyze the obstructions to extending maps $X^{(K)} \to K(\pi, n)$ to $X^{(K+1)}$ or homotopies $X^{(K)} \times I \to K(\pi, n)$ to $X^{(K+1)} \times I$, like what was done in the proof of case 1. However in these cases the obstructions will always lie is spaces of cochains with coefficients in $\pi_q(K(\pi, n))$ with $q = K \,\text{or}\, K+1$, and so $q \geq n+1$. But then $\pi_q(K(\pi, n)) = 0$ and so these obstructions will always vanish. We leave the details of carrying out this argument to the reader. □

7.5 Spectral Sequences

One of the great achievements of Algebraic Topology was the development of spectral sequences. They were originally invented by Leray in the late 1940's and since that time have become fundamental calculational tools in many areas of Geometry, Topology, and Algebra. One of the earliest and most important applications of spectral sequences was the work of Serre [84] for the calculation of the homology of a fibration. We divide our discussion of spectral sequences in these notes into three parts. In the first section we develop the notion of a spectral sequence of a filtration. In the next section we discuss the Leray - Serre spectral sequence for a fibration. In the final two sections we discuss applications: we prove the Hurewicz theorem, calculate the cohomology of the Lie groups $U(n)$, and $O(n)$, and of the loop spaces ΩS^n, and we discuss *Spin* and $Spin_{\mathbb{C}}$ - structures on manifolds. We refer the reader to [67] for a more complete discussion of spectral sequences.

7.5.1 The spectral sequence of a filtration

A spectral sequence is the algebraic machinery for studying sequences of long exact sequences that are interelated in a particular way. We begin by illustrating this with the example of a filtered complex.

Let C_* be a chain complex, and let $A_* \subset C_*$ be a subcomplex. The short exact sequence of chain complexes

$$0 \longrightarrow A_* \hookrightarrow C_* \longrightarrow C_*/A_* \longrightarrow 0$$

leads to a long exact sequence in homology:

$$\longrightarrow \cdots \longrightarrow H_{q+1}(C_*, A_*) \longrightarrow H_q(A_*) \longrightarrow H_q(C_*) \longrightarrow H_q(C_*, A_*) \longrightarrow H_{q-1}(A_*) \longrightarrow \cdots$$

This is useful in computing the homology of the big chain complex, $H_*(C_*)$ in terms of the homology of the subcomplex $H_*(A_*)$ and the homology of the quotient complex $H_*(C_*, A_*)$. A spectral sequence is the machinery used to study the more general situation when one has a *filtration* of a chain complex C_* by subcomplexes

$$0 = F_0(C_*) \hookrightarrow F_1(C_*) \hookrightarrow \cdots \hookrightarrow F_k(C_*) \hookrightarrow F_{k+1}(C_*) \hookrightarrow \cdots \hookrightarrow C_* = \bigcup_k F_k(C_*).$$

Let D_*^k be the subquotient complex $D_*^k = F_k(C_*)/F_{k-1}(C_*)$ and so for each k there is a long exact sequence in homology

$$\longrightarrow H_{q+1}(D_*^k) \longrightarrow H_q(F_{k-1}(C_*)) \longrightarrow H_q(F_k(C_*)) \longrightarrow H_q(D_*^k) \longrightarrow \cdots$$

By putting these long exact sequences together, in principle one should be able to use information about $\oplus_k H_*(D_*^k)$ in order to obtain information about

$$H_*(C_*) = \varinjlim_k H_*(F_k(C_*)).$$

A spectral sequence is the bookkeeping device that allows one to do this. To be more specific, consider the following diagram.

$$
\begin{array}{ccccccccc}
0 & & & & 0 & & \\
\downarrow{\scriptstyle i} & & & & \downarrow{\scriptstyle i} & & \\
H_q(F_1(C_*)) & & & & H_{q-1}(F_1(C_*)) & \xrightarrow{=} & H_{q-1}(D_*^1) \\
\downarrow{\scriptstyle i} & & & & \downarrow{\scriptstyle i} & & \\
\vdots & & & & \vdots & & \\
\downarrow{\scriptstyle i} & & & & \downarrow{\scriptstyle i} & & \\
H_q(F_{k-p}(C_*)) & \xrightarrow{j} & H_q(D_*^{k-p}) & \xrightarrow{\partial} & H_{q-1}(F_{k-p-1}(C_*)) & \xrightarrow{j} & H_{q-1}(D_*^{k-p-1}) \\
\downarrow{\scriptstyle i} & & & & \downarrow{\scriptstyle i} & & \\
\vdots & & & & H_{q-1}(F_{k-p}(C_*)) & \xrightarrow{j} & H_{q-1}(D_*^{k-p}) \\
\downarrow{\scriptstyle i} & & & & \downarrow{\scriptstyle i} & & \\
\vdots & & & & \vdots & & \\
\downarrow{\scriptstyle i} & & & & \downarrow{\scriptstyle i} & & \\
H_q(F_{k-2}(C_*)) & & & & H_{q-1}(F_{k-3}(C_*)) & & \\
\downarrow{\scriptstyle i} & & & & \downarrow{\scriptstyle i} & & \\
H_q(F_{k-1}(C_*)) & \xrightarrow{j} & H_q(D_*^{k-1}) & \xrightarrow{\partial} & H_{q-1}(F_{k-2}(C_*)) & \xrightarrow{j} & H_{q-1}(D_*^{k-2})) \\
\downarrow{\scriptstyle i} & & & & \downarrow{\scriptstyle i} & & \\
H_q(F_k(C_*)) & \xrightarrow{j} & H_q(D_*^{k}) & \xrightarrow{\partial} & H_{q-1}(F_{k-1}(C_*)) & \xrightarrow{j} & H_{q-1}(D_*^{k-1})) \\
\downarrow{\scriptstyle i} & & & & \downarrow{\scriptstyle i} & & \\
\vdots & & & & \vdots & & \\
\downarrow{\scriptstyle i} & & & & \downarrow{\scriptstyle i} & & \\
H_q(C_*) & & & & H_{q-1}(C_*) & &
\end{array}
\tag{7.2}
$$

The columns represent the homology filtration of $H_*(C_*)$ and the three maps ∂, j, and i combine to give long exact seqences at every level.

Let $\alpha \in H_q(C_*)$. We say that α has *algebraic filtration* k, if α is in the image of a class $\alpha_k \in H_q(F_k(C_*))$ but is not in the image of $H_q(F_{k-1}(C_*))$. In such a case we say that the image $j(\alpha_k) \in H_q(D_*^k)$ is a *representative* of α. Notice that this representative is not unique. In particular we can add any class in the image of

$$d_1 = j \circ \partial : H_{q+1}(D_*^{k+1}) \longrightarrow H_q(D_*^k)$$

to $j(\alpha_k)$ and we would still have a representative of $\alpha \in H_q(C_*)$ under the above definition.

Conversely, let us consider when an arbitrary class $\beta \in H_q(D_*^k)$ represents a class in $H_q(C_*)$. By the exact sequence this occurs if and only if the image $\partial(\beta) = 0$, for this is the obstruction to β being in the image of $j : H_q(F_k(C_*)) \to H_q(D_*^k)$. Furthermore if $j(\tilde{\beta}) = \beta$ then β represents the image

$$i \circ \cdots \circ i(\tilde{\beta}) \in H_q(C_*).$$

Now $\partial(\beta) = 0$ if and only if it lifts all the way up the second vertical tower in diagram 7.2. The first obstruction to this lifting, (i.e the obstruction to lifting $\partial(\beta)$ to $H_{q-1}(F_{k-2}(C_*))$ is that the composition

$$d_1 = j \circ \partial : H_q(D_*^k) \longrightarrow H_{q-1}(D_*^{k-1})$$

maps β to zero. That is elements of $H_q(C_*)$ are represented by elements in the subquotient

$$ker(d_1)/Im(d_1)$$

of $H_q(D_*^k)$. We use the following notation to express this. We define

$$E_1^{r,s} = H_{r+s}(D_*^r)$$

and define

$$d_1 = j \circ \partial : E_1^{r,s} \longrightarrow E_1^{r-1,s}.$$

r is said to be the *algebraic filtration* of elements in $E_1^{r,s}$ and $r+s$ is the *total degree* of elements in $E_1^{r,s}$. Since $\partial \circ j = 0$, we have that

$$d_1 \circ d_1 = 0$$

and we let

$$E_2^{r,s} = Ker(d_1 : E_1^{r,s} \to E_1^{r-1,s})/Im(d_1 : E_1^{r+1,s} \to E_1^{r,s})$$

be the resulting homology group. We can then say that the class $\alpha \in H_q(C_*)$ has as its representative, the class $\alpha_k \in E_2^{k,q-k}$.

Now let us go back and consider further obstructions to an arbitrary class $\beta \in E_2^{k,q-k}$ representing a class in $H_q(C_*)$. Represent β as a cycle in E_1: $\beta \in Ker(d_1 = j \circ \partial \in H_q(D_*^k))$. Again, β represents a class in $H_q(C_*)$ if and only if $\partial(\beta) = 0$. Now since $j \circ \partial(\beta) = 0$, $\partial(\beta) \in H_{q-1}(F_{k-1}(C_*))$ lifts to a class, say $\tilde{\beta} \in H_{q-1}F_{k-2}(C_*)$. Remember that the goal was to lift $\partial(\beta)$ all the way up the vertical tower (so that it is zero). The obstruction to lifting it the next stage, i.e to $H_{q-1}(F_{k-3}(C_*))$ is that $j(\tilde{\beta}) \in H_{q-1}(D_*^{k-2})$ is zero. Now the fact that a d_1 cycle β has the property that $\partial(\beta)$ lifts to $H_{q-1}F_{k-2}(C_*))$ allows to define a map

$$d_2 : E_2^{k,q-k} \longrightarrow E_2^{k-2,q-k+1}$$

and more generally,

$$d_2 : E_2^{r,s} \longrightarrow E_2^{r-2,s+1}$$

by composing this lifting with

$$j : H_{s+r-1}(F_{r-2}(C_*)) \longrightarrow H_{s+r-1}(D_*^{r-2}).$$

That is, $d_2 = j \circ i^{-1} \circ \partial$. It is straightforward to check that $d_2 : E_2^{r,s} \longrightarrow E_2^{r-2,s+1}$ is well defined, and that elements of $H_q(C_*)$ are actually represented by elements in the subquotient homology groups of $E_2^{*,*}$:

$$E_3^{r,s} = Ker(d_2 : E_2^{r,s} \to E_2^{r-2,s+1})/Im(d_2 : E_2^{r+2,s-1} \to E_1^{r,s})$$

Inductively, assume the subquotient homology groups $E_j^{r,s}$ have been defined for $j \leq p-1$ and differentials

$$d_j : E_j^{r,s} \longrightarrow E_j^{r-j,s+j-1}$$

defined on representative classes in $H_{r+s}(D_*^r)$ to be the composition

$$d_j = j \circ (i^{j-1} = i \circ \cdots \circ i)^{-1} \circ \partial$$

so that $E_{j+1}^{*,*}$ is the homology $Ker(d_j)/Im(d_j)$. We then define

$$E_p^{r,s} = Ker(d_{p-1} : E_{p-1}^{r,s} \to E_{p-1}^{r-p+1,s+p-2})/Im(d_{p-1} : E_{p-1}^{r+p-1,s-p+2} \to E_{p-1}^{r,s}).$$

Thus $E_p^{k,q-k}$ is a subquotient of $H_q(D_*^k)$, represented by elements β so that $\partial(\beta)$ lifts to $H_q(F_{k-p}(C_*))$. That is, there is an element $\tilde{\beta} \in H_q(F_{k-p}(C_*))$ so that

$$i^{p-1}(\tilde{\beta}) = \partial(\beta) \in H_{q-1}(F_{k-1}(C_*)).$$

The obstruction to $\tilde{\beta}$ lifting to $H_{q-1}(F_{k-p-1}(C_*))$ is $j(\beta) \in H_q(D_*^{k-p})$. This procedure yields a well defined map

$$d_p : E_p^{r,s} \longrightarrow E_p^{r-p,s+p-1}$$

given by $j \circ (i^{p-1})^{-1} \circ \partial$ on representative classes in $H_q(D_*^k)$. This completes the inductive step. Notice that if we let

$$E_\infty^{r,s} = \varinjlim_p E_p^{r,s}$$

then $E_\infty^{k,q-k}$ is a subquotient of $H_q(D_*^k)$ consisting of precisely those classes represented by elements $\beta \in H_q(D_*^k)$ so that $\partial(\beta)$ lifts all the way up the vertical tower i.e $\partial(\beta)$ is in the image of i^p for all p. This is equivalent to the condition that $\partial(\beta) = 0$ which as observed above is precisely the condition necessary for β to represent a class in $H_q(C_*)$. These observations can be made more precise as follows.

Theorem 7.28. *Let* $I^{r,s} = Image(H_{r+s}(F_r(C_*)) \longrightarrow H_{r+s}(C_*))$. *Then* $E_\infty^{r,s}$ *is isomorphic to the quotient group*

$$E_\infty^{r,s} \cong I^{r,s}/I^{r-1,s+1}.$$

Thus the $E_\infty^{*,*}$ *determines* $H_*(C_*)$ *up to extensions. In particular, if all homology groups are taken with field coefficients we have*

$$H_q(C_*) \cong \bigoplus_{r+s=q} E_\infty^{r,s}.$$

In this case we say that $\{E_p^{r,s}, d_p\}$ is a *spectral sequence* starting at $E_1^{r,s} = H_{r+s}(D_*^r)$, and converging to $H_{r+s}(C_*)$.

Often times a filtration of this type occurs when one has a topological space X filtered by subspaces,

$$* = X_0 \hookrightarrow X_1 \hookrightarrow \cdots \hookrightarrow X_k \hookrightarrow X_{k+1} \hookrightarrow \cdots \hookrightarrow X.$$

An important example is the filtration of a CW - complex X by its skeleta, $X_k = X^{(k)}$. We get a spectral sequence as above by applying the homology of the chain complexes to this topological filtration. This spectral sequence converges to $H_*(X)$ with E_1 term $E_1^{r,s} = H_{r+s}(X_r, X_{r-1})$. From the construction of this spectral sequence one notices that chain complexes are irrelevant in this case; indeed all one needs is the fact that each inclusion $X_{k-1} \hookrightarrow X_k$ induces a long exact sequence in homology.

Exercise. Show that in the case of the filtration of a CW - complex X by its skeleta, that the E_1 -term of the corresponding spectral sequence is the cellular chain complex, and the E_2 - term is the homology of X,

$$E_2^{r,s} = \begin{cases} H_r(X), & \text{if } s = 0 \\ 0 & \text{otherwise} \end{cases}$$

Furthermore, show that this spectral sequence "collapses" at the E_2 level, in the sense that

$$E_p^{r,s} = E_2^{r,s} \quad \text{for all } p \geq 2$$

and hence

$$E_\infty^{r,s} = E_2^{r,s}.$$

Now if $h_*(-)$ is any *generalized* homology theory (that is, a functor that obeys all the Eilenberg - Steenrod axioms but dimension) then the inclusions of a filtration as above $X_{k-1} \hookrightarrow X_k$ induce long exact sequences in $h_*(-)$, and one gets, by a procedure completely analogous to the above, a spectral sequence converging to $h_*(X)$ with E_1 term

$$E_1^{r,s} = h_{r+s}(X_r, X_{r-1}).$$

Again, for the skeletal filtration of a CW complex, this spectral sequence is called the Atiyah - Hirzebruch spectral sequence for the generalized homology h_*.

Exercise. Show that the E_2 -term of the Atiyah - Hirzebruch spectal sequence for the generalized homology theory h_* is

$$E_2^{r,s} = h_{r+s}(S^r) \otimes H_r(X).$$

Particularly important examples of such generalized homology theories include stable homotopy ($\cong$ framed bordism), other bordism theories, and K - homology theory. Similar spectral sequences also exist for cohomology theories. The reader is referred to [67] for a good general reference on spectral sequences with many examples of those most relevant in Algebraic Topology.

7.5.2 The Leray - Serre spectral sequence for a fibration

The most important example of a spectral sequence from the point of view of these notes is the Leray - Serre spectral sequence of a fibration. Given a fibration $F \to E \to B$, the goal is to understand how the homology of the three spaces (fiber, total space, base space) are related. In the case of a trivial fibration, $E = B \times F \to B$, the answer to this question is given by the Kunneth formula, which says, that when taken with field coefficients,

$$H_*(B \times F; k) \cong H_*(B; k) \otimes_k H_*(F : k),$$

where k is the field.

When $p : E \to B$ is a nontrivial fibration, one needs a spectral sequence to study the homology. The idea is to construct a filtration on a chain complex $C_*(E)$ for computing the homology of the total space E, in terms of the skeletal filtration of a CW - decomposition of the base space B.

Assume for the moment that $p : E \to B$ is a fiber bundle with fiber F. For the purposes of our discussion we will assume that the base space B is simply connected. Let $B^{(k)}$ be the k - skeleton of B, and define

$$E(k) = p^{-1}(B^{(k)}) \subset E.$$

We then have a filtration of the total space E by subspaces

$$* \hookrightarrow E(0) \hookrightarrow E(1) \hookrightarrow \cdots \hookrightarrow E(k) \hookrightarrow E(k+1) \hookrightarrow \cdots \hookrightarrow E.$$

To analyze the E_1 - term of the associated homology spectral spectral sequence we need to compute the E_1 - term, $E_1^{r,s} = H_{r+s}(E(r), E(r-1))$. To do this, write the skeleta of B in the form

$$B^{(r)} = B^{(r-1)} \cup \bigcup_{j \in J_r} D_j^r.$$

Now since each cell D_r is contractible, the restriction of the fibration E to the cells is trivial, and so

$$E(r) - E(r-1) \cong \bigcup_{j \in J_r} D^r \times F.$$

Moreover the attaching maps are via the maps

$$\tilde{\alpha}_r : \bigvee_{j \in J_r} S_j^{r-1} \times F \to E(r-1)$$

induced by the cellular attaching maps $\alpha_k : \bigvee_{j \in J_k} S_j^{k-1} \to B^{(k-1)}$. Using the Mayer - Vietoris sequence, one then computes that

$$\begin{aligned} E_1^{r,s} = H_{r+s}(E(r), E(r-1)) &= H_{r+s}(\bigcup_{j \in J_r} D^r \times F, \bigcup_{j \in J_r} S^{r-1} \times F) \\ &= H_{r+s}(\bigvee_{j \in J_r} S^r \times F, F) \\ &= H_r(\bigvee_{j \in J_r} S^r) \otimes H_s(F) \\ &= C_r(B; H_s(F)). \end{aligned}$$

These calculations indicate the following result, due to Serre in his thesis [84]. We refer the reader to that paper for details. It is one of the great pieces of mathematics literature in the last 75 years.

Theorem 7.29. *Let $p : E \to B$ be a fibration with fiber F. Assume that F is connected and B is simply connected. Then there are chain complexes $C_*(E)$ and $C_*(B)$ computing the homology of E and B respectively, and a filtation of $C_*(E)$ leading to a spectral sequence converging to $H_*(E)$ with the following properties:*

1. $E_1^{r,s} = C_r(B) \otimes H_s(F)$

2. $E_2^{r,s} = H_r(B; H_s(F))$

3. *The differential d_j has bidegree $(-j, j-1)$:*

$$d_j : E_j^{r,s} \to E_j^{r-j, s+j-1}.$$

4. *The inclusion of the fiber into the total space induces a homomorphism*

$$i_* : H_n(F) \to H_n(E)$$

which can be computed as follows:

$$i_* : H_n(F) = E_2^{0,n} \to E_\infty^{0,n} \subset H_n(E)$$

where $E_2^{0,n} \to E_\infty^{0,n}$ *is the projection map which exists because all the differentials* d_j *are zero on* $E_j^{0,n}$*.*

5. *The projection map induces a homomorphism*

$$p_* : H_n(E) \to H_n(B)$$

which can be computed as follows:

$$H_n(E) \to E_\infty^{n,0} \subset E_2^{n,0} = H_n(B)$$

where $E_\infty^{n,0}$ *includes into* $E_2^{n,0}$ *as the subspace consisting of those classes on which all differentials are zero. This is well defined because no class in* $E_j^{n,0}$ *can be a boundary for any* j*.*

Remark. The theorem holds when the base space is not simply connected also. However in that case the E_2 -term is homology with "twisted coefficients". This has important applications in many situations, however we will not consider this issue in these notes. Again, we refer the reader to Serre's thesis [84] or McCleary's text [67] for details.

We will finish this chapter by describing several applications of this important spectal sequence. The first, due to Serre himself [84], is the use of this spectral sequence to prove that even though fibrations do not, in general, admit long exact sequences in homology, they do admit exact sequences in homology through a range of dimensions depending on the connectivity of the base space and fiber.

Theorem 7.30. *Let* $p : E \to B$ *be a fibration with connected fiber* F*, where* B *is simply connected and* $H_i(B) = 0$ *for* $0 < i < n$*, and* $H_i(F) = 0$ *for* $0 < i < m$*. Then there is an exact sequence*

$$H_{n+m-1}(F) \xrightarrow{i_*} H_{n+m-1}(E) \xrightarrow{p_*} H_{n+m-1}(B) \xrightarrow{\tau} H_{n+m-2}(F) \to \cdots \to H_1(E) \to 0.$$

Proof. The E_2 -term of the Serre spectral sequence is given by

$$E_2^{r,s} = H_r(B; H_s(F))$$

which, by hypothesis is zero for $0 < r < n$ or $0 < j < m$. Let $q < n + m$. Then this implies that the composition series for $H_q(E)$, given by the filtration defining the spectral sequence, reduces to the short exact sequence

$$0 \to E_\infty^{0,q} \to H_q(E) \to E_\infty^{q,0} \to 0.$$

Now in general, for these "edge terms", we have

$$E_\infty^{q,0} = kernel\{d_q : E_q^{q,0} \to E_q^{0,q-1}\} \quad \text{and}$$
$$E_\infty^{0,q} = coker\{d_q : E_q^{q,0} \to E_q^{0,q-1}\}.$$

But when $q < n + m$, we have $E_q^{q,0} = E_2^{q,0} = H_q(B)$ and $E_q^{0,q-1} = E_2^{0,q-1} = H_{q-1}(F)$ because there can be no other differentials in this range. Thus if we define

$$\tau : H_q(B) \to H_{q-1}(F)$$

to be $d_q : E_q^{q,0} \to E_q^{0,q-1}$, for $q < n + m$, we then have that $p_* : H_q(E) \to H_q(B)$ maps surjectively onto the kernel of τ, and if $q < n + m - 1$, then the kernel of p_* is the cokernel of $\tau : H_{q+1}(B) \to H_q(F)$. This establishes the existence of the long exact sequence in homology in this range. □

Remark. The homomorphism $\tau : H_q(B) \to H_{q-1}(F)$ for $q < n + m$ in the proof of this theorem is called the "transgression" homomorphism.

7.5.3 Applications I: The Hurewicz theorem

As promised earlier in this chapter, we now use the Serre spectral sequence to prove the Hurewicz theorem. The general theorem is a theorem comparing relative homotopy groups with relative homology groups. We begin by proving the theorem comparing homotopy groups and homology of a single space.

Theorem 7.31. *Let X be an $n-1$ - connected space, $n \geq 2$. That is, we assume $\pi_q(X) = 0$ for $q \leq n-1$. Then $H_q(X) = 0$ for $q \leq n-1$ and the previously defined "Hurewicz homomorphism"*

$$h : \pi_n(X) \to H_n(X)$$

is an isomorphism.

Proof. We assume the reader is familiar with the analogue of the theorem when $n = 1$, which says that for X connected, the first homology group $H_1(X)$ is given by the abelianization of the fundamental group

$$h : \pi_1(X)/[\pi_1, \pi_1] \cong H_1(X)$$

where $[\pi_1, \pi_1] \subset \pi_1(X)$ is the commutator subgroup. We use this preliminary result to begin an induction argument to prove this theorem. Namely we assume that the theorem is true for $n-1$ replacing n in the statement of the theorem. We now complete the inductive step. By our inductive hypotheses, $H_i(X) = 0$ for $i \leq n-2$ and $\pi_{n-1}(X) \cong H_{n-1}(X)$. But we are assuming that $\pi_{n-1}(X) = 0$. Thus we need only show that $h : \pi_n(X) \to H_n(X)$ is an isomorphism.

Consider the path fibration $p : PX \to X$ with fiber the loop space ΩX. Now $\pi_i(\Omega X) \cong \pi_{i+1}(X)$, and so $\pi_i(\Omega X) = 0$ for $i \leq n-2$. So our inductive assumption applied to the loop space says that

$$h : \pi_{n-1}(\Omega X) \to H_{n-1}(\Omega X)$$

is an isomorphism. But $\pi_{n-1}(\Omega X) = \pi_n(X)$. Also, by the Serre exact sequence applied to this fibration, using the facts that

1. the total space PX is contractible, and

2. the fiber ΩX is $n-2$ - connected and the base space X is $(n-1)$ - connected

we then conclude that the transgression,

$$\tau : H_n(X) \to H_{n-1}(\Omega X)$$

is an isomorphism. Hence the Hurewicz map $h : \pi_{n-1}(\Omega X) \to H_{n-1}(\Omega X)$ is the same as the Hurewicz map $h : \pi_n(X) \to H_n(X)$, which is therefore an isomorphism. □

We are now ready to prove the more general relative version of this theorem 7.21

Theorem 7.32. *Let X be simply connected, and let $A \subset X$ be a simply connected subspace. Suppose that the pair (X, A) is $(n-1)$ - connected, for $n > 2$. That is,*

$$\pi_k(X, A) = 0 \quad \text{if} \quad k \leq n-1.$$

Then the Hurewicz homomorphism $h_* : \pi_n(X, A) \to H_n(X, A)$ is an isomorphism.

Proof. . Replace the inclusion

$$\iota : A \hookrightarrow X.$$

by a homotopy equivalent fibration $\tilde{\iota} : \tilde{A} \to X$ as in 7.7. Let F_ι be the fiber. Then $\pi_i(F_\iota) \cong \pi_{i+1}(X, A)$, by comparing the long exact sequences of the pair (X, A) to the long exact sequence in homotopy groups for the fibration $\tilde{A} \to X$. So by the Hurewicz theorem 7.31 we know that $\pi_i(F) = H_i(F) = 0$ for $i \leq n-2$ and

$$h : \pi_{n-1}(F) \to H_{n-1}(F)$$

is an isomorphism. But as mentioned, $\pi_{n-1}(F) \cong \pi_n(X, A)$ and by comparing the homology long exact sequence of the pair (X, A) to the Serre exact sequence for the fibration $F \to \tilde{A} \to B$, one has that $H_{n-1}(F) \cong H_n(X, A)$. The theorem follows. □

As a corollary, we obtain the following strengthening of the Whitehead theorem 7.16 which is quite useful in calculations.

Corollary 7.33. *Suppose X and Y are simply connected CW - complexes and $f : X \to Y$ a continuous map that induces an isomorphism in homology groups,*

$$f_* : H_k(X) \xrightarrow{\cong} H_k(Y) \quad \textit{for all } k \geq 0$$

Then $f : X \to Y$ is a homotopy equivalence.

Proof. Replace $f : X \to Y$ by the inclusion into the mapping cylinder

$$\bar{f} : X \hookrightarrow \bar{Y}$$

where $\bar{Y} = Y \cup_f X \times I$ which is homotopy equivalent to Y, and $\bar{f}$ includes X into $\bar{Y}$ as $X \times \{1\}$.

Since X and Y are simply connected, we have that $\pi_2(X) \cong H_2(X)$ and $\pi_2(Y) \cong H_2(Y)$. Thus $f_* : \pi_2(X) \to \pi_2(Y)$ is an isomorphism. Again, since X and Y are simply connected, this implies that $\pi_q(\bar{Y}, X) = 0$ for $q = 1, 2$. Thus we can apply the relative Hurewicz theorem. However since $f_* : H_k(X) \cong H_k(Y)$ for all $k \geq 0$, we have that $H_k(\bar{Y}, X) = 0$ for all $k \geq 0$. But then the Hurewicz theorem implies that $\pi_k(\bar{Y}, X) = 0$ for all k, which in turn implies that $f_* : \pi_k(X) \to \pi_k(Y)$ is an isomorphism for all k. The theorem follows from the Whitehead theorem 7.16. □

7.5.4 Applications II: $H_*(\Omega S^n)$ and $H^*(U(n))$

In this section we will use the Serre spectral sequence to compute the homology of the loop space ΩS^n and the cohomology ring of the Lie groups, $H^*(U(n))$.

Theorem 7.34.

$$H_q(\Omega S^n) = \begin{cases} \mathbb{Z} & \text{if } q \text{ is a multiple of } n-1, \text{ i.e } q = k(n-1) \\ 0 & \text{otherwise} \end{cases}$$

Proof. ΩS^n is the fiber of the path fibration $p : PS^n \to S^n$. Since the total space of this fibration is contractible, the Serre spectral sequence converges to 0 in positive dimensions. That is,

$$E_\infty^{r,s} = 0$$

for all r, s, except that $E_\infty^{0,0} = \mathbb{Z}$. Now since the base space, S^n has nonzero homology only in dimensions 0 and n (when it is $\mathbb{Z}$), then

$$E_2^{r,s} = H_r(S^n; H_s(\Omega S^n))$$

is zero unless $r = 0$ or n. In particular, since $d_q : E_q^{r,s} \to E_q^{r-q,s+q-1}$, we must have that for $q < n$, $d_q = 0$. Thus $E_2^{r,s} = E_n^{r,s}$ and the only possible nonzero differential d_n occurs in dimensions

$$d_n : E_n^{n,s} \to E_n^{0,s+n-1}.$$

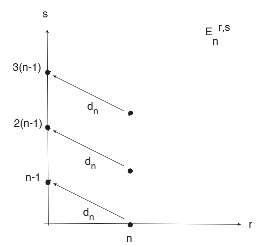
It is helpful to picture this spectral sequence as in the following diagram, where a dot in the (r, s) - entry denotes a copy of the integers in $E_n^{r,s} = H_r(S^n; H_s(\Omega S^n))$.

Notice that if the generator $\sigma_{n,0} \in E_n^{n,0}$ is in the kernel of d_n, then it would represent a nonzero class in $E_{n+1}^{n,0}$. But d_{n+1} and all higher differentials on $E_{n+1}^{n,0}$ must be zero, for dimensional reasons. That is, $E_{n+1}^{n,0} = E_\infty^{n,0}$. But we saw that $E_\infty^{n,0} = 0$. Thus we must conclude that $d_n(\sigma_{n,0}) \neq 0$. For the same reasoning, (i.e the fact that $E_{n+1}^{n,0} = 0$) we must have that $d_n(k\sigma_{n,0}) \neq 0$ for all integers k. This means that the image of

$$d_n : E_n^{n,0} \to E_n^{0,n-1}$$

is $\mathbb{Z} \subset E_n^{0,n-1} = H_{n-1}(\Omega S^n)$. On the other hand, we claim that $d_n : E_n^{n,0} \to E_n^{0,n-1}$ must be surjective. For if $\alpha \in E_n^{0,n-1}$ is not in the image of d_n, then it represents a nonzero class in $E_{n+1}^{0,n-1} = E_\infty^{0,n-1}$. But as mentioned earlier $E_\infty^{0,n-1} = 0$. So d_n is surjective as well. In fact we have proven that

$$d_n : \mathbb{Z} = H_n(S^n) = E_n^{n,0} \to E_n^{0,n-1} = E_2^{0,n-1} = H_{n-1}(\Omega S^n)$$



is an isomorphism. Hence $H_{n-1}(\Omega S^n) \cong \mathbb{Z}$, as claimed. Now notice this calculation implies a calculation of $E_2^{n,n-1}$, namely,

$$E_2^{n,n-1} = H_n(S^n; H_{n-1}(\Omega S^n) = bz.$$

Repeating the above argument shows that $E_2^{n,n-1} = E_n n, n-1$ and that

$$d_n : E_n^{n,n-1} \to E_n^{0,2(n-1)}$$

must be an isomorphism. This yields that

$$\mathbb{Z} = E_2^{0,2(n-1)} = H_{2(n-1)}(\Omega S^n).$$

Repeating this argument shows that for every q, $\mathbb{Z} = E_2^{n,q(n-1)} = E_n^{n,q(n-1)}$ and that

$$d_n : E_n^{n,q(n-1)} \to E_n 0, (q+1)(n-1) \cong H_{(q+1)(n-1)}(\Omega S^n)$$

is an isomorphism. And so $H_{k(n-1)}(\Omega S^n) = \mathbb{Z}$ for all k.

We can also conclude that in dimensions j not a multiple of $n-1$, then $H_j(\Omega S^n)$ must be zero. This is true by the following argument. Assume the contrary, so that there is a smallest $j > 0$ not a multiple of $n-1$ with $H_j(\Omega S^n) = E_2^{0,j} \neq 0$. But for dimensional reasons, this group cannot be in the image of any differential, because the only $E_q^{r,s}$ that can be nonzero with $r > 0$ is when $r = n$. So the only possibility for a class $\alpha \in E_2^{0,j}$ to represent a class which is in the image of a differential is $d_n : E_n^{n,s} \to E_n^{0,s+n-1}$. So $j = s+n-1$.

But since j is the smallest positive integer not of the form a multiple of $n-1$ with $H_j(\Omega S^n)$ nonzero, then for $s < j$, $E_n^{n,s} = H_n(S^n, H_s(\Omega S^n)) = H_s(\Omega S^n)$ can only be nonzero if s is a multiple of $(n-1)$, and therefore so is $s+n-1 = j$. This contradiction implies that if j is not a multiple of $n-1$, then $H_j((\Omega S^n))$ is zero. This completes our calculation of$H_*(\Omega S^n)$. □

We now use the cohomology version of the Serre spectral sequence to compute the cohomology of the unitary groups. We first give the cohomological analogue of 7.29. Again, the reader should consult [84] for details.

Theorem 7.35. *Let $p : E \to B$ be a fibration with fiber F. Assume that F is connected and B is simply connected. Then there is a cohomology spectral sequence converging to $H^*(E)$, with $E_2^{r,s} = H^r(B; H^s(F))$, having the following properties.*

1. *The differential d_j has bidegree $(j, -j+1)$:*

$$d_j : E_j^{r,s} \to E_j^{r+j,s-j+1}.$$

2. *For each j, $E_j^{*,*}$ is a bigraded ring. The ring multiplication maps*

$$E_j^{p,q} \otimes E_j^{i,j} \to E_j^{p+i,q+j}.$$

3. *The differential $d_j : E_j^{r,s} \to E_j^{r+j,s-j+1}$. is an antiderivation in the sense that it satisfies the product rule:*

$$d_j(ab) = d_j(a) \cdot b + (-1)^{u+v} a \cdot d_j(b)$$

   where $a \in E_j^{u,v}$.

4. *The product in the ring E_{j+1} is induced by the product in the ring E_j, and the product in E_∞ is induced by the cup product in $H^*(E)$.*

We apply this to the following calculation.

Theorem 7.36. *There is an isomorphism of graded rings,*

$$H^*(U(n)) \cong \Lambda[\sigma_1, \sigma_3, \cdots, \sigma_{2n-1}],$$

the graded exterior algebra on one generator σ_{2k-1} in every odd dimension $2k-1$ for $1 \leq k \leq n$.

Proof. We prove this by induction on n. For $n = 1$, $U(1) = S^1$ and we know the assertion is correct. Now assume that $H^*(U(n-1)) \cong \Lambda[\sigma_1, \cdots \sigma_{2n-3}]$. Consider the Serre cohomology spectral sequence for the fibration

$$U(n-1) \subset U(n) \to U(n)/U(n-1) \cong S^{2n-1}.$$

Then the E_2 - term is given by

$$E_2^{*,*} \cong H^*(S^{2n-1}; H^*(U(n-1)) = H^*(S^{2n-1}) \otimes H^*(U(n-1))$$

and this isomorphism is an isomorphism of graded rings. But by our inductive assumption we have that

$$\begin{aligned} H^*(S^{2n-1}) \otimes H^*(U(n-1)) &\cong \Lambda[\sigma_{2n-1}] \otimes \Lambda[\sigma_1, \cdots \sigma_{2n-3}] \\ &\cong \Lambda[\sigma_1, \sigma_3, \cdots, \sigma_{2n-1}]. \end{aligned}$$

Thus

$$E_2^{**} \cong \Lambda[\sigma_1, \sigma_3, \cdots, \sigma_{2n-1}]$$

as graded algebras. Now since all the nonzero classes in $E_2^{*,*}$ have odd total degree (where the total degree of a class $\alpha \in E_2^{r,s}$ is $r+s$), and all differentials increase the total degree by one, we must have that all differentials in this spectral sequence are zero. Thus

$$E_\infty^{*,*} = E_2^{*,*} \cong \Lambda[\sigma_1, \sigma_3, \cdots, \sigma_{2n-1}].$$

We then conclude that $H^*(U(n)) =\cong \Lambda[\sigma_1, \sigma_3, \cdots, \sigma_{2n-1}]$ which completes the inductive step in our proof. □

7.5.5 Applications III: $H_*(K(\mathbb{Q}, n))$

We will use the Serre spectral sequence to compute the homology of the rational Eilenberg-MacLane spaces, $K(\mathbb{Q}; n)$.

Theorem 7.37. *The homology of the Eilenberg-MacLane spaces $K(\mathbb{Q}, n)$ is given as follows:*

$$\tilde{H}_q(K(\mathbb{Q}, 2m); \mathbb{Z}) = \begin{cases} \mathbb{Q}, & \text{if } q \text{ is a positive multiple of } 2m \,, \\ 0 & \text{otherwise.} \end{cases}$$
$$\tilde{H}_q(K(\mathbb{Q}, 2m+1); \mathbb{Z}) = \begin{cases} \mathbb{Q}, & \text{if } q = 2m+1, \\ 0 & \text{otherwise.} \end{cases} \quad .$$

Proof. Consider the path-loop fibration, $\Omega K(\mathbb{Q}, n) \to PK(\mathbb{Q}, n) \to K(\mathbb{Q}, n)$. Notice that the based loop space, $\Omega K(\mathbb{Q}, n)$ is an Eilenberg-MacLane space of type $K(\mathbb{Q}, n-1)$. We now prove the theorem by induction on n. For $n = 0$, the statement is obvious. Inductively assume the theorem is true for $n-1$ and we want to prove it for n. We consider the Serre spectral sequence for this fibration. Since the path space $PK(\mathbb{Q}, n)$ is contractible, the spectral sequence must converge to zero in positive dimensions. For this to happen, the spectral sequences must have the following form, depending on whether n is even or odd. The argument is very similar to that which was carried out in the calculation of $H^*(\Omega S^n)$ (Theorem 7.34). We leave the verification of these descriptions as an exercise for the reader.

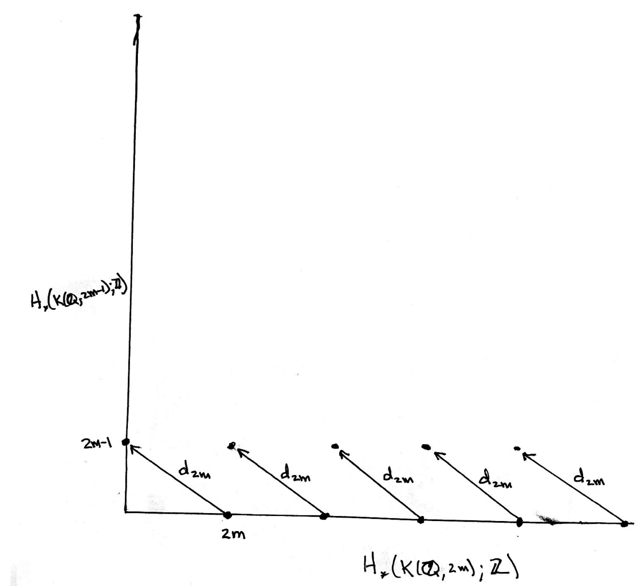


FIGURE 7.1
The Serre spectral sequence for the homology of the fibration $K(\mathbb{Q}, 2m-1) \to PK(\mathbb{Q}, 2m) \to K(\mathbb{Q}, 2m)$.

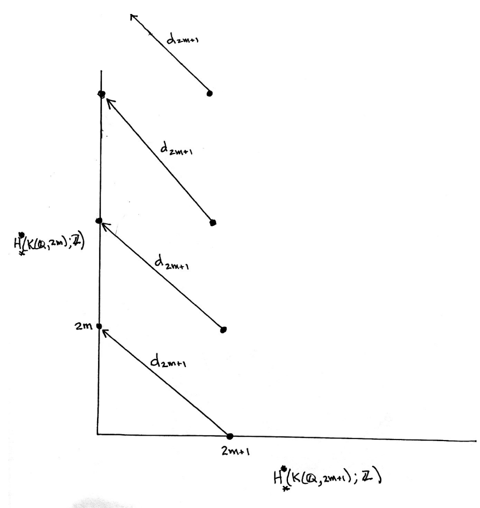


FIGURE 7.2
The Serre spectral sequence for the homology of the fibration $K(\mathbb{Q}, 2m) \to PK(\mathbb{Q}, 2m+1) \to K(\mathbb{Q}, 2m+1)$.

The result now follows from these spectral sequences.

□

7.5.6 Applications IV: *Spin* and $Spin_{\mathbb{C}}$ structures

In this section we describe the notions of *Spin* and $Spin_{\mathbb{C}}$ structures on vector bundles. We then use the Serre spectral sequence to identify characteristic class conditions for the existence of these structures. These structures are particularly important in geometry, geometric analysis, and geometric topology.

Recall from chapter 4 that an n - dimensional vector bundle ζ over a space X is orientable if and only if it has a $SO(n)$ - structure, which exists if and only if the classifying map $f_\zeta : X \to BO(n)$ has a homotopy lifting to $BSO(n)$. In chapter 4 we proved the following property as well.

Proposition 7.38. *The n - dimensional bundle ζ is orientable if and only if its first Stiefel - Whitney class is zero,*

$$w_1(\zeta) = 0 \in H^1(X; \mathbb{Z}_2).$$

A *Spin* structure on ζ is a refinement of an orientation. To define it we need to further study the topology of $SO(n)$.

The group $O(n)$ has two path components, i.e $\pi_0(O(n)) \cong \mathbb{Z}_2$ and $SO(n)$ is the path component of the identity map. In particular $SO(n)$ is connected, so $\pi_0(SO(n)) = 0$. We have the following information about $\pi_1(SO(n))$.

Proposition 7.39. *$\pi_1(SO(2)) = \mathbb{Z}$. For $n \geq 3$, we have*

$$\pi_1(SO(n)) = \mathbb{Z}_2.$$

Proof. $SO(2)$ is topologically a circle, so the first part of the theorem follows. $SO(3)$ is topologically the projective space

$$SO(3) \cong \mathbb{RP}^3$$

which has a double cover $\mathbb{Z}_2 \to S^3 \to \mathbb{RP}^3$. Since S^3 is simply connected, this is the universal cover of $\mathbb{RP}^3$ and hence $\mathbb{Z}_2 = \pi_1(\mathbb{RP}^3) = \pi_1(SO(3))$.

Now for $n \geq 3$, consider the fiber bundle $SO(n) \to SO(n+1) \to SO(n+1)/SO(n) = S^n$. By the long exact sequence in homotopy groups for this fibration we see that $\pi_1(SO(n)) \to \pi_1(SO(n+1))$ is an isomorphism for $n \geq 3$. The result follows by induction on n. □

Since $\pi_1(SO(n)) = \mathbb{Z}_2$, the universal cover of $SO(n)$ is a double covering. The group $Spin(n)$ is defined to be this universal double cover:

$$\mathbb{Z}_2 \to Spin(n) \to SO(n).$$

Exercise. Show that $Spin(n)$ is a group and that the projection map $p : Spin(n) \to SO(n)$ is a group homomorphism with kernel $\mathbb{Z}_2$.

Now the group $Spin(n)$ has a natural $\mathbb{Z}_2$ action, since it is the double cover of $SO(n)$. Define the group $Spin_{\mathbb{C}}(n)$ using this $\mathbb{Z}_2$ - action in the following way.

Definition 7.8. *The group $Spin_{\mathbb{C}}(n)$ is defined to be*

$$Spin_{\mathbb{C}}(n) = Spin(n) \times_{\mathbb{Z}_2} U(1).$$

where $\mathbb{Z}_2$ acts on $U(1)$ by $z \to -z$ for $z \in U(1) \subset \mathbb{C}$.

Notice that there is a principal $U(1)$ - bundle,

$$U(1) \to Spin_{\mathbb{C}}(n) = Spin(n) \times_{\mathbb{Z}_2} U(1) \to Spin(n)/\mathbb{Z}_2 = SO(n).$$

$Spin_{\mathbb{C}}$ - structures have been shown to be quite important in the Seiberg - Witten theory approach to the study of smooth structures on four dimensional manifolds [53].

The main theorem of this section is the following:

Theorem 7.40. *Let ζ be an oriented n - dimensional vector bundle over a CW - complex X. Let $w_2(\zeta) \in H^2(X;\mathbb{Z}_2)$ be the second Stiefel - Whitney class of ζ. Then*

1. *ζ has a $Spin(n)$ structure if and only if $w_2(\zeta) = 0$.*

2. *ζ has a $Spin_{\mathbb{C}}(n)$ - structure if and only if $w_2(\zeta) \in H^2(X;\mathbb{Z}_2)$ comes from an integral cohomology class. That is, if and only if there is a class $c \in H^2(X;\mathbb{Z})$ which maps to $w_2(\zeta)$ under the projection map*

$$H^2(X;\mathbb{Z}) \to H^2(X;\mathbb{Z}_2).$$

Proof. The question of the existence of a $Spin$ or $Spin_{\mathbb{C}}$ structure is equivalent to the existence of a homotopy lifting of the classifying map $f_\zeta : X \to BSO(n)$ to $BSpin(n)$ or $BSpin_{\mathbb{C}}(n)$. To examine the obstructions to obtaining such liftings we first make some observations about the homotopy type of $BSO(n)$.

We know that $BSO(n) \to BO(n)$ is a double covering (the orientation double cover of the universal bundle). Furthermore $\pi_1(BO(n)) = \pi_0(O(n)) = \mathbb{Z}_2$, so this is the universal cover of $BO(n)$. In particular this says that $BSO(n)$ is simply connected and

$$\pi_i(BSO(n)) \to \pi_i(BO(n))$$

is an isomorphism for $i \geq 2$.

Recall that for n odd, say $n = 2m+1$, then there is an isomorphism of groups

$$SO(2m+1) \times \mathbb{Z}_2 \cong O(2m+1).$$

Exercise. Prove this!

This establishes a homotopy equivalence

$$BSO(2m+1) \times B\mathbb{Z}_2 \cong BO(2m+1).$$

The following is then immediate from our knowledge of $H^*(BO(2m+1);\mathbb{Z}_2) \cong \mathbb{Z}_2[w_1, \cdots, w_{2m+1}]$ and $H^*(B\mathbb{Z}_2;\mathbb{Z}_2) \cong \mathbb{Z}_2[w_1]$.

Lemma 7.41.

$$H^*(BSO(2m+1);\mathbb{Z}_2) \cong \mathbb{Z}_2[w_2, \cdots, w_{2m+1}]$$

where $w_i \in H^i(BSO(2m+1);\mathbb{Z}_2)$ is the i^{th} Stiefel - Whitney class of the universal oriented $(2m+1)$ - dimensional bundle classified by the natural map $BSO(2m+1) \to BO(2m+1)$.

Corollary 7.42. *For $n \geq 3$, $H^2(BSO(n); \mathbb{Z}_2) \cong \mathbb{Z}_2$, with nonzero class w_2.*

Proof. This follows from the lemma and the fact that for $n \geq 3$ the inclusion $BSO(n) \to BSO(n+1)$ induces an isomorphism in H^2, which can be seen by looking at the Serre exact sequence for the fibration $S^n \to BSO(n) \to BSO(n+1)$. □

This allows us to prove the following.

Lemma 7.43. *The classifying space $BSpin(n)$ is homotopy equivalent to the homotopy fiber F_{w_2} of the map*

$$w_2 : BSO(n) \to K(\mathbb{Z}_2, 2)$$

classifying the second Stiefel - Whitney class $w_2 \in H^2(BSO(n); \mathbb{Z}_2)$.

Proof. The group $Spin(n)$ is the universal cover of $SO(n)$, and hence is simply connected. This means that $BSpin(n)$ is 2 - connected. By the Hurewicz theorem this implies that $H^2(BSpin(n); \mathbb{Z}_2) = 0$. Thus the composition

$$BSpin(n) \xrightarrow{p} BSO(n) \xrightarrow{w_2} K(\mathbb{Z}_2, 2)$$

is null homotopic. Convert the map w_2 to a homotopy equivalent fibration, $\tilde{w}_2 : \tilde{B}SO(n) \to K(\mathbb{Z}_2, 2)$. The map p defines a map (up to homotopy) $\tilde{p} : BSpin(n) \to \tilde{B}SO(n)$, and the composition $\tilde{p} \circ w_2$ is still null homotopic. A null homotopy $\Phi : BSpin(n) \times I \to K(\mathbb{Z}_2, 2)$ between $\tilde{p} \circ w_2$ and the constant map at the basepoint, lifts, due to the homotopy lifting property, to a homotopy $\tilde{\Phi} : BSpin(n) \times I \to \tilde{B}SO(n)$ between $\tilde{p}$ and a map $\bar{p}$ whose image lies entirely in the fiber over the basepont, F_{w_2},

$$\bar{p} : BSpin(n) \to F_{w_2}.$$

We claim that $\bar{p}$ induces an isomorphism in homotopy groups. To see this, observe that the homomorphism $p_q : \pi_q(BSpin(n)) \to \pi_q(BSO(n))$ is equal to the homomorphism $\pi_{q-1}(Spin(n)) \to \pi_{q-1}(SO(n))$ which is an isomorphism for $q \geq 3$ because $Spin(n) \to SO(n)$ is the universal cover. But similarly $\pi_q(F_{w_2}) \to \pi_q(BSO(n))$ is also an isomorphism for $q \geq 3$ by the exact sequence in homotopy groups of the fibration $F_{w_2} \to BSO(n) \xrightarrow{w_2} K(\mathbb{Z}_2, 2)$, since w_2 induces an isomorphism on π_2. $BSpin(n)$ and F_{w_2} are also both 2 - connected. Thus they have the same homotopy groups, and we have a commutative square for $q \geq 3$,

$$\begin{array}{ccc} \pi_q(BSpin(n)) & \xrightarrow{\bar{p}_*} & \pi_q(F_{w_2}) \\ {\scriptstyle p}\downarrow\cong & & \downarrow\cong \\ \pi_q(BSO(n)) & \xrightarrow[=]{} & \pi_q(BSO(n)). \end{array}$$

Thus $\bar{p} : BSpin(n) \to F_{w_2}$ induces an isomorphism in homotopy groups, and by the Whitehead theorem is a homotopy equivalence. $\square$

Notice that we are now able to complete the proof of the first part of the theorem. If ζ is any oriented, n - dimensional bundle with $Spin(n)$ structure, its classifying map$f_\zeta : X \to BSO(n)$ lifts to a map $\tilde{f}_\zeta : X \to BSpin(n)$, and hence by this lemma, $w_2(\zeta) = f_\zeta^*(w_2) = f_\zeta^* \circ p^*(w_2) = 0$. Conversely, if $w_2(\zeta) = 0$, then the classifying map $f_\zeta : X \to BSO(n)$ has the property that $f_\zeta^*(w_2) = 0$. This implies that the composition

$$X \xrightarrow{f_\zeta} BSO(n) \xrightarrow{w_2} K(\mathbb{Z}_2, 2)$$

is null homotopic. A null homotopy lifts to give a homotopy between f_ζ and a map whose image lies in the homotopy fiber F_{w_2}, which, by the above lemma is homotopy equivalent to $BSpin(n)$. Thus $f_\zeta : X \to BSO(n)$ has a homotopy lift $\tilde{f}_\zeta : X \to BSpin(n)$, which implies that ζ has a $Spin(n)$ - structure.

We now turn our attention to $Spin_{\mathbb{C}}$ - structures.

Consider the projection map

$$p : Spin_{\mathbb{C}}(n) = Spin(n) \times_{\mathbb{Z}_2} U(1) \to U(1)/\mathbb{Z}_2 = U(1).$$

p is a group homomorpism with kernel $Spin(n)$. p therefore induces a map on classifying spaces, which we call c,

$$c : BSpin_{\mathbb{C}}(n) \to BU(1) = K(\mathbb{Z}, 2)$$

which has homotopy fiber $BSpin(n)$. But clearly we have the following homotopy commutative diagram

$$\begin{array}{ccccc}
BSpin(n) & \xrightarrow{\subset} & B(Spin(n) \times_{bz_2} U(1)) & \xrightarrow{=} & BSpin_{\mathbb{C}}(n) \\
{=}\downarrow & & \downarrow & & \downarrow p \\
BSpin(n) & \longrightarrow & B(Spin(n)/\mathbb{Z}_2) & \xrightarrow[=]{} & BSO(n)
\end{array}$$

Therefore we have the following diagram between homotopy fibrations

$$\begin{array}{ccccc}
BSpin(n) & \longrightarrow & BSpin_{\mathbb{C}}(n) & \xrightarrow{c} & K(\mathbb{Z}, 2) \\
{=}\downarrow & & \downarrow & & \downarrow p \\
BSpin(n) & \longrightarrow & BSO(n) & \xrightarrow{w_2} & K(\mathbb{Z}_2, 2)
\end{array}$$

where $p : K(\mathbb{Z}, 2) \to K(\mathbb{Z}_2, 2)$ is induced by the projection $\mathbb{Z} \to \mathbb{Z}_2$. As we've done before we can assume that $p : K(\mathbb{Z}, 2) \to K(\mathbb{Z}_2, 2)$ and $w_2 : BSO(n) \to K(\mathbb{Z}_2, 2)$ have been modified to be fibrations. Then this means that $BSpin_{\mathbb{C}}(n)$ is homotopy equivalent to the pull - back along w_2 of the fibration $p : K(\mathbb{Z}, 2) \to K(\mathbb{Z}_2, 2)$:

$$BSpin_{\mathbb{C}}(n) \simeq w_2^*(K(\mathbb{Z}, 2)).$$

But this implies that the map $f_\zeta : X \to BSO(n)$ homotopy lifts to $BSpin_{\mathbb{C}}(n)$ if and only if there is a map $u : X \to K(\mathbb{Z}, 2)$ such that $p \circ u : X \to K(\mathbb{Z}_2, 2)$ is homotopic to $w_2 \circ f_\zeta : X \to K(\mathbb{Z}_2, 2)$. Interpreting these as cohomology classes, this says that f_ζ lifts to $BSpin_{\mathbb{C}}(n)$ (i.e ζ has a $Spin_{\mathbb{C}}(n)$ - structure) if and only if there is a class $u \in H^2(X; \mathbb{Z})$ so that the $\mathbb{Z}_2$ reduction of u, $p_*(u)$ is equal to $w_2(\zeta) \in H^2(X; \mathbb{Z}_2)$. This is the statement of the theorem. □

8

Tubular Neighborhoods, more on Transversality, and Intersection Theory

8.1 The tubular neighborhood theorem

We begin this chapter by proving another important, and basic result in differential topology: the "Tubular Neighborhood Theorem".

Theorem 8.1. *Suppose M^n is an n-dimensional smooth manifold, and suppose the $N^k \subset M^n$ is a k-dimensional submanifold. Then there exists an open neighborhood η of N^k in M^n that satisfies the following properties:*

1. *There is a neighborhood deformation retract*

$$p : \eta \to N^k.$$

That is, p is a smooth map with the property that $p \circ \iota = id_{N^k}$ and $\iota \circ p : \eta \to \eta$ is homotopic to id_η. Here $\iota : N^k \hookrightarrow M^n$ is the inclusion.

2. *Let $\pi : \nu \to N^k$ be the normal bundle of N^k in M^n. Then there is a diffeomorphism $\Phi : \eta \to \nu$ making the following diagram commute:*

$$\begin{array}{ccc} \eta & \xrightarrow{\Phi} & \nu \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} \\ N^k & \xrightarrow[=]{} & N^k \end{array}$$

Remark. The open set η in this theorem is referred to as a "tubular neighborhood" because, as the theorem states, it is diffeomorphic to the total space of a vector bundle, ν which locally looks like a "tube", $N^k \times \mathbb{R}^{n-k}$.

Observe that the statement of this theorem can be made in another way, which is often quite useful.

Theorem 8.2. *(Tubular neighborhood theorem, equivalent formulation.) Suppose $e : N^k \hookrightarrow M^n$ is an embedding of smooth manifolds with normal bundle ν. Assume that N^k is closed. Consider the inclusion of the zero section,*

$\zeta : N^k \hookrightarrow \nu$. Then the embedding e extends to an embedding $\tilde{e} : \nu \hookrightarrow M^n$ which is a diffeomorphism onto an open subset of M^k. By $\tilde{e}$ "extending" e we mean that the composition

$$N^k \xrightarrow{\zeta} \nu \xrightarrow{\tilde{e}} M^n$$

is equal to the embedding $e : N^k \hookrightarrow M^n$.

We leave it to the reader to check that this formulation is indeed equivalent to Theorem 8.1. We begin the proof of Theorem 8.1 by first proving it in the case where the ambient manifold is Euclidean spaces.

Theorem 8.3. *Let $e : N^k \hookrightarrow \mathbb{R}^n$ be an embedding of a closed manifold N^k. Then N^k has a "tubular neighborhood".*

Proof. Observe that it suffices to show that there is an open neighborhood V of the zero section $N^k \hookrightarrow \nu$ that supports an embedding into $\mathbb{R}^n$ that extends $e : N^k \to \mathbb{R}^n$. This is because, by the vector bundle structure of ν there is clearly an embedding of ν into any neighborhood of the zero section that fixes the zero section.

Let m be the codimension, $m = n - k$. Consider the map to the Grassmannian,

$$g : N^k \to Gr_m(\mathbb{R}^n)$$

defined by $g(x) = \nu_x \subset \mathbb{R}^n$. That is, $g(x)$ is the normal space to x in $\mathbb{R}^n$. More precisely,

$$\nu_x = \left(D_x e(T_x N^k)\right)^\perp.$$

Notice that the normal bundle $\nu \to N^k$ is the pullback, $\nu = g^*(\gamma_m)$, where $\gamma_m \to Gr_m(\mathbb{R}^n)$ is the canonical bundle. Specifically,

$$g^*(\gamma_m) = \{(x, v) \in N^k \times \mathbb{R}^n \ : \ v \in \nu_x\}.$$

Define a map $\phi : \nu \to \mathbb{R}^n$ by $\phi(x, v) = x + v \in \mathbb{R}^n$. As above, identify ν with $g^*(\gamma_m)$. Then notice that the tangent space to ν at $(x, 0)$ is given by

$$T_{(x,0)}\nu = T_x M \oplus \nu_x.$$

Furthermore, one immediately sees that the derivative of ϕ at $(x, 0)$,

$$D_{(x,0)}\phi : T_{(x,0)}\nu \to T_x\mathbb{R}^n$$

is the identity on on both $T_x M$ and on ν_x. Therefore $D\phi$ has rank n at all points on the zero section. It follows that ϕ is an immersion of a neighborhood U of the zero section in ν. Since the restriction of ϕ to the zero section itself is the given by the identity of $N^k \subset \mathbb{R}^n$, it implies that the restriction of ϕ to a perhaps smaller neighborhood V of the zero section in ν is an embedding.

□

We now proceed with the proof of Theorem 8.1.

Proof. By Whitney's embedding theorem 6.1 we can assume that $M^n \subset \mathbb{R}^N$ for some sufficiently large N. Let $W \subset \mathbb{R}^N$ be a tubular neighborhood of M^n, and $r : W \to M^n$ a retraction. Give M^n a metric induced by the Euclidean metric on $\mathbb{R}^n$. Notice we have an inclusion of vector bundles over N^k,

$$\nu \hookrightarrow TM^n_{|_{N^k}} \hookrightarrow T\mathbb{R}^N_{|_{N^k}} = N^k \times \mathbb{R}^N.$$

For $x \in N^k$, let $U_x = \{(x, v) \in \nu_x : x + v \in W\}$. Then the set $U = \bigcup_{x \in N^k} U_x$ can be viewed as a subset of $N^k \times \mathbb{R}^N$ and can then be given the subspace topology. Notice that by definition, $U \subset \nu$ and is an open subspace, because it is the inverse image of W under the map

$$\nu \to \mathbb{R}^N$$
$$(x, v) \to x + v.$$

The map

$$\phi : U \to M^n$$
$$\phi(x, v) = r(x + v)$$

is then easily checked to be a tubular neighborhood of $e : N^k \hookrightarrow M^n$. □

The tubular neighborhood theorem is extremely important in differential topology, and is used quite often. For example, it is crucial in knot theory, where one studies embeddings of S^1 in $\mathbb{R}^3 \subset S^3$. Let K be such a knot. That is, it is the image of such an embedding. Let $\eta(K)$ be a tubular neighborhood of K in S^3. Then the fundamental group of the complement, $S^3 - \eta(K)$ is an extremely important invariant of the isotopy class of the knot, and has been the main tool in studying knot theory for a century. This group is most often not abelian, but has abelianization $= \mathbb{Z}$. This is seen using the fact that the abelianiization of $\pi_1(S^3 - K)$ is equal to the first homology, $H_1((S^3 - K)$, and then using Alexander duality.

8.2 The genericity of transversality

In Chapter 3 we discussed the notions of regular values and transversality. In this section we will return to these notions and prove that they are *generic* in a sense that we will make precise. We will be following the discussion of these results given in Bredon's book [13] which is a very good reference for these concepts.

Recall that if $\phi : M^n \to N^n$ is smooth, then $p \in M$ is a *critical point* of ϕ

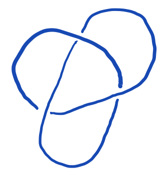


FIGURE 8.1
The trefoil knot

if the derivative $D_p\phi$ has rank strictly smaller than n. If p is critical, $\phi(p) \in N$ is a critical value. If $x \in N$ is not a critical value, it is called a *regular* value. So in particular, $x \in N$ is regular

- if $m \geq n$ and $D_p\phi$ is surjective for all $p \in M$ with $\phi(p) = x$, or
- $m < n$ and x is not in the image of ϕ.

The following theorem is well known in Analysis and Topology, and its proof is given in many texts, including the appendix of Bredon's book [13], as well as in Hirsch's book [44].

Theorem 8.4. *(Sard's theorem) If $\phi : M^m \to \mathbb{R}^n$ is C^∞, then the set of critical values has measure zero in $\mathbb{R}^n$.*

Before we state an important corollary to this theorem, which we will rely on heavily, we recall some terms from measure theory.

Definition 8.1. *A* **nowhere dense** subspace of a topological space is one whose set theoretic closure has empty interior. A subspace $E \subset X$ is **first category** if E is the countable union of subspaces that are nowhere dense. A **residual subspace** is the complement of a first category subspace. That is, it's complement is the countable union of nowhere dense subspaces. A residual subspace is sometimes called "everywhere dense".

Corollary 8.5. *(A. B Brown's theorem) If $\phi : M^m \to N^n$ is a C^∞ map, then the set of regular values of ϕ is residual in N^n.*

Proof. If C is the set of critical points of ϕ, and $K \subset M^m$ is compact, then $\phi(C \cap K)$ is a compact subspace of N^n, and its interior is empty by Sard's theorem. Therefore $\phi(C \cap K)$ is nowhere dense. Since M^m is covered by a countable union of such compace subspaces, $\phi(C)$ is first category and thus it's complement is residual. □

We note that if $m = 1$, then Sard's theorem says that there aren't any smooth, space-filling curves, unlike in the continuous setting.

We now apply Sard's theorem to the setting of transversality theory. We first show that zero sections of vector bundles can be perturbed to be transverse to any map.

Theorem 8.6. *Let $\xi \to Y$ be a smooth vector bundle over a smooth, compact manifold. Let X be a smooth manifold and $f : X \to \xi$ a smooth map. Then there is a smooth cross section $s : Y \to \xi$ as close to the zero section as desired, so that $f \pitchfork s(Y)$.*

Proof. Since Y is compact, we know that there exists a smooth vector bundle $\eta \to Y$ such that $\xi \oplus \eta$ is trivial. That is, there is an isomorphism of vector bundles over Y,

$$\Psi : \xi \oplus \eta \xrightarrow{\cong} Y \times \mathbb{R}^n,$$

which we can take to be smooth. Let $p : \xi \oplus \eta \to \mathbb{R}^n$ be the projection of Ψ onto the $\mathbb{R}^n$ factor. We then have a commutative diagram

$$\begin{array}{ccc} f^*(\xi \oplus \eta) & \xrightarrow{\bar{f}} & \xi \oplus \eta \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow[f]{} & \xi \end{array}$$

Here π is the projection, and $\bar{f}$ and π' are the obvious maps induced by f and π, respectively.

Let $z \in \mathbb{R}^n$ be a regular value of the composition

$$f^*(\xi \oplus \eta) \xrightarrow{\bar{f}} \xi \oplus \eta \xrightarrow{p} \mathbb{R}^n.$$

By Sard's theorem z can be chosen to be arbitrarily close to the origin. Since z is regular, the composition of the derivatives

$$Dp \circ D\bar{f} : T_v f^*(\xi \oplus \eta) \to \mathbb{R}^n$$

is surjective for any $v \in f^*(\xi \oplus \eta)$ such that $p \circ \bar{f}(v) = z$. Using the trivialization $\Psi : \xi \oplus \eta \xrightarrow{\cong} Y \times \mathbb{R}^n$, we may conclude that the image of $D\bar{f}$ must span the complement of that tangent space to $Y \times \{z\}$ at $(p_Y \bar{f}(v), z)$, where p_Y is the projection of the trivialization Ψ onto the Y factor. This means that $\bar{f}$ is transverse to the section $s' : Y \to \xi \oplus \eta$ given in terms of the trivialization Ψ, by $s'(y) = (y, z)$. Define the section $s : Y \to \xi$ by $s(y) = \pi(s'(y)$. Notice that the following diagram commutes:

$$
\begin{array}{ccccc}
f^*(\xi \oplus \eta) & \xrightarrow{\bar{f}} & \xi \oplus \eta & \xleftarrow{s'} & Y \\
{\scriptstyle \pi'}\big\downarrow & & \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle =} \\
X & \xrightarrow[f]{} & \xi & \xleftarrow{s} & Y
\end{array}
$$

We claim that f is transverse to $s(Y)$. To see this, let $x \in X$, $y \in Y$ be such that $f(x) = s(y)$. Then

$$\pi(s'(y)) = s(y) = f(x).$$

By the definition of the pullback bundle, $(x, s'(y)) \in f^*(\xi \oplus \eta)$ has $\bar{f}(x, s'(y)) = s'(y)$. Since $\bar{f} \pitchfork s'(Y)$, the images of $D_{(x,s'(y))}\bar{f}$ and $D_y s'$ span $T_{\bar{f}(x,s'(y))=s'(y)}(\xi \oplus \eta)$. Since π is a submersion, we may conclude that the images of $D_x f$ and $D_y s$ span $T_{f(x)=s(y)}\xi$. That is, $f \pitchfork s(Y)$. Notice that by Sard's theorem, the section s may be taken to be arbitrarily close to the zero section by choosing $z \in \mathbb{R}^n$ sufficiently close to the $0 \in \mathbb{R}^n$. □

Corollary 8.7. *Let $f : M \to W$ be a smooth map between smooth manifolds. Assume that M is compact. Let N be another compact, smooth manifold and suppose $g_0 : N \hookrightarrow W$ is a smooth embedding. Then there is an arbitrarily small isotopy of g_0 to a smooth embedding $g_1 : N \hookrightarrow W$ with the property that $f \pitchfork g_1(N)$.*

Proof. Let $\nu \to N$ be the normal bundle of the embedding $g_0 : N \hookrightarrow W$. By the tubular neighborhood theorem (8.2) g_0 extends to an embedding $g : \nu \hookrightarrow W$ which is a diffeomorphism onto an open subspace (the tubular neighborhood). Notice that if we define $M' = f^{-1}(g(\nu))$, then $M' \subset M$ is an open submanifold. We now apply the Theorem 8.6 to the restriction of f, $f_{|_{M'}} : M' \to \nu$. □

We will actually need another version of this corollary that says that transversal intersections are generic with respect to perturbations of the map f. But first we need the following:

Lemma 8.8. *Let N be a compact smooth submanifold of a smooth manifold W. Let T be a tubular neighborhood of N. It is equipped with a retraction $p : T \to N$¿ If $s : N \to T$ is any section (i.e $p \circ s = id$) then there is a diffeomorphism $h : T \to T$ that preserves fibers, extends continuously to the identity on the boundary ∂T, and takes s to the zero section. Moreover the diffeomorphism h can be taken to be homotopic to the identity of T.*

Proof. By the tubular neighborhood theorem, it suffices to work in the vector bundle setting. Let $p : \nu \to N$ be the normal bundle of N in W. Let $s : N \to \nu$ be any section, and let $z : N \to \nu$ be the zero section. Define

$$H : \nu \to \nu$$
$$H(v) = v - s(p(v)).$$

Notice that H is a map of fiber bundles in that it preserves fibers (i.e $p \circ H(v) = p(v)$). But notice also that H is not a map of vector bundles since it is not linear on each fiber. Rather, H is affine on each fiber. In any case, H is clearly a diffeomorphism.

Notice that $H \circ s(x) = z(x)$. Moreover H is homotopic to the identity through diffeomorphisms. To see this, define for $t \in [0,1]$ $H_t(v) = v - ts(p(v))$. Notice that H_0 is the identity, and $H_1 = H$. □

Corollary 8.9. *Let M be a closed, smooth manifold and $f_0 : M \to W$ a smooth map between smooth manifolds. Let $N \subset W$ be a smooth, closed submanifold and let T be any tubular neighborhood of N. Then there is a smooth map $f_1 : M \to W$ with the following properties.*

1. *$f_1 \pitchfork N$,*
2. *$f_1 = f_0$ outside of $f^{-1}(T)$,*
3. *f_1 is homotopic to f_0 on all of M via a homotopy that is constant outside of $f_0^{-1}(T)$.*

Proof. By Theorem 8.6 we know there exists a section s of a tubular neighborhood of N such that $f_0 \pitchfork s(N)$. Composing f_0 with the homotopy h described in the above lemma defines f_1. This f_1 may not be smooth at the boundary of the tubular neighborhood, but it can be smoothly approximated without changing it near the intersection with N, where f_1 is already smooth. □

Remarks. 1. This corollary says that one can perturb any map f_0 with as small of a perturbation as one would like, to make it transverse to N.

2. There exist strengthenings of this result saying that the set $\{f : M \to W$ such that $f \pitchfork N\}$ is "generic" (i.e a countable intersection of open, dense subsets) in the space of all smooth maps $C^\infty(M, W)$. Hirsch's book [44] gives a good exposition of this. For our purposes we only need that the space of transverse maps is *dense in the space of all smooth maps,* which is what the above results show.

8.3 Applications to intersection theory

One immediate application of transversality theory says that one can pull low dimensional submanifolds of a large dimensional manifolds apart, so that the do not interesect. More precisely, we have the following.

Proposition 8.10. *Let P^p and Q^q be closed submanifolds of M^n where $p+q < n$. Then one can perturb either P^p or Q^q by an arbitrarily small amount so that they do not intersect.*

More precisely, suppose $M^n \subset \mathbb{R}^N$. Let $e : P^p \hookrightarrow M^n$ be an embedding whose image is the submanifold in question. Then for any choice of $\epsilon > 0$, there exists another embedding $\tilde{e} : P^p \hookrightarrow M^n$, isotopic to e, so that for any $x \in P^p$, $\|e(x) - \tilde{e}(x)\| < \epsilon$ and $\tilde{e}(P^p) \cap Q^q = \emptyset$.

Proof. This follows from Corollary 8.7 and the fact that from Theorem 3.7 we see that the only transversal intersections of p-dimensional and q-dimensional submanifolds of an n-dimensional manifold when $p + q < n$, is the empty intersection. □

Here is another easy consequence of transversality theory. It is a statement about the homotopy groups of complements of submanifolds of Euclidean space.

Proposition 8.11. *Suppose M^m is a smooth, closed manifold, equipped with an embedding $e : M^m \hookrightarrow \mathbb{R}^n$. Then any smooth map of a sphere to the complement,*

$$f : S^k \to \mathbb{R}^n - M^m$$

can be extended to a map of the closed disk, $\tilde{f} : D^{k+1} \to \mathbb{R}^n - M^m$ if $k < n - m - 1$.

Proof. $f : S^k \to \mathbb{R}^n$ is null homotopic, since $\mathbb{R}^n$ is contractible. So there exists an extension $\tilde{f}_0 : D^{k+1} \to \mathbb{R}^n$. Perturb $\tilde{f}_0$ if necessary, to a map $\tilde{f}_1 : D^{k+1} \to \mathbb{R}^n$ that is homotopic to $\tilde{f}_0$ relative to its boundary, and such that $\tilde{f}_1 \pitchfork M^m$. But since $(k+1)+m < n$, this means that $\tilde{f}_1(D^{k+1}) \cap M^m = \emptyset$. In particular this means that the original map $f : S^k \to \mathbb{R}^n - M^m$ is null homotopic, and can therefore be extended to a map $\tilde{f} : D^{k+1} \to \mathbb{R}^n - M^m$. □

Another important application of transversality to intersection theory is when the sum of the dimensions of the submanifolds *equals* the dimension of the ambient manifold. So let P^p and Q^q be closed submanifolds of M^n, where $n = p+q$. Then basic transversality theory says that one can perturb either P^p or Q^q so that they intersect transversally. (At this point the reader should be able to make this statement precise.) In this setting the intersection $P^p \cap Q^q$ is a manifold of dimension $p + q - n = 0$. By compactness $P^p \cap Q^q$ is a finite number of points. When P^q, Q^q, and M^n are all oriented, $P^p \cap Q^q$ will inherit an orientation, and so each of the points making it up will have an orientation. This will just be a sign (± 1) and so one can count these points according to sign to obtain the "*intersection*" number. We now make this more precise.

Consider the following commutative diagram of embeddings:

$$\begin{array}{ccc} P^p & \xrightarrow{\subset} & M^n \\ \cup\uparrow & & \uparrow\cup \\ P^p \cap Q^q & \xrightarrow[\subset]{} & Q^q \end{array}$$

When P^p, Q^q, and M^n are all oriented, the normal bundle of $Q^q \hookrightarrow M^n$ has an induced orientation. Furthermore it restricts to give the (oriented) normal bundle of $P^p \cap Q^q \hookrightarrow P^p$. Since $P^p \cap Q^q$ is a finite set of points, $\{x_1, \cdots, x_k\}$, its normal bundle in P^p, being diffeomorphic to its tubular neighborhood, is just a finite collection of disjoint disks, $D_i \subset P^p$, $i = 1, \cdots, k$ each of which is oriented. In particular each tangent space $T_{x_i}D_i$ is oriented. But notice that $T_{x_i}D_i = T_{x_i}P^p$, which has an orientation coming from the original orientation of P^p. If these two orientations agree we say that $sgn(x_i) = +1$. If these orientations disagree we say that $sgn(x_i) = -1$. We can now make the following definition.

Definition 8.2. *Define the intersection number*

$$[P^p \cap Q^q] = \sum_{i=1}^{k} sgn(x_i) \in \mathbb{Z}$$

It is important to know that the intersection number is well defined. Of course we had to choose orientations and that can affect the ultimate sign of the intersection number. But it is important to also know that the intersection number does not depend on the particular perturbation (small isotopy) used in order to achieve transversal intersections. Once we know that we will be able to conclude the following:

Proposition 8.12. *Let P^p and Q^q be closed submanifolds of M^n, where $n = p+q$. Suppose these manifolds are all oriented. Then if the intersection number $[P^p \cap Q^q] \neq 0$, the neither P^p nor Q^q can be isotoped so that the resulting embeddings are disjoint. That is, P^p and Q^q cannot be "pulled off of each other" in M^n.*

To show that the intersection number is well defined, and to generalize it to study more complicated intersections, we will employ the use of Poincaré duality to develop the intersection theory homologically.

9

Poincaré Duality, Intersection theory, and Linking numbers

Our goal in this chapter is to use Poincaré duality to do intersection theory rigorously. A particular goal will be to prove that the intersection number of two submanifolds, the sum of whose dimensions equals the dimension of the ambient manifold, is well defined (see Definition 8.2). Along the way we relate intersection theory with such constructions as the "shriek" or "umkehr" map, the Pontrjagin-Thom "collapse map", and the Thom isomorphism.

9.1 Poincaré Duality, the "shriek map", and the Thom isomorphism

Let M^m and N^n be closed, oriented manifolds of dimensions m and n respectively. Their orientations determine (and are determined by) choices of fundamental classes $[M^m] \in H_m(M^m;\mathbb{Z})$ and $[N^n] \in H_n(N^n;\mathbb{Z})$ that determine Poincaré duality isomorphisms

$$\cap[M^m] : H^q(M;\mathbb{Z}) \xrightarrow{\cong} H_{m-q}(M;\mathbb{Z}) \quad \text{and} \quad \cap[N^n] : H^q(N;\mathbb{Z}) \xrightarrow{\cong} H_{n-q}(N;\mathbb{Z}) \tag{9.1}$$

We refer to their inverse isomorphisms as

$$D_M : H_r(M;\mathbb{Z}) \xrightarrow{\cong} H^{m-r}(M;\mathbb{Z}) \quad \text{and} \quad D_N : H_r(N;\mathbb{Z}) \xrightarrow{\cong} H^{n-r}(N;\mathbb{Z}). \tag{9.2}$$

Given a map $f : M^m \to N^n$, we of course have the induced homomorphisms in both homology and cohomology, which would exist even if M and N were replaced by any topological spaces. However, given that they are closed, oriented manifolds, the existence of Poincaré duality allows one to define a "shriek" or "umkehr" map.

Definition 9.1. *Define the homomorphism* $f^! : H^q(M^m;\mathbb{Z}) \to H^{n-m+q}(N^n;\mathbb{Z})$ *to be the unique map making the following diagram commute:*

$$\begin{array}{ccc} H^q(M^m;\mathbb{Z}) & \xrightarrow{f^!} & H^{n-m+q}(N^n;\mathbb{Z}) \\ {\scriptstyle \cap[M^m]}\downarrow \cong & & \cong \downarrow {\scriptstyle \cap[N^n]} \\ H_{m-q}(M;\mathbb{Z}) & \xrightarrow[f_*]{} & H_{m-q}(N;\mathbb{Z}) \end{array}$$

Now suppose that M^m is a closed, oriented m-dimnesional manifold and N^n is a compact, oriented, n-dimensional manifold with boundary. Then a map $f : M \to N$ defines a shriek map with values in *relative* cohomology,

$$f^! : H^q(M^m;\mathbb{Z}) \to H^{n-m+q}(N^n, \partial N;\mathbb{Z}). \tag{9.3}$$

This is defined by using the relative version of Poincaré duality ("Poincaré - Lefschetz duality"). We leave the details to the reader.

This relative version of the shriek map is important in many settings, but particularly so when one has an oriented vector bundle over a closed, oriented manifold

$$p : \xi \to M^m.$$

Assume the fiber dimension of this vector bundle is k. Give ξ a Euclidean structure, and as before, let $D(\xi)$ and $S(\xi)$ denote the associated unit disk bundle and sphere bundle respectively. Notice that the orientation on ξ as well as the orientation on the base manifold M^m gives $D(\xi)$ the structure of a compact $m+k$-dimensional oriented manifold, whose boundary is $\partial D(\xi) = S(\xi)$.

Now let $\zeta : M^m \to D(\xi)$ be the the zero section. Then as discussed above, this defines a shriek map

$$\begin{aligned} \zeta^! : H^q(M^n;\mathbb{Z}) \to H^{q+k}(D(\xi), \partial D(\xi);\mathbb{Z}) &\xrightarrow{=} H^{q+k}(D(\xi), S(\xi);\mathbb{Z}) \\ &= H^{q+k}(T(\xi);\mathbb{Z}) \end{aligned}$$

where $T(\xi)$ is the Thom space of the bundle ξ.

The following result relates this shriek map, which is defined via Poincaré duality, with the Thom isomorphism.

Proposition 9.1. *Given an oriented, k-dimensional vector bundle over a closed, oriented manifold, $p : \xi \to M^m$ the shriek map of the zero section*

$$\zeta^! : H^q(M^m;\mathbb{Z}) \to H^{q+k}(D(\xi), \partial D(\xi);\mathbb{Z}) = H^{q+k}(T(\xi);\mathbb{Z})$$

is equal to the Thom isomorphism

$$\cup u : H^q(M^m;\mathbb{Z}) \xrightarrow{\cong} H^{q+k}(T(\xi);\mathbb{Z}).$$

Here $u \in H^k(T(\xi);\mathbb{Z})$ is the Thom class.

Proof. The shriek map $\zeta^!$ is defined to be the composition,

$$\zeta^! : H^q(M^n;\mathbb{Z}) \xrightarrow{\cap[M]} H_{m-q}(M^m;\mathbb{Z}) \xrightarrow{\zeta_*} H_{m-q}(D(\xi);\mathbb{Z}) \xrightarrow{D_{D(\xi)}} H^{k+q}(D(\xi), \partial D(\xi)).$$

Here, as above, $D_{D(\xi)}$ is the inverse to the Poincaré duality isomorphism given by capping with the fundamental class. Since $\cap[M]$ and $D_{D(\xi)}$ are both isomorphisms, and because ζ_* is an isomorphism since the zero section ζ is a homotopy equivalence, we may conclude that the composition $\zeta^!$ is an isomorphism. Of course we know that cupping with the Thom class $\cup u : H^q(M^m;\mathbb{Z}) \xrightarrow{\cong} H^{q+k}(T(\xi);\mathbb{Z})$. is also an isomorphism. So we need only show that they are the same isomorphism.

Notice that when $q = 0$, $H^q(M^m;\mathbb{Z}) \cong \mathbb{Z}$ and so the two isomorphisms $\zeta^!$ and $\cup u$ must agree in this dimension, at least up to sign. We leave it to the reader to check that the signs in fact agree given the compatibility of the orientation of $D(\xi)$ with the orientation of the bundle $p : \xi \to M^n$ and the orientation of M.

In general dimensions, let $\beta \in H^q(M)$. Since the zero section ζ is a homotopy equivalence we may write $\beta = \zeta^*(\alpha)$ for a unique class $\alpha \in H^q(D(\xi);\mathbb{Z})$.

$$\begin{aligned}\zeta^!(\beta) &= D_{D(\xi)}(\zeta_*(\beta \cap [M]) \\ &= D_{D(\xi)}(\zeta_*(\zeta^*(\alpha) \cap [M]) \\ &= D_{D(\xi)}(\alpha \cap \zeta_*[M]) \quad \text{by the naturality of the cap product.}\end{aligned} \tag{9.4}$$

Now the Thom isomorphism in *homology* is given by capping with the Thom class $\cap u : H_r(D(\xi), S(\xi)) \xrightarrow{\cong} H_{r-k}(M)$. In particular $[M] \in H_m(M;\mathbb{Z}) \cong \mathbb{Z}$ is equal to $u \cap [D(\xi), \partial D(\xi)]$ where $[D(\xi), \partial D(\xi)] \in H_{m+k}(D(\xi), \partial D(\xi);\mathbb{Z})$ is the (relative) fundamental class. Thus

$$\begin{aligned}\zeta^!(\beta) = D_{D(\xi)}(\alpha \cap \zeta_*[M]) &= D_{D(\xi)}(\alpha \cap z_*(u \cap [D(\xi), \partial D(\xi)]) \\ &= D_{D(\xi)}((\zeta^*(\alpha) \cup u) \cap [D(\xi), \partial D(\xi)]) \\ &= \zeta^*(\alpha) \cup u \quad \text{since } D_{D(\xi)} \text{ is inverse to} \\ &\quad \text{capping with the fundamental class} \\ &= \beta \cup u.\end{aligned}$$

□

As a result of this proposition we will be able to prove a result relating the shriek map to so-called "Thom collapse map", which is crucial in intersection theory.

The Thom collapse map can be described as follows. Let $e : N^n \hookrightarrow M^m$ be a smooth embedding of closed, oriented, smooth manifolds. Let ν be tubular neighborhood of $e(N^n)$ in M^m. Notice that the quotient space, $M/(M-\nu)$ is the one point compactification $\nu \cup \infty$, which is in turn homeomorphic to the Thom space $T(\nu)$.

Definition 9.2. *The "Thom collapse map" $\tau : M^m \to T(\nu)$ is the projection*

$$\tau : M^m \to M/(M-\nu) \cong T(\nu).$$

Theorem 9.2. *As above let $e : N^n \hookrightarrow M^m$ be a smooth embedding of closed, oriented, smooth manifolds. Let ν be tubular neighborhood of $e(N^n)$ in M^m. Let $k = m - n$ be the codimension of the embedding. Then the composition in cohomology*

$$H^q(N) \xrightarrow[\cong]{\cup u} H^{q+k}(T(\nu)) \xrightarrow{\tau^*} H^{q+k}(M)$$

is equal to the shriek map $e^! : H^q(N) \to H^{q+k}(M)$. Here $u \in H^k(T(\nu))$ is the Thom class.

Proof. Notice that the disk bundle, $D(\nu)$, is an oriented $m = n+k$-dimensional manifold with boundary $\partial D(\nu) = S(\nu)$. Let $[D(\nu), S(\nu)] \in H_m(D(\nu), S(\nu)) = H_m(T(\nu))$ be the relative fundamental class.

Observe first that $\tau_*[M] = [D(\nu), S(\nu)] \in H_m(D(\nu), S(\nu))$. This is because the diagrams

$$\begin{array}{ccc}
H_m(M) & \xrightarrow{\tau_*} & H_m(D(\nu), S(\nu)) \\
\cong \downarrow & & \downarrow \cong \\
H_m(M, M-x) & \xrightarrow[=]{} & H_m(D(\nu), D(\nu)-x)
\end{array}$$

commute for every $x \in D(\nu) \subset M$. Now the fundamental class $[M] \in H_m(M)$ is the unique class that maps to the generator of $H_m(M, M-x) \cong \mathbb{Z}$ determined by the orientation. Therefore $\tau_*([M]) \in H_m(D(\nu), S(\nu))$ is a class that maps to the generator of $H_n(D(\nu), D(\nu)-x)) \cong \mathbb{Z}$ determined by the orientation. But this property characterizes $[D(\nu), S(\nu)] \in H_m(D(\nu), S(\nu))$.

Secondly, observe that the following diagram commutes:

$$\begin{array}{ccc}
\tilde{H}^*(D(\nu)/S(\nu)) & \xleftarrow{=} & H^*(D(\nu), S(\nu)) \\
\cap[D(\nu)/S(\nu)] \downarrow & & \cong \downarrow \cap[D(\nu), S(nu)] \\
\tilde{H}_{m-*}(D(\nu)/S(\nu)) & \longleftarrow & H_{m-*}(D(\nu)) \\
\tau_* \uparrow & & \downarrow \tilde{e}_* \\
H_{m-*}(M) & \xleftarrow[=]{} & H_{m-*}(M).
\end{array}$$

Here $[D(\nu)/S(\nu)]$ is the image of the (relative) fundamental class $[D(\nu), S(\nu)]$ under the isomorphism $H_m(D(\nu), S(nu)) \xrightarrow{\cong} \tilde{H}_m(D(\nu)/S(\nu))$. $\tilde{e} : D(\nu) \hookrightarrow M$ is the extension of the embedding e to its tubular neighborhood.

By the naturality of the cap product this diagram expands to the following commutative diagram.

$$\begin{array}{ccccc}
H^*(D(\nu), S(\nu)) & \xrightarrow{=} & H^*(D(\nu), S(\nu)) & \xrightarrow[\cong]{\cap[D(\nu),S(\nu)]} & H_{m-*}(D(\nu)) \\
{\scriptstyle \tau^*}\downarrow & & {\scriptstyle \cap[D(\nu)/S(\nu)]}\downarrow & & \downarrow{\scriptstyle \tilde{e}_*} \\
H^*(M) & & H_{m-*}(D(\nu)/S(\nu)) & \xleftarrow[\tau_*]{} & H_{m-*}(M) \\
{\scriptstyle =}\downarrow & & & & \downarrow{\scriptstyle =} \\
H^*(M) & & \xrightarrow[\cap[M]]{} & & H_{m-*}(M)
\end{array}$$

By the above proposition we can now add to the exterior of this diagram:

$$\begin{array}{ccc}
H^{*-k}(N) & \xrightarrow[\cong]{\cap[N]} & > H_{m-*}(N) \\
{\scriptstyle \cup u = \zeta^!}\downarrow & & \downarrow{\scriptstyle \zeta_*} \\
H^r(D(\nu), S(\nu)) & \xrightarrow[\cong]{\cap[D(\nu),S(\nu)]} & H_{m-*}(D(\nu)) \\
{\scriptstyle \tau^*}\downarrow & & \downarrow{\scriptstyle \tilde{e}_*} \\
H^*(M) & \xrightarrow[\cap[M]]{\cong} & H_{m-*}M
\end{array}$$

Thus $\tau^* \circ \cup u = D_M \circ \tilde{e}_* \circ \zeta_* \circ \cap[N]$. (Recall that the duality isomorphism $D_M = (\cap[M])^{-1}$.) But $\tilde{e} \circ \zeta = e$, so we have that

$$\tau^* \circ \cup u = D_M \circ e_* \circ \cap[N] = e^!, \quad \text{by definition.}$$

□

The following corollary gives a clear relation between the Thom collapse map and Poincaré duality. In particular it says that the Thom class of a normal bundle of an embedded submanifold is dual to the fundamental class of the submanifold.

Corollary 9.3. *Let M be a closed, oriented manifold, with oriented, closed submanifold $e : N \hookrightarrow M$ of codimension k. Let ν be a tubular neighborhood of N, which we identify with the normal bundle. Let $\tau : M \to M/M - \nu \cong T(\nu)$ be the Thom collapse map, and let $u \in H^k(T(\nu)$ be the Thom class. Then*

$$\tau^*(u) = D(N).$$

Said another way, $\tau^*(u) \cap [M] = [N]$.

Proof. By Theorem 9.2, $\tau^*(u) = e^!(1)$. But recall that $e^! : H^0(N) \to H^k(M)$ is defined to be the unique homomorphism that makes the following diagram commute:

$$\begin{array}{ccc} H^0(N) & \xrightarrow{e^!} & H^k(M) \\ {\scriptstyle \cap[N]}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle \cap[M]} \\ H_n(M) & \xrightarrow[e_*]{} & H_n(M). \end{array}$$

Thus $\tau^*(u) \cap [M] = e^!(1) \cap [M] = e_*([N])$. □

9.2 The intersection product

One can define the "intersection product" in the homology of a closed, oriented manifold both geometrically, using transversality theory, and algebraically, using Poincaré duality and the cup product. Our goal in this section is to show that these constructions define the same homological product. The intersection number, defined earlier (Definition 8.2), will be shown to be a special example of this product, and the consequence of these results will show that this number does not depend on the various geometric choices one makes in defining it.

Definition 9.3. *Let M^m be a closed, oriented m-dimensional manifold. The intersection product* is the pairing

$$H_p(M) \times H_n(M) \to H_{p+n-m}(M)$$
$$\alpha \times \beta \to \alpha \cdot \beta$$

defined to be the unique homomorphism making the following diagram commute:

$$\begin{array}{ccc} H_p(M) \times H_n(M) & \xrightarrow{\cdot} & H_{p+n-m}(M) \\ {\scriptstyle \cap[M]\times\cap[M]}\uparrow{\scriptstyle \cong} & & {\scriptstyle \cong}\uparrow{\scriptstyle \cap[M]} \\ H^{m-p}(M) \times H^{m-n}(M) & \xrightarrow[\cup]{} & H^{2m-p-n}(M). \end{array}$$

That is, the intersection product is Poincaré dual to the cup product.

The following is the main result of this section.

Again, let M^m be a closed, oriented m-dimensional manifold. Suppose it has two oriented, closed submanifolds P^p of dimension p and N^n of dimension n that intersect transversally. (Otherwise perturb one of them so that the interesection becomes transverse.) By abuse of notation we let $[P] \in H_p(M)$ and $[N^n] \in H_n(M)$ be the homology classes given by the images of the fundamental classes of these submanifolds under the homomorphisms induced by their embeddings. We say that these submanifolds *represent* these homology classes.

Theorem 9.4. *Under these assumptions the homology class represented by the intersection*

$$[P \cap N] \in H_{p+n-m}(M)$$

represents the intersection product of the classes represented by the submanifolds P^p and N^n:

$$[P^p] \cdot [N^n] = [P \cap N].$$

This theorem actually has a generalization, whose proof requires only small adjustments to the proof of Theorem 9.4. We leave the details to the reader.

Theorem 9.5. *Let M^n, P^p, and N^n be closed, oriented manifolds. Let $f : P^p \to M^n$ be a smooth map and $g : N^n \hookrightarrow M^n$ a smooth embedding. Assume that $f \pitchfork g(N^n)$. That is for every $x \in P$ and $y \in N$ with $f(x) = g(y) = z \in M$, then $Df_x(T_xP) \oplus Dg_y(T_yN) = T_zM$. Consider the submanifold $f^{-1}(g(N)) \subset P$. Then this is a closed, oriented submanifold of dimension $p+n-m$ and the image of its fundamental class in homology $f_*[f^{-1}(g(N))] \in H_{p+n-m}(M)$ is Poincaré dual to the cup product $D_M(f_*[P]) \cup D_M(g_*([N]) \in H^{2m-p-n}(M)$.*

Before we prove Theorem 9.4 we make a couple remarks:

Remarks.

- Let's generalize our notion of "representing" a homology class in closed oriented manifold by a submanifold, to a homology class $\alpha \in H_q(M)$ being represented by a manifold if there exists a closed, oriented manifold Q^q and a map $\phi : Q \to M$ with $\phi_*([Q]) = \alpha$. Then we will see in Chapter 12 below, that not every integral homology class is represented by such a manifold. However, as we will see below, a consequence of Thom's calculation of the unoriented cobordism ring is that in homology with $\mathbb{Z}/2$-coefficients, indeed every homology class is represented by a manifold. In the presence of such representations, (in integral or $\mathbb{Z}/2$ homology), this theorem says that the Poincaré dual of the cup product is represented by (transversal) intersections of manifolds. This gives a rather remarkable geometric interpretation of the cup product.

- Historically, there is reason to believe that the development of cohomology and the cup product was motivated by goal of representing intersections of submanifolds. S. Lefshetz, who did seminal work in the development of intersection theory in both algebraic geometry and algebraic topology, was instrumental in developing the cup product in singular cohomology.

Proof of Theorem 9.4.

Proof. Consider the following commutative diagram, where the maps are all embeddings:

$$\begin{array}{ccc} N & \xrightarrow[\subset]{e_N} & M \\ \cup\big\uparrow e_{P\cap N,N} & & \cup\big\uparrow e_P \\ P\cap N & \xrightarrow[e_{P\cap N,P}]{\subset} & P \end{array}$$

By examining this diagram one sees that when one restricts the normal bundle of N in M to $P \cap N$, one gets the normal bundle of $P \cap N$ in P:

$$(\nu_{e_N})_{|P\cap N} = \nu_{e_{P\cap N,P}}.$$

Equivalently, the intersection of a tubular neighborhood of e_N with P is a tubular neighborhood of $e_{P\cap N,P}$. We represent these tubular neighborhoods by η's. We therefore have a commutative diagram involving Thom collapse maps:

$$\begin{array}{ccc} M & \xrightarrow{\tau_N} & M/(M-\eta_N) \cong T(\nu_{e_N}) \\ e_P\big\uparrow & & \big\uparrow T(e_P) \\ P & \xrightarrow[\tau_{P\cap N,P}]{} & P/(P-\eta_{P\cap N,P}) = T(\nu_{e_{P\cap N,P}}). \end{array}$$

Here $T(\nu)$ denotes the Thom space of the corresponding normal bundle, and $T(e_P)$ denotes the map of Thom spaces induced by the embedding e_P.

In particular this means that on the level of Thom classes,

$$T(e_P)^*(u_N) = u_{P\cap N,P} \in H^{m-n}(T(\nu_{e_{P\cap N,P}})).$$

Now by Corollary 9.3

$$\tau^*_{P\cap N,P}(u_{P\cap N,P}) \cap [P] = [P\cap N] \in H_{p+n-m}(P).$$

So therefore

$$\tau^*_{P\cap N,P}(T(e_P)^*(u_N)) \cap [P] = [P\cap N] \in H_{p+n-m}(P),$$

and by the commutativity of the above diagram, this means

$$e_P^*(\tau_N^*(u_N)) \cap [P] = [P\cap N] \in H_{p+n-m}(P).$$

So we may conclude that

$$(e_P)_*(e_P^*(\tau_N^*(u_N)) \cap [P]) = (e_P)_*[P\cap N] \in H_{p+n-m}(M).$$

By the definition of the intersection product, this says that

$$[P]\cdot[N] = [P\cap N] \in H_{p+n-m}(M).$$

□

An immediate consequence of this theorem is that the (homological) intersection pairing gives an obstruction to separating two submanifolds. By "separating", we mean that there is an isotopy of one or both of the embeddings of the two submanifolds, so that the resulting submanifolds do not intersect. That is, we have the following immediate corollary.

Corollary 9.6. *Let M^m be a closed, oriented m-dimensional manifold. Suppose it has two oriented, closed submanifolds P^p of dimension p and N^n of dimension n, such that the intersection product, $[P] \cdot [N] \in H_{p+n-m}(M)$ is nonzero. Then P and N cannot be separated in M.*

Exercises.

(1). Let M^m be a C^∞ closed manifold, and let $N^n \subset M^m$ be a smooth embedded submanifold, where N^n is also assumed to be compact with no boundary. We say that N^n can be *"moved off of itself" in M* if a tubular neighborhood η of N with retraction map $\rho : \eta \to N$ admits a section $\sigma : N \to \eta$ that is disjoint from N. That is, $N \cap \sigma(N) = \emptyset \subset \eta \subset M$.

(a). Suppose the dimensions of the manifolds satisfy $2n < m$. Prove that N can be moved off of itself in M.

(b). To see that the dimension requirement above is necessary in general, show that

$$\mathbb{RP}^1 \subset \mathbb{RP}^2$$

cannot be moved off of itself. *Hint:* Compute the self intersection number (mod 2) of $\mathbb{RP}^1 \subset \mathbb{RP}^2$.

(2). Write $\mathbb{CP}^n$ in projective coordinates. $\mathbb{CP}^2 = \{[z_0, z_1, z_2] \in \mathbb{C}^{n+1} - \{0\}/\mathbb{C}^\times\}$. That is $\mathbb{CP}^{n+1}$ is the quotient of $\mathbb{C}^{n+1} - \{0\}$ by the action, via scalar multiplication, of the nonzero complex numbers $\mathbb{C}^\times$.

There are two natural copies of $\mathbb{CP}^1$ inside $\mathbb{CP}^2$ given by $\{[z_0, z_1, 0]\}$ and by $\{[0, z_1, z_2]\}$. If we call one of these N and the other K,

Show that the intersection product $[N] \cdot [K] = 1 \in H_0(\mathbb{CP}^2)$. Conclude that each of these classes represent a generator of $H_2(\mathbb{CP}^2)$.

(3). Let M^n be a closed oriented n-dimensional manifold, and let $\Delta : M \to M \times M$ be the diagonal map. Let $\Delta_! : H_q(M \times M) \to H_{q-n}(M)$ be the shriek map in homology. Show that for any homology classes α and β of M, then $\alpha \cdot \beta = \pm\Delta_!(\beta \times \alpha)$.

9.2.1 Intersection theory via Differential Forms

We end this section by pointing out how to compute the intersection number of two submanifolds of complementary dimension using differential forms.

Let M^m be a closed oriented manifold, with submanifolds Q^q of dimension q and P^p of dimension p where $p + q = m$. Let η_Q and η_P be tubular neighborhoods of these submanifolds. These can be viewed as open manifolds of dimension n. The (DeRham) cohomology with compact supports, $H^*_{cpt}(\eta_Q)$ is equal to the cohomology of the one-point compactification, which is homeomorphic to the Thom space of the normal bundle. Therefore there is a Thom class $u_Q \in H^p_{cpt}(\eta_Q)$, and similarly $u_P \in H^q_{cpt}(\eta_P)$. The Thom collapse map gives classes $\nu_Q \in H^p(M)$ and $\nu_P \in H^q(M)$. By abuse of notation we let ν_Q and ν_P denote differential forms on M of dimension q and p respectively that represent these cohomology classes.

These "Thom forms" can be viewed a differential forms on M whose support lies in the relevant tubular neighborhood which yield the orientation forms of the corresponding normal bundles.

The following is a reinterpretation of Theorem 9.4 in this setting, using the DeRham theorem. We leave the job of filling in the details of its proof as an exercise to the reader,

Theorem 9.7. *In the setting described above,*

$$\begin{aligned}[Q]\cdot[P] &= \int_M \nu_Q \wedge \nu_P \\ &= \langle u_Q \cup u_P; [M]\rangle \\ &= \int_P \nu_Q = \pm \int_Q \nu_P.\end{aligned}$$

9.3 Degrees, Euler numbers, and Linking numbers

In this section we will discuss interesting applications of the results about intersection theory developed in the last section.

9.3.1 The Degree of a map

Let $f : N^n \to M^n$ be a smooth map between closed, oriented, connected smooth manifolds of the same dimension ($= n$). The *degree* of f is an oriented (signed) count of the number of elements in the preimage of a generic point. More specifically we make the following definition:

Definition 9.4. *The degree of f, written $Deg(f)$ is defined to be the intersection number of $f : N^n \to M^n$ and a regular value $x \in M^n$, viewed as a zero-dimensional submanifold. That is, $Deg(f) = f_*[N^n]\cdot[x] \in \mathbb{Z}$.*

.

Notice that the intersection number, as defined in Definition 8.2, in this setting is given by

$$
\begin{aligned}
f_*[N]\cdot[x] &= \sum_{i=1}^{k} sgn(x_i) \in \mathbb{Z}, \quad \text{where the sum is taken over all points in } f^{-1}(x) \in N \\
&= [f^{-1}(x)] \in H_0(N^n) = \mathbb{Z} \quad \text{by Theorem 9.5.}
\end{aligned}
$$

Now by Theorem 9.5,

$$
\begin{aligned}
f^{-1}(x)] &= f_*[N]\cdot[x] = f^*(D_M[x]) \cap [N] \\
&= D_M[x] \cap f_*[N].
\end{aligned}
$$

Since the fundamental class $[M] \in H_n(M) \cong \mathbb{Z}$ is a generator, we may interpret this as the following corollary to Theorem 9.5

Corollary 9.8. *Write $f_*[N] = d[M] \in H_n(M)$. Then $d = Deg(f)$.*

This corollary allows for easier calcuations of degree, and also shows that the notion of degree does not depend on the choice of regular value $x \in M$. Moreover it allows the extension of the notion of degree to any continuous (not necessarily smooth) map.

9.3.2 The Euler class and self intersections

Recall from Definition 5.5 that if $\xi \to N$ is an oriented vector bundle of fiber dimension k, the *Euler class*

$$\chi(\xi) \in H^k(N)$$

is defined to be the image of the Thom class under the composition

$$H^k(T(\xi)) = H^k(D(\xi), S(\xi)) \to H^k(D(\xi)) \xrightarrow{\zeta^*} H^k(N)$$

where $\zeta : N \to D(\xi) \subset \xi$ is the zero section.

In the setting when N is a n-dimensional, closed, oriented manifold, we can relate the Euler class to the self intersection of the zero section.

First, we explain what we mean by "self intersection". If $e : N^n \hookrightarrow M^m$ is an embedding of N into a compact, connected, oriented manifold M^n (with or without boundary), then we can perturb (i.e find an isotopy) of the embedding e to an embedding $\tilde{e} : N^n \hookrightarrow M^m$ so that $e(N) \pitchfork \tilde{e}(N)$. By Theorem 9.4, the resulting intersection, $e(N) \cap \tilde{e}(N)$ represents the class $[N]\cdot[N] \in H_{2n-m}(M)$. This class is called the "self intersection class". In particular, if $m = 2n$, this is a zero dimensional homology class, and therefore an integer, which represents a (signed) count of the number of points in the intersection $e(N) \cap \tilde{e}(N)$.

In the setting of a k-dimensional, oriented, smooth vector bundle $p : \xi \to$

N^n we may view the disk bundle $D(\xi)$ as an $(n+k)$-dimensional, oriented, compact manifold with boundary, and the zero section $\zeta : N \hookrightarrow D(\xi)$ as an embedding. We then have the following result.

Theorem 9.9. *The self intersection class of the zero section*

$$[\zeta(N)] \cdot [\zeta(N)] \in H_{n-k}(D(\zeta)) \cong H_{n-k}(N)$$

is Poincaré dual to the Euler class. That is,

$$\chi(\xi) \cap [N] = [\zeta(N)] \cdot [\zeta(N)].$$

In particular, when $k = n$, the evaluation of the Euler class on the fundamental class $\langle \chi(\xi); [N] \rangle$ is equal to the self intersection number of the zero section.

Before we prove this theorem we observe the following corollary.

Corollary 9.10. *If a smooth vector bundle $p : \xi \to N^n$ over a closed, oriented manifold has a nowhere zero section, then the Euler class $\chi(\xi)$ is zero.*

Proof. Notice that any section $\sigma : N \to \xi$ is a homotopy equivalence, and is homotopic, as a map of spaces, to the zero section ζ. Such a homotopy can be taken to be $(x, t) \to (1-t)\sigma(x)$. Therefore the homology classes represented by these sections, $[\sigma(N)]$ and $[\zeta(n)]$ are equal. If $\sigma(x)$ is never zero, then $\sigma(N) \cap \zeta(N) = \emptyset$. Therefore by Theorem 9.4

$$0 = [\zeta(n)] \cdot [\sigma(N)] = [\zeta(n)] \cdot [\zeta(n)].$$

By Theorem 9.9, the Euler class $\chi(\xi) = 0$. $\square$

We now prove Theorem 9.9.

Proof. By Proposition 9.1 the following diagram commutes:

$$\begin{array}{ccc} H^q(N) & \xrightarrow[\cong]{\cup u} & H^{q+k}(D(\xi), S(\xi)) \\ {\scriptstyle \cap[N]}\downarrow{\scriptstyle\cong} & & {\scriptstyle\cong}\downarrow{\scriptstyle \cap[D(\xi),S(\xi)]} \\ H_{n-q}(N) & \xrightarrow[\zeta_*]{\cong} & H_{n-q}(D(\xi)). \end{array}$$

We now insert this into a larger diagram:

$$\begin{array}{ccccc} H^k(D(\xi), S(\xi)) \times H^k(D(\xi), S(\xi)) & \xrightarrow{\cup} & H^{2k}(D(\xi), S(\xi)) & \xleftarrow[\cong]{\cup u} & H^k(N) \\ {\scriptstyle \cap[D(\xi),S(\xi)] \times \cap[D(\xi),S(\xi)]}\downarrow{\scriptstyle\cong} & & {\scriptstyle\cong}\downarrow{\scriptstyle \cap[D(\xi),S(\xi)]} & & {\scriptstyle\cong}\downarrow{\scriptstyle \cap[N]} \\ H_n(D(\xi)) \times H_n(D(\xi)) & \xrightarrow[\cdot]{} & H_{n-k}(D(\xi)) & \xleftarrow[\zeta_*]{\cong} & H_{n-k}(N) \end{array}$$

Notice that the left hand square defines the intersection product in $H_*(D(\xi))$. Now by Theorem 9.9, the product of the Thom classes $u \times u$ in the upper left corner of this diagram, maps to $\zeta_*([N]) \times \zeta_*([N])$ in the lower left corner. But this class in turn maps to the intersection product $\zeta_*([N]) \cdot \zeta_*([N])$ in the lower middle of the diagram $(H_{n-k}(D(\xi)))$.

Furthermore, by definition, the Euler class $\chi(\xi) \in H^k(N)$ in the upper right corner of the diagram, maps to $u \cup u \in H^{2k}(D(\xi), S(\xi))$, and so

$$(\chi(\xi) \cup u) \cap [D(\xi), S(\xi)] = \zeta_*([N]) \cdot \zeta_*([N]) \in H_{n-k}(D(\xi)).$$

By the commutativity of the right hand square we conclude that

$$\zeta_*(\chi(\xi) \cap [N]) = \zeta_*([N]) \cdot \zeta_*([N]) \in H_{n-k}(D(\xi)).$$

This is the statement of the theorem. □

We now turn our attention to the case when the bundle we are considering is the tangent bundle, $p : TN \to N$. A section of the tangent bundle is a *vector field* on N. Applying Corollary 9.10 to this situation gives us the following:

Proposition 9.11. *If a smooth, closed, orientable manifold N has a nowhere zero vector field, then the Euler class of its tangent bundle, $\chi(TN)$, which we denote by $\chi(N)$, is zero.*

We end this subsection with a well known result which relates the Euler class of a manifold (i.e of its tangent bundle), with its Euler characteristic.

Theorem 9.12. *Let N be a closed, oriented, n-dimensional smooth manifold. Then the evaluation of its Euler class on the fundamental class is the Euler characteristic of the manifold:*

$$\langle \chi(N), [N] \rangle = \sum_{i=0}^{n} (-1)^i rank\, H_i(N).$$

Proof. The proof of this theorem involves a few steps. First, consider the diagonal embedding,

$$\Delta : N \to N \times N.$$

We first observe that the normal bundle $\nu(\Delta)$ of this embedding is the tangent bundle TN. We leave the verification of this fact to the reader. In order not to confuse notation we now adopt the "exponential" notation for the Thom space of a bundle. That is if $\xi \to X$ is a vector bundle, we now use the notation X^ξ to denote its Thom space.

Let $\tau : N \times N \to N^\nu(\Delta) = N^{TN}$ be the Thom collapse map. We now compute this Thom collapse map in cohomology. To do this, notice that Poincaré duality defines a nonsingular pairing

$$\begin{aligned} \langle\, ,\, \rangle : H^*(N; k) \times H^*(N; k) &\to k \\ \langle \alpha, \beta \rangle &= (\alpha \cup \beta)([N]) \end{aligned}$$

Let $\{\alpha_i\}$ be a basis for $H^*(N;k)$. Since this pairing is nondegenerate, there is a corresponding *dual* basis $\{\alpha_i^*\}$. That is, $(\alpha_i^* \cup \alpha_j)([N]) = \delta_{i,j}$, the Kronecker delta. In particular notice that if $\alpha_i \in H^q(N;k)$, then $\alpha_i^* \in H^{n-q}(N;k)$.

Lemma 9.13. *Let $u \in H^n(N^{TN};k)$ be the Thom class of the Tangent bundle. Then*

$$\tau^*(u) = \sum_i (-1)^{|\alpha_i|}\alpha_i^* \times \alpha_i \in H^n(N \times N;k),$$

where $|\alpha_i|$ denotes the degree of α_i.

Proof. We take the following computation from Bredon [13], proof of Theorem 12.4.

By the Kunneth theorem we can write

$$\tau^*(u) = \sum_{i,j} c_{i,j}\, \alpha_i^* \times \alpha_j$$

for some coefficients $c_{i,j}$. Notice that we need only add over those terms where $|\alpha_i^*| + |\alpha_j| = n$. Since $|\alpha_i^*| = n - |\alpha_i|$, we assume $|\alpha_j| = |\alpha_i|$. For the following calculation take basis elements α_i and α_j of degree p. We compute $((\alpha_i \times \alpha_j^*) \cup \tau^*(u))([N \times N])$ in two different ways.

$$\begin{aligned}
((\alpha_i \times \alpha_j^*) \cup \tau^*(u))([N \times N]) &= (\alpha_i \times \alpha_j^*)(\tau \cap [N \times N]) \\
&= (\alpha_i \times \alpha_j^*)(\Delta_*([N]), \quad \text{by Corollary 9.3} \\
&= \Delta^*(\alpha_i \times \alpha_j^*)([N]) \\
&= (\alpha_i \cup \alpha_j^*)([N]) \\
&= (-1)^{p(n-p)}(\alpha_j^* \cup \alpha_i)([N]) \\
&= (-1)^{p(n-p)}\delta_{i,j}
\end{aligned}$$

On the other hand

$$\begin{aligned}
&((\alpha_i \times \alpha_j^*) \cup \tau^*(u))([N \times N]) \\
&\quad = ((\alpha_i \times \alpha_j^*) \cup (\sum_{r,s} c_{r,s}\, \alpha_r^* \times \alpha_s))([N \times N]) \\
&\quad = (-1)^{n-p} c_{i,j}((\alpha_i \cup \alpha_i^*) \times (\alpha_j^* \cup \alpha_j)([N] \times [N]) \\
&\quad \text{since one gets zero for } \alpha_i, \alpha_j \neq \alpha_r, \alpha_s, \text{ all of degree } p \\
&\quad = (-1)^{n-p+p(n-p)+n} c_{i,j}((\alpha_i \cup \alpha_i^*)([N]))((\alpha_j^* \cup \alpha_j)([N])) \\
&\quad = (-1)^{p(n-p)-p} c_{i,j}.
\end{aligned}$$

So we conclude that $c_{i,j} = (-1)^p \delta_{i,j}$. ∎

To complete the proof of Theorem 9.12, we make the following observation about the relation of the Thom collapse map and the Euler class. Let $e : N \hookrightarrow M$ be a codimension k embedding of oriented manifolds, with normal bundle ν_e, and let $\tau : M \to N^{\nu_e}$ is the Thom collapse map. The following comes from a quick check of definitions, which we leave for the reader.

Lemma 9.14. *If $u \in H^k(N^{\nu_e})$ be the Thom class. Then the Euler class of the normal bundle ν_e can be described by*

$$\chi(\nu_e) = e^*\tau^*(u) \in H^k(N).$$

Applying this lemma to the diagonal embedding $\Delta : N \to N \times N$, we have that $\Delta^*(\tau^*(u)) = \chi(N)$. Applying Lemma 9.3.2 with rational coefficients we have that

$$\begin{aligned} \chi(N)([N]) = \Delta^*(\tau^*(u))([N]) &= \sum_i (-1)^{|}\alpha_i|(\alpha_i^* \cup \alpha_i)([N]) \\ &= \sum_i (-1)^{|}\alpha_i|\langle \alpha_i^*, \alpha_i \rangle \\ &= \sum_i (-1)^{|}\alpha_i| \\ &= \text{ Euler characteristic of } N \end{aligned}$$

□

Notice that as an application of this theorem and of Proposition 9.11 we get the following classical result:

Proposition 9.15. *If a closed, oriented manifold N has nonzero Euler characteristic, then every vector field on N must contain a zero.*

In particular every vector field on an even dimensional sphere must contain a zero. This famous result, when applied to S^2 is often referred to as the "Hairy billiard ball" theorem.

9.3.3 Linking Numbers

We now discuss one more application of intersection theory. This is the classical notion of *linking numbers*.

In the general setting, suppose we have embeddings of closed, oriented manifolds in Euclidean space,

$$\begin{array}{ccc} M^m & \xrightarrow[\subset]{K_1} & \mathbb{R}^{n+m+1} \\ & & \cup \uparrow K_2 \\ & & N^n. \end{array}$$

We will assume that these manifolds intersect transversally, which in these dimensions means that they have disjoint images. Consider the composition

$$\begin{aligned}\Psi_{M,N} : & M^m \times N^n \to \mathbb{R}^{n+m+1} - \{0\} \to S^{n+m} \\ & (x,y) \longrightarrow (K_1(x) - K_2(y)) \longrightarrow \frac{K_1(x) - K_2(y)}{|K_1(x) - K_2(y)|}\end{aligned}$$

Giving S^{n+m} the orientation coming from viewing it as the boundary of the ball D^{n+m+1} inside $\mathbb{R}^{n+m+1}$, we can make the following definition.

Definition 9.5. *Define the linking number, $Lk(K_1, K_2)$ to be the degree*

$$Lk(K_1, K_2) = Deg(\Psi_{M,N}).$$

This is an algebraic-topological definition based on the homological properties of the map $\Psi_{M,N}$. However this notion has important geometric significance as well, as we will see in considering the classical case when we have the link of two disjointly embedded circles in S^3. We have the following diagram of embeddings:

$$\begin{array}{ccc} S^1 & \xrightarrow[\subset]{K_1} & \mathbb{R}^3 \\ & & \cup \uparrow K_2 \\ & & S^1. \end{array}$$

For $p \in S^2$, let

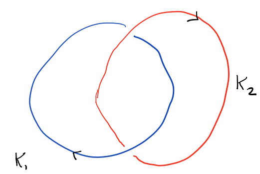
$$I(p) = \{(q_1, q_2) \in K_1 \times K_2 \, : \, q_2 - q_1 = \lambda p, \text{ where } \lambda > 0\}.$$

Notice that for $p \in S^2$, $I(p) = \Psi^{-1}_{K_1,K_2}(p)$.

Observation. Assume that $p = (0, 0, 1)$ is a regular value of Ψ_{K_1,K_2}. (If it is not, compose Ψ_{K_1,K_2} with a rotation of S^2 so that this condition is satisfied.) Project $K_1 \cup K_2$ onto $\mathbb{R}^2 = (x_1, x_2)-$ plane in $\mathbb{R}^3$, keeping track of the over and under-crossings:

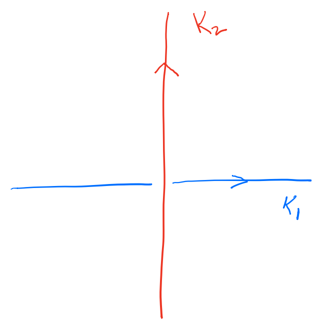
We claim that there is one element of $I(p)$ for every place that K_2 crosses *over* K_1. To see this, observe that if $(q_1, q_2) \in K_1 \times K_2$ is in $I(p)$, then the projections of q_1 and q_2 on $\mathbb{R}^2$ agree. This means that the first two coordinates of q_1 and of q_2 agree. Now since $q_2 - q_2 = \lambda p = (0, 0, \lambda)$ with $\lambda > 0$, we must have that the third coordinate (the "z-coordinate") of q_2 is larger than the third coordinate of z_1. That is, K_2 crosses *over* K_1 at this point.

By Definition 9.5 of the linking number as the degree of Ψ_{K_1,K_2}, we can calculate this invariant either homologically, or, as seen after the discussion of the definition of degree (Definition 9.4) as the signed count of the points in the preimage of a regular value of Ψ_{K_1,K_2}. That is, it is a signed count of the

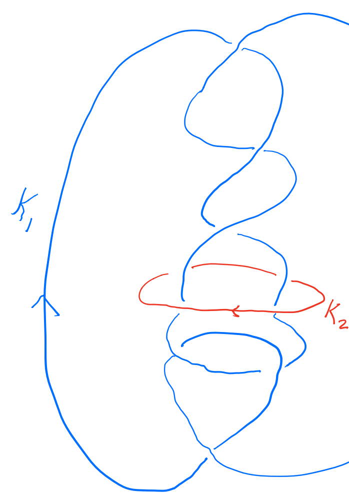


points of $I(0,01)$. If $(q_1, q_2) \in I(0,0,1)$, then the sign $sgn(q_1, q_2)$ is determined by comparing the orientations of the curves, and the standard orientation of the plane. In the above example of the Hopf link, $I(0,0,1)$ consists of a single point, and the local orientations of the curves K_1 an K_2 at this point looks like the following. Therefore the linking number of the Hopf link is

$$Lk(K_1, K_2) = -1.$$



We now turn our attention to the following, more complicated link (figure 9.3.3).

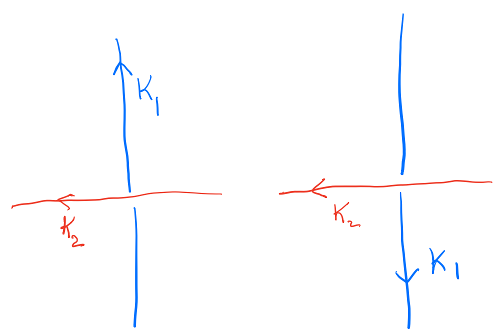


Notice that there are two places where K_2 crosses over K_1, and thus $I(0,0,1)$ has consists of two points.

The crossing on the left has $sgn = -1$ and the crossing on the right has $sgn = +1$. This means that the linking number,

$$Lk(K_1, K_2) = 0,$$

even though evidently the two embedded circles cannot be unlinked. This shows that while the linking number is a useful, computable invariant, it is not a *complete* invariant of a link of two embedded circles in $\mathbb{R}^3$.


K1
K2
K2
K1

10

Stable Homotopy

Throughout this chapter all spaces will equipped with basepoints, and will be assumed to be of the homotopy type of (based) CW-complexes.

Given a based space X, let ΣX denote its (reduced) suspension, and let

$$\Sigma : \pi_k(X) \to \pi_{k+1}(\Sigma X)$$

be the suspension homomorphism in homotopy groups. It is defined as follows: If $\alpha : S^k \to X$ is a basepoint preserving map representing an element of $\pi_k(X)$, then define its suspension

$$\Sigma\alpha : S^{k+1} = \Sigma S^k = S^1 \wedge S^k \xrightarrow{1\wedge\alpha} S^1 \wedge X = \Sigma X. \tag{10.1}$$

The roots of stable homotopy theory go back to the following classical theorem of Freudenthal:

Theorem 10.1. *("Freudenthal Suspension Theorem" [33]) Let X be an n-connected based space, then the suspension homomorphism*

$$\Sigma : \pi_k(X) \to \pi_{k+1}(\Sigma X)$$

is an isomorphism if $k \leq 2n$ and an epimorphism if $k = 2n+1$.

Many textbooks contain proofs of this theorem. A traditional reference is [101]. Below we will sketch a proof of this theorem using the Serre spectral sequence.

The Freudenthal Suspension Theorem naturally leads to the notion of the "stable range" for homotopy groups, i.e twice the connectivity of a space, in which homotopy data is preserved by suspension. An important example of this is the excision property. It is well known that homotopy groups, viewed as a functor from the category of pairs of based topological spaces to the category of groups, does not satisfy excision. For example, the fact that homology *does* satisfy excision implies that

$$\tilde{H}_*(X \vee Y) \xrightarrow{\cong} \tilde{H}_*(X) \oplus \tilde{H}_*(Y)$$

and more generally it provides the tremendously important calculational tool of the Mayer-Vietoris sequence. But notice that the analogous homomorphism at the level of homotopy groups, is not, in general, an isomorphism. For example, $\pi_1(S^1 \vee S^1)$ is the free group on two generators, which is not commutative. On the other hand, $\pi_1(S^1) \oplus \pi_1(S^1) \cong \mathbb{Z} \oplus \mathbb{Z}$, which certainly is commutative.

Nonetheless, as we will see later in this chapter, the Freudenthal theorem implies that the homotopy groups functor *does* satisfy excision in the stable range. This leads to the following definition.

Definition 10.1. *. Let X be a based space. Define its k^{th}-stable homotopy group $\pi_k^s(Z)$ to be*

$$\pi_k^s(X) = \lim_{n \to \infty} \pi_{k+n}(\Sigma^n X).$$

where the colimit is taken over the suspension homomorphisms,

$$\Sigma : \pi_{k+n-1}(\Sigma^{n-1}(X)) \to \pi_{k+n}(\Sigma^n X).$$

Notice that by the Freudenthal Suspension Theorem, the limit in the definition of stable homotopy groups is achieved at a finite stage. More specifically, $\pi_k^s(X) \cong \pi_{k+q}(\Sigma^q X)$ for $q \geq k - 2c(X)$, where $c(X)$ is the connectivity of X (i.e the maximal nonnegative integer such that $\pi_r(X) = 0$ for $r \leq c(X)$).

Exercise Verify this claim. That is, show that $\pi_k^s(X) \cong \pi_{k+q}(\Sigma^q X)$ for $q \geq k - 2c(X)$, where $c(X)$ is the connectivity of X.

Stable homotopy groups are an invariant of the collection of spaces, $\{\Sigma^k X\}$, together with maps between the spaces in this collection $\Sigma(\Sigma^k X) \xrightarrow{=} \Sigma^{k+1} X$. More generally, a collection of spaces $\{X_m\}$ together with maps $\epsilon_m : \Sigma X_m \to X_{m+1}$ is called a *spectrum.* Spectra are the objects of study of stable homotopy theory, and they were originally introduced and studied by Lima [57] and G. Whitehead [100]. We will introduce them and discuss some of their properties in this chapter. An important classical feature of spectra is that described by work of Brown [15] and Whitehead [100], where they classify *"generalized (co)homology theories"*. These are theories that satisfy all the Eilenberg-Steenrod except "dimension" (but including excision).We will discuss this classification in this chapter and discuss many important examples such as stable homotopy groups and K-theory. We mention Bott periodicity, one of the great theorems of the twentieth century, but put off it's proof until an addendum when we can use Morse theory, as Bott did in his original proof. We then discuss and apply the Atiyah-Hirzebruch spectra sequence for computing generalized (co)homology.

Another important, and more modern aspect of the study of spectra are their categorical aspects. We introduce symmetric spectra, describe ring spectra and module spectra, and then describe the Thom spectrum, viewed as a functor from the category of "spaces over BO" to the category of symmetric spectra. We discuss various products and generalized orientations of manifolds and then we have a discussion of Spanier-Whitehead duality and Atiyah duality for manifolds. We end this chapter with a discussion of the special properties of Eilenberg-MacLane spectra and the Steenrod algebras of cohomology operations.

10.0.1 Sketch of proof of the Freudenthal suspension theorem

We end this introductory section with a sketch of a proof of the Freudenthal suspension theorem. As you will see we rely on the Hurewicz theorem and the Serre spectral sequence.

Let X be an n-connected space of the homotopy type of a CW complex. If $n = 0$ then the theorem is trivial since both X and ΣX are path connected, and ΣX is simply connected. So we assume $n \geq 1$.

In our proof we will be considering loop spaces, ΩY, where Y has the homotopy type of a based CW complex. We first recall that there is an adjunction isomorphism,

$$\pi_q(\Omega Y) \xrightarrow{\cong} \pi_{q+1}(Y).$$

This isomorphism is given as follows: Let $\alpha : S^q \to \Omega Y$ represent an element of $\pi_q(\Omega Y)$. So for every $x \in S^q$, $\alpha(x) : S^1 \to Y$ is a basepoint preserving loop. We may then consider the adjoint

$$\bar{\alpha} : S^{q+1} = S^q \wedge S^1 \to Y$$
$$\bar{\alpha}(x \wedge t) = \alpha(x)(t) \in Y$$

Exercises.

1. Show that the correspondence

$$\pi_q(\Omega Y) \to \pi_{q+1}(Y)$$
$$\alpha \to \bar{\alpha}$$

defines an isomorphism.

2. Consider the adjoint map

$$j : X \to \Omega\Sigma X$$
$$x \to j_x : S^1 \to S^1 \wedge X$$

defined by $j_x(t) = t \wedge x$. Show that the induced composition

$$\pi_q(X) \xrightarrow{j} \pi_q(\Omega\Sigma X) \xrightarrow{\cong} \pi_{q+1}(\Sigma X)$$

is equal to the suspension homomorphism $\Sigma : \pi_q(X) \to \pi_{q+1}(\Sigma X)$ defined above (10.1).

Lemma 10.2. *Let X be an n-connected space. The map $j : X \to \Omega\Sigma X$ induces an isomorphism in homology,*

$$j_* : H_q(X) \xrightarrow{\cong} H_q(\Omega\Sigma X)$$

for $q \leq 2n$.

Proof. For a based space Y let PY be the space of basepoint preserving paths. Such paths are maps $\gamma : [0,1] \to Y$ such that $\gamma(0) = y_0$, where $y_0 \in Y$ is the basepoint. Consider the fibration

$$\Omega\Sigma X \to P\Sigma X \xrightarrow{\epsilon} \Sigma X$$

defined by $\epsilon(\gamma) = \gamma(1)$. We will consider the Serre spectral sequence for this fibration. Notice that the base space ΣX is simply connected and the total space $P\Sigma X$ is contractible. Therefore the E_∞ term of this spectral sequence must be identically zero, with the exception that $E_\infty^{0,0} = \mathbb{Z}$. This means that for every nonzero, positive dimensional class x in the E_2 term that is an infinite cycle, i.e $d_r(x) = 0$ for all r, then there must exist an element $y \in E_q$ for some q, with $d_q(y) = x$.

Consider the E_2-term. Recall that $E_2^{p,q} = H_p(\Sigma X; H_q(\Omega\Sigma X))$. Since X is n-connected, ΣX is $(n+1)$-connected, so $E_2^{p,q} = 0$ for $0 < p \leq n$. Also, since $\Omega\Sigma X$ is n-connected, we also have that $E_2^{p,q} = 0$ for $0 < q < n$.

Consider the differentials of the form

$$d_m : E_m^{n+q,0} \to E_m^{n+q-m,m-1}$$

for $q \leq n+1$. We claim that these differentials are all zero except when $m = n+q$, in which case

$$\begin{aligned} d_{n+q} : E_{n+q}^{n+q,0} &\to E_{n+q}^{0,n+q-1} \\ H_{n+q}(\Sigma X) &\to H_{n+q-1}(\Omega\Sigma X) \end{aligned}$$

is an isomorphism. To see this notice that $E_m^{n+q,0}$ is a subquotient of $H_{n+q}(\Sigma X)$ and $E_m^{n+q-m,m-1}$ is a subquotient of $H_{n+q-m}(\Sigma X; H_{m-1}(\Omega\Sigma X)$. When $m \leq n$, $H_{m-1}(\Omega\Sigma X) = 0$ since $\Omega\Sigma X$ is n-connected. If $n+q-1 \geq m > n$ then $H_{n+q-m}(\Sigma X) = 0$ since $n+q-m < q \leq n+1$ and ΣX is $(n+1)$-connected.

Since the spectral sequence converges to the zero E_∞-term (except $E_\infty^{0,0} = \mathbb{Z}$), we therefore must have that

$$\begin{aligned} d_{n+q} : E_{n+q}^{n+q,0} &\to E_{n+q}^{0,n+q-1} \\ H_{n+q}(\Sigma X) &\to H_{n+q-1}(\Omega\Sigma X) \end{aligned}$$

must be an isomorphism for $0 \leq q \leq n+1$. We leave it for the reader to check that the composition

$$H_{n+q-1}(X) \xrightarrow{\cong} H_{n+q}(\Sigma X) \xrightarrow{d_{n+q}} H_{n+q}(\Omega\Sigma X)$$

is the homology homomorphism induced by the map $j : X \to \Omega\Sigma X$. $\square$

We now complete the proof of the Freudenthal suspension theorem.

Proof. We continue to assume that X is n-connected. We now further consider the map $j : X \to \Omega\Sigma X$. As defined in Chapter 7 (Definition 7.5), we may consider the homotopy fiber of this map

$$F_j = \{(x, \alpha) \in X \times (\Omega\Sigma X)^I \quad \text{such that} \quad \alpha(0) = j(x) \quad \text{and} \quad \alpha(1) = y_0\}$$

where $y_0 \in \Omega\Sigma X$ is the basepoint. Since the sequence

$$F_j \to X \xrightarrow{j} \Omega\Sigma X$$

is, up to homotopy, a fibration sequence we may consider its Serre spectral sequence. Recall we are assuming that X (and therefore $\Omega\Sigma X$) is n-connected, with $n \geq 1$. Therefore the long exact sequence in homotopy groups ends with

$$\to \pi_2(X) \xrightarrow{j_*} \pi_2(\Omega\Sigma X) \xrightarrow{\partial_*} \pi_1(F_j) \to 0.$$

Since X and $\Omega\Sigma X$ are both simply connected, the Hurewicz theorem says that $\pi_2(X) \cong H_2(X)$ and $\pi_2(\Omega\Sigma X) \cong H_2(\Omega\Sigma X)$. But $j_* : H_2(X) \to H_2(\Omega\Sigma X)$ is an isomorphism by the above lemma. Thus $j_* : \pi_2(X) \to \pi_2(\Omega\Sigma X)$ is an isomorphism, and so we may conclude that $\pi_1(F_j) = 0$.

Now by examining the Serre spectral sequence for the homotopy fibration sequence $F_j \to X \xrightarrow{j} \Omega\Sigma X$, we see that since, by the above lemma, $j_* : H_p(X) \to H_p(\Omega\Sigma X)$ is an isomorphism for $p \leq 2n + 1$, then along the horizontal axis of this spectral sequence we have that

$$H_p(\Omega\Sigma X) = E_2^{p,0} \cong E_\infty^{p,0} = H_p(X)$$

for $p \leq 2n + 1$. Now along the vertical axis we have $H_q(F_j) = E_2^{0,q}$ which is equal to $E_\infty^{0,q}$ for $q \leq 2n$ because in this range there are no possible nonzero differentials. By the convergence of the spectral sequence to $H_*(X)$, and the fact that every element of $H_*(X)$ is represented in this spectral sequence by an element on the horizontal axis $E_\infty^{*,0}$ (in this range), we must conclude that $E_2^{0,q}$ must be zero for $q \leq 2n$. That is,

$$H_q(F_j) = 0$$

for $q \leq 2n$. Since, as observed above, $\pi_1(F_j) = 0$, then by the Hurewicz theorem we may conclude that $\pi_q(F_j) = 0$ for $q \leq 2n$. By the long exact sequence in homotopy groups for a fibration, this says that

$$j_* : \pi_q X \to \pi_q(\Omega\Sigma X)$$

is an isomorphism for $q \leq 2n$ and surjective for $q = 2n + 1$. Equivalently, the suspension homomorphism, $\pi_q X \to \pi_{q+1}(\Sigma X)$ is an isomorphism for $q \leq 2n$ and is surjective for $q = 2n + 1$. □

10.1 Spectra

The basic definition of a spectrum is the following.

Definition 10.2. *([57], [100]) A spectrum* is a sequence of (based) spaces $\{X_n,\, n \in \mathbb{Z}\}$ together with maps $\epsilon_n : \Sigma X_n \to X_{n+1}$. These maps are known as "structure maps". These structure maps can equivalently be given as maps $\bar{\epsilon}_n : X_n \to \Omega X_{n+1}$, where, as above, ΩY denotes the based loop space of a based space Y. The relation between these types of structure maps is given by the *adjunction* between a map $f : \Sigma X \to Y$ and the map $\bar{f} : X \to \Omega Y$, defined by $\bar{f}(x)(t) = f(t \wedge x) \in Y$.

In some settings one is only required to have spaces X_n for $n \geq 0$. This fits with the situation above, since we can simply define for $n < 0$, $X_n = point$.

Examples.

1. The *sphere spectrum* $\mathbb{S}$ is defined by

$$\mathbb{S}_n = S^n,$$

and $\epsilon_n : \Sigma \mathbb{S}_n = \Sigma S^n = S^{n+1} \to S^{n+1} = \mathbb{S}_{n+1}$ is the identity map. We will see that the sphere spectrum plays a crucial role in stable homotopy theory, analogous of the role the integers play in the theory of rings and modules.

2. The archetypical example of a spectrum is the *suspension spectrum* of a space, X. We denote the suspension spectrum by $\Sigma^\infty X$. Its definition is

$$(\Sigma^\infty X)_n = \Sigma^n X$$

and

$$\epsilon_n : \Sigma(\Sigma^n X) = \Sigma^{n+1} X \to \Sigma^{n+1} X$$

is the identity map for each n. Notice that the sphere spectrum $\mathbb{S}$ is a suspension spectrum, $\mathbb{S} = \Sigma^\infty S^0$.

3. An Eilenberg-MacLane spectrum $\mathbb{H}G$ for an abelian group G is a collection of other important examples. A spectrum $\mathbb{H}G$ is an Eilenberg-MacLane spectrum of type HG, if $\mathbb{H}G_n$ is an Eilenberg-MacLane space of type $K(G,n)$, and the structure maps $\epsilon_n : \Sigma \mathbb{H}G_n \to \mathbb{H}G_{n+1}$ are maps whose homotopy class in

$$\begin{aligned}
[\Sigma K(G,n), K(G,n+1)] &\cong H^{n+1}(\Sigma K(G,n); G) \\
&\cong H^n(K(G,n); G) \\
&\cong Hom(H_n(K(G,n)); G) \\
&\cong Hom(G,G)
\end{aligned}$$

corresponds to the identity homomorphism.

4. Let $U = \lim_{n\to\infty} U(n)$ be the colimit of the unitary groups as described in Chapter 4. A famous theorem of R. Bott [11] known as *"Bott Periodicity"* says that there is a homotopy equivalence,

$$\beta : \mathbb{Z} \times BU \xrightarrow{\simeq} \Omega U. \tag{10.2}$$

Furthermore, as we saw in Theorem 7.18, for any topological group G, there is a homotopy equivalence $\gamma : G \xrightarrow{\simeq} \Omega BG$. So in particular we have an equivalence $\gamma : U \xrightarrow{\simeq} \Omega BU = \Omega(\mathbb{Z} \times BU)$. (We take the basepoint of $\mathbb{Z} \times BU$ to be the basepoint of BU in $\{0\} \times BU \subset \mathbb{Z} \times BU$.)

One can then define the complex K-theory spectrum $\mathbb{KU}$ by

$$\mathbb{KU}_n = \begin{cases} \mathbb{Z} \times BU & \text{if } n \text{ is even} \\ U & \text{if } n \text{ is odd} \end{cases}$$

The structure maps in the K-theory spectrum are given by the homotopy equivalences

$$\bar{\epsilon}_{2m} = \beta : \mathbb{Z} \times BU \xrightarrow{\simeq} \Omega U$$
$$\bar{\epsilon}_{2m+1} = \gamma : U \xrightarrow{\simeq} \Omega(\mathbb{Z} \times BU)$$

This spectrum is sometimes referred to as the "Bott spectrum" or the "Bott periodicity" spectrum. Notice in particular that the spectrum $\mathbb{KU}$ has the feature that the adjoints of the structure maps , $\bar{\epsilon}_n : \mathbb{KU}_n \xrightarrow{\simeq} \Omega(\mathbb{KU}_{n+1})$ are homotopy equivalences for every $n \in \mathbb{Z}$.

Definition 10.3. *Given a spectrum $\mathbb{X}$, we define its homotopy groups by $\pi_k(\mathbb{X}) = \lim_{q\to\infty} \pi_{k+q}(X_q)$. This colimit is defined via maps*

$$\pi_{k+q}(X_q) \xrightarrow{\Sigma} \pi_{k+q+1}(\Sigma X_q) \xrightarrow{(\epsilon_k)_*} \pi_{k+q+1}(X_{q+1}).$$

Its homology groups are defined similarlly:

$$H_k(\mathbb{X}) = \lim_{q\to\infty} H_{k+q}(X_q).$$

Notice that a spectrum may have nonzero negatively graded homotopy groups and homology groups. For example, $\pi_{-3}(\mathbb{X}) = \lim_{q\to\infty} \pi_{q-3} X_q$ which may not be zero.

Exercise. Show that for k any integer (positive, negative, or zero),

$$\pi_k(\mathbb{KU}) \cong \begin{cases} \mathbb{Z} & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd} \end{cases}$$

Hint. Use the fact that $\pi_k(\Omega Y) \cong \pi_{k+1}(Y)$ for any based space Y.

One important feature of spectra is that they can be suspended, or *desuspended* an arbitrary number of times. If $\mathbb{X}$ is a spectrum, then for $k \in \mathbb{Z}$ define the k-fold suspension. $\Sigma^k \mathbb{X}$ by letting

$$(\Sigma^k \mathbb{X})_m = \mathbb{X}_{m+k}, \tag{10.3}$$

and the structure maps for $\Sigma^k bx$ are defined in terms of the structure maps of $\mathbb{X}$.

Exercise. Show that there are suspension isomorphisms

$$\begin{aligned} \pi_q(\mathbb{X}) &\cong \pi_{q+k}(\Sigma^k \mathbb{X}) \quad \text{and} \\ H_q(\mathbb{X}) &\cong H_{q+k}(\Sigma^k \mathbb{X}) \end{aligned} \tag{10.4}$$

10.1.1 Morphisms

If we think categorically, we now have objects in a category of spectra. But what about morphisms? Naively, one might expect a morphism between two spectra $\mathbb{X}$ and $\mathbb{Y}$ should be a collection of maps $f_n : \mathbb{X}_n \to \mathbb{Y}_n$ that respect the structure maps. That is, the following diagrams should commute:

$$\begin{array}{ccc} \Sigma \mathbb{X}_n & \xrightarrow{\Sigma f_n} & \Sigma \mathbb{Y}_n \\ {\scriptstyle \epsilon_n}\downarrow & & \downarrow{\scriptstyle \epsilon_n} \\ \mathbb{X}_{n+1} & \xrightarrow[f_{n+1}]{} & \mathbb{Y}_n. \end{array}$$

Certainly such collections of maps $\{f_n\}$ will constitute a morphism (or map) of spectra, but here is an important example of what such a definition would exclude.

Consider the Hopf map $\eta : S^3 \to S^2$. We can suspend the Hopf map an arbitrary number of times to produce maps

$$\Sigma^{n-2}\eta : S^{n+1} \to S^n$$

for all $n \geq 2$. In terms of the spaces making up sphere spectra we have maps

$$\begin{aligned} \eta_n : (\Sigma \mathbb{S})_n &\to \mathbb{S}_n \\ \Sigma^{n-2}\eta : S^{n+1} &\to S^n \end{aligned}$$

for $n \geq 2$ that preserve the structure maps. But notice that no such maps exist for $n = 0$ or 1, because there is no map $S^2 \to S^1$ whose suspension is the Hopf map $\eta : S^3 \to S^2$. But surely we want the collection of maps defined by suspending η to define a map between spectra,

$$\eta : \Sigma \mathbb{S} \to \mathbb{S}.$$

More generally, we would like to have a definition of morphisms between spectra such that every element of the stable homotopy groups of any spectrum $\mathbb{X}$, $\alpha \in \pi_n^s(\mathbb{X})$ is represented by a map of spectra

$$\alpha : \Sigma^n \mathbb{S} \to \mathbb{X}$$

for any $n \in \mathbb{Z}$.

To produce such an appropriate definition of morphism of spectra, we use the notion of an "ω - spectrum".

Definition 10.4. *An "ω - spectrum" is a spectrum $\mathbb{Y}$ such that the adjoints of the structure maps*

$$\epsilon_n : \mathbb{Y}_n \to \Omega \mathbb{Y}_{n+1}$$

are homeomorphisms.

This might seem like a very restrictive definition, but we observe that each spectrum in the sense of Definition 10.2 naturally has an associated ω-spectrum.

Definition 10.5. *Let $\mathbb{X}$ be a spectrum. Define its associated ω-spectrum $\mathbb{X}^\omega$ by*

$$\mathbb{X}^\omega_n = \lim_{k \to \infty} \Omega^k \mathbb{X}_{n+k}.$$

The maps used in this limit are

$$\Omega^{k-1}\mathbb{X}_{n+k-1} \xrightarrow{\Omega^{k-1}\bar{\epsilon}_{n+k-1}} \Omega^{k-1}\Omega \mathbb{X}_{n+k} = \Omega^k \mathbb{X}_{n+k}.$$

The structure maps for the ω-spectrum $\mathbb{X}^\omega$ are given by

$$\begin{aligned} \epsilon^\omega_n : \Sigma \mathbb{X}^\omega_n = \Sigma \lim_{k\to\infty} \Omega^k \mathbb{X}_{n+k} \to \lim_{k\to\infty} \Sigma\Omega^k \mathbb{X}_{n+k} &= \lim_{q\to\infty} \Sigma\Omega(\Omega^q \mathbb{X}_{n+1+q}) \\ &\xrightarrow{ev} \lim_{q\to\infty} \Omega^q \mathbb{X}_{n+1+q} = \mathbb{X}^\omega_{n+1}. \end{aligned}$$

In this description q was substituted for $k-1$ and $ev : \Sigma\Omega Y \to Y$ is the evaluation map, $ev(t \wedge \theta) = \theta(t) \in Y$.

Exercise. Check that $\mathbb{X}^\omega$ is indeed an ω-spectrum, and that there are natural isomorphisms

$$\pi_k \mathbb{X} \xrightarrow{\cong} \pi_k(\mathbb{X}^\omega) \quad \text{and} \quad H_k \mathbb{X} \xrightarrow{\cong} H_k(\mathbb{X}^\omega)$$

for any spectrum $\mathbb{X}$.

We may now define what we mean by a map or morphism between spectra.

Definition 10.6. *Let $\mathbb{X}$ and $\mathbb{Y}$ be spectra, as in Definition 10.2. We define a map of spectra $\phi : \mathbb{X} \to \mathbb{Y}$ to be a collection of maps between the spaces making up their associated ω-spectra,*

$$\phi_n : \mathbb{X}^\omega = \lim_{k \to \infty} \Omega^k X_{n+k} \to \lim_{k \to \infty} \Omega^k Y_{n+k} = \mathbb{Y}^\omega_n$$

that preserve the structure maps

$$\begin{array}{ccc} \Sigma \mathbb{X}^\omega_n & \xrightarrow{\Sigma \phi_n} & \Sigma \mathbb{Y}^\omega_n \\ {\scriptstyle \epsilon^\omega_n} \downarrow & & \downarrow {\scriptstyle \epsilon^\omega_n} \\ \mathbb{X}^\omega_{n+1} & \xrightarrow[\phi_{n+1}]{} & \mathbb{Y}^\omega_n . \end{array}$$

Remark. Since morphisms between spectra are made up out of maps between spaces in their corresponding ω-spectra, some texts refer to an object satisfying Definition 10.2 as a *"prespectrum"* and reserve the term "spectrum" to an object that we call an ω-spectrum.

Exercise. Let $\mathbb{X}$ be a spectrum. Show that every element of its homotopy groups $\alpha \in \pi_m \mathbb{X}$ represents, and is represented by a map of spectra

$$\alpha : \Sigma^m \mathbb{S} \to \mathbb{X}.$$

10.2 Generalized (co)homology and Brown's Representability Theorem

One of the most important applications of spectra over the many years since their original definition, has been to *generalized (co)homology theories*. Such a theory is a functor from the category of pairs of spaces to the category of graded abelian groups that satisfy all the Eilenberg-Steenrod axioms with the possible exception of the dimension axiom. The *Brown Representability theorem* states that any such generalized cohomology theory is represented by a spectrum, and conversely, any spectrum represents a generalized cohomology theory. Considering the homological perspective, Whitehead [100] showed how spectra give rise to generalized homology theories as well, and he studied manifold orientations and Poincaré duality in the setting of these generalized theories. In this section we describe these major advances in homotopy theory.

10.2.1 Brown's Representability Theorem

We begin by describing Brown's representability theorem [15] [16]. We actually describe a variant of Brown's theorem proved by Adams in [4].

Let $\mathcal{CW}$ be the category whose objects are finite CW-complexes with basepoints, and whose morphisms are basepoint preserving maps. Let $\mathcal{G}$ be the category of abelian groups and homomorphisms. We consider contravariant functors

$$\mathcal{H} : \mathcal{CW} \to \mathcal{G}.$$

For ease of notation, if $f : X \to Y$ is a basepoint preserving map between finite CW complexes, we denote the homomorphism induced by applying $\mathcal{H}$ to this map by

$$f^* : \mathcal{H}(Y) \to \mathcal{H}(X).$$

We consider three interesting axioms on such contravariant functors. The first is the *homotopy axiom.*

Homotopy Axiom. If $f, g : X \to Y$ are basepoint preserving maps between finite CW-complexes that are homotopic via a basepoint preserving homotopy, then

$$f^* = g^* : \mathcal{H}(Y) \to \mathcal{H}(X).$$

For our next axiom let X and $Y \in \mathcal{CW}$, and let $X \vee Y$ be their wedge. Let $\iota_X : X \hookrightarrow X \vee Y$ and $\iota_Y : Y \hookrightarrow X \vee Y$ be the natural inclusions. A contravariant functor $\mathcal{H} : \mathcal{CW} \to \mathcal{G}$ defines a homomorphism

$$\iota_X^* \times \iota_Y^* : \mathcal{H}(X \vee Y) \to \mathcal{H}(X) \times \mathcal{H}(Y).$$

Wedge Axiom. $\iota_X^* \times \iota_Y^* : \mathcal{H}(X \vee Y) \to \mathcal{H}(X) \times \mathcal{H}(Y)$ is an isomorphism.

In order to state the third axiom, consider the following diagram, in which the homomorphisms are induced by the obvious inclusion maps.

$$\begin{array}{ccc} \mathcal{H}(X \cup Y) & \xrightarrow{c^*} & \mathcal{H}(X) \\ {\scriptstyle d^*}\big\downarrow & & \big\downarrow{\scriptstyle a^*} \\ \mathcal{H}(Y) & \xrightarrow[b^*]{} & .\mathcal{H}(X \cap Y) \end{array}$$

Mayer-Vietoris Axiom. . Suppose $x \in \mathcal{H}(X)$ and $y \in \mathcal{H}(Y)$ are such that $a^*(x) = b^*(y)$. Then there exists an element $z \in \mathcal{H}(X \cup Y)$ such that $c^*(z) = x$ and $d^*(z) = y$.

The following is the variant of Brown's representability theorem, proved in this form by Adams, that will be most useful to us.

Theorem 10.3. *[15][16][4] Let $\mathcal{H} : \mathcal{CW} \to \mathcal{G}$ be a contravariant functor satisfying the Homotopy Axiom, the Wedge Axiom, and the Mayer-Vietoris Axiom. Then $\mathcal{H}$ is "representable". That is, there is a based (not necessarily finite) CW-complex B and a natural bijection of sets*

$$T : [X, B] \xrightarrow{\cong} \mathcal{H}(X)$$

defined for all finite, based CW complexes X.

Comment. By "natural bijection" we mean that if $f : X \to Y$ is a morphism in $\mathcal{CW}$ (i.e a basepoint preserving map), then the following diagram commutes:

$$\begin{array}{ccc} [Y, B] & \xrightarrow[\cong]{T} & \mathcal{H}(Y) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ [X, B] & \xrightarrow[\cong]{T} & \mathcal{H}(X). \end{array}$$

We will now sketch a proof of Brown's theorem. We will actually describe the proof of something stronger. Namely we will show that the "representing space" B is a "weak, group-like H-space", to be properly defined below, but it implies that B has a product map, $B \times B \to B$ which gives the set of homotopy classes of maps $[X, B]$ a group structure. We will then show that the set bijection

$$T : [X, B] \xrightarrow{\cong} \mathcal{H}(X)$$

is actually an isomorphism of groups.

The first step is to consider an extension of a contravariant functor $\mathcal{H} : \mathcal{CW} \to \mathcal{G}$ to *all* CW-complexes (i.e not necessarily finite), as described Adams [4]. He defined a contravariant functor

$$\hat{\mathcal{H}}(X) = \varprojlim_{\alpha} \mathcal{H}(X_\alpha)$$

where the inverse limit runs over all finite subcomplexes $X_\alpha \subset X$. (All of our subcomplexes are assumed to contain the basepoint.). Of course if X is a finite CW-complex, $\hat{\mathcal{H}}(X) = \mathcal{H}(X)$. In order to understand the properties of the extended functor $\hat{\mathcal{H}}$, Adams introduced the notion of "weak homotopy".

Definition 10.7. *Two basepoint preserving maps between based CW-complexes $f, g : X \to Y$ are "weakly homotopic", written $f \sim_w g$ if fh is homotopic to gh for every map $h : K \to X$ where K is a based finite CW-complex and h is basepoint preserving.*

The following result is an easy exercise that we leave to the reader.

Lemma 10.4. . *Let $\mathcal{H} : \mathcal{CW} \to \mathcal{G}$ be a contravariant functor satisfying the Homotopy, Wedge, and Mayer-Vietoris axioms. Then if $f, g : X \to Y$ are basepoint preserving maps between (not necessarily finite) CW-complexes that are weakly homotopic, then*

$$f^* = g^* : \hat{\mathcal{H}}(Y) \to \hat{\mathcal{H}}(X).$$

Here is a rather straightforward result that the reader can verify or look up in Brown's papers [15][16].

Lemma 10.5. *Let $\mathcal{H} : \mathcal{CW} \to \mathcal{G}$ satisfy the Homotopy, Wedge, and Mayer-Vietoris axioms. Let*

$$K \xrightarrow{f} L \xrightarrow{i} L \cup_f c(K)$$

be a cofibration sequence of finite complexes. Then the sequence

$$\mathcal{H}(K) \xleftarrow{f^*} \mathcal{H}(L) \xleftarrow{i^*} \mathcal{H}(L \cup_f c(K))$$

is exact.

For the next result we assume that K is a finite complex, containing subcomplexes L and M. We continue to assume that $\mathcal{H} : \mathcal{CW} \to \mathcal{G}$ is a contravariant functor satisfying the Homotopy, Wedge, and Mayer-Vietoris axioms.

Lemma 10.6. *Let $\iota_1 : L \to L \cup M$ and $\iota_2 : M \to L \cup M$ be the inclusion maps. Then there is an exact sequence*

$$\mathcal{H}(L) \times \mathcal{H}(M) \xleftarrow{\iota_1^* \times \iota_2^*} \mathcal{H}(L \cup M) \leftarrow \mathcal{H}(\Sigma(L \cap M) \xleftarrow{g^*} \mathcal{H}(\Sigma(L \vee M)$$

which is natural with respect to maps $K, L, M \to K', L', M'$. Furthermore the homomorphism g^* is induced by a map of spaces, $g : \Sigma(L \cap M) \to \Sigma(L \vee M)$.

Proof. Consider the obvious map

$$L \vee M \to L \cup M$$

and the resulting cofibration sequence

$$L \vee M \to L \cup M \to (L \cup M) \cup c(L \vee M) \to \Sigma(L \vee M) \to \cdots$$

Notice that the third term is homotopy equivalent to $\Sigma(L \cap M)$. The lemma now follows from Lemma 10.5. □

Exercise. Prove the assertion made in this proof that $(L \cup M) \cup c(L \vee M)$ is homotopy equivalent to $\Sigma(L \cap M)$.

The next two results are immediate, and we leave their verifications to the reader. In both cases we continue to assume that $\mathcal{H} : \mathcal{CW} \to \mathcal{G}$ is a contravariant functor satisfying the Homotopy, Wedge, and Mayer-Vietoris axioms.

Lemma 10.7. *$\hat{\mathcal{H}}$ satisfies the Wedge axiom.*

Lemma 10.8. *Let X be a CW-complex and $\{X_\alpha\}$ a directed set of subcomplexes whose union is X. Then the natural map*

$$\hat{\mathcal{H}}(X) \to \varprojlim_{\alpha} \hat{\mathcal{H}}(X_\alpha)$$

is an isomorphism.

The following is quite a reasonable result, whose proof is in Adams's paper [4]. We refer the reader to that paper for the argument.

Proposition 10.9. *Let X be a based CW-complex with subcomplexes $U, V \subset X$, each containing the basepoint. Suppose furthermore that $U \cap V$ is a finite complex. Let $\mathcal{H} : \mathcal{CW} \to \mathcal{G}$ be a contravariant functor satisfying the Homotopy, Wedge, and Mayer-Vietoris axioms. Then the square*

$$\begin{array}{ccc} \mathcal{H}(U \cap V) = \hat{\mathcal{H}}(U \cap V) & \longleftarrow & \hat{\mathcal{H}}(U) \\ \uparrow & & \uparrow \\ \hat{\mathcal{H}}(V) & \longleftarrow & \hat{\mathcal{H}}(U \cup V) \end{array}$$

satisfies the Mayer-Vietoris axiom.

We now apply these results to prove Brown's representability theorem. For the remainder of this section we continue to assume that $\mathcal{H} : \mathcal{CW} \to \mathcal{G}$ is a contravariant functor satisfying the Homotopy, Wedge, and Mayer-Vietoris axioms.

Let Y be a CW complex (not necessarily finite) and let $y \in \hat{\mathcal{H}}(Y)$. Given any CW-complex X, let $[X, Y]_w$ denote the set of weak homotopy clases of basepoint preserving maps, as defined in Definition 10.7. Consider the natural transformation

$$\hat{T} : [X, Y]_w \to \hat{\mathcal{H}}(Y)$$

given by

$$\hat{T}^*(f) = f^*(y).$$

Notice that $\hat{T}$ is well-defined by Lemma 10.4 and is natural for all CW-complexes X. By restricting to finite complexes one as a natural transformation

$$T : [K, Y] \to \mathcal{H}(Y).$$

Let $NatTrans(A, B)$ be the set of natural transformations between functors A and B. The following lemma is not difficult, and is Adams's interpretation ([4], Lemma 4.1) of a result of Brown ([15], p. 478).

Lemma 10.10. *The above construction gives bijective correspondences*

$$\begin{aligned} \hat{\mathcal{H}}(Y) &\cong NatTrans([X, Y]_w, \hat{\mathcal{H}}(X)) \\ &\cong NatTrans([K, Y], \mathcal{H}(K)). \end{aligned}$$

Furthermore these correspondences are natural with respect to maps of Y.

The following is the basic constructive idea in forming the representing space for a functor $\mathcal{H}$.

Lemma 10.11. *Let Y_n be a CW-complex provided with an element $y_n \in \hat{\mathcal{H}}(Y_n)$. Then there exists a complex Y_{n+1} with an element $y_{n+1} \in \hat{\mathcal{H}}(Y_{n+1})$ and an embedding $i : Y_n \hookrightarrow Y_{n+1}$ satisfying the following properties:*

1. $i^* y_{n+1} = y_n$
2. *If K is any finite CW-complex and $f, g : K \to Y_n$ are maps such that $f^*(y_n) = g^*(y_n)$, then $i \circ f$ is homotopic to $i \circ g$ as maps $K \to Y_{n+1}$.*

Proof. (Sketch) For each finite complex K and pair of homotopy classes of maps $f : K \to Y_n$ and $g : K \to Y_n$ such that $f^*(y_n) = g^*(y_n)$ choose representatives for f and g. Furthermore we let K range over representatives of all homotopy classes of finite complexes. Thus we have chosen a countable set of indices A and maps $f_\alpha, g_\alpha : K_\alpha \to Y_n$ for $\alpha \in A$. One then forms

$$Y_{n+1} = Y_n \cup \bigcup_{\alpha \in A} (I \times K_\alpha)/I \times point)$$

where. "point" refers to the basepoint in K_α and the reduced cylinder $I \times K_\alpha)/I \times point$ is attached to Y_n by the map f_α and one end and g_α at the other end. (This construction is called a "mapping cylinder".).

The embedding $i : Y_n \hookrightarrow Y_{n+1}$ is the obvious inclusion map. Clearly $i \circ f_\alpha$ is homotopic to $i \circ g_\alpha$ for each $\alpha \in A$. It remains to define the class $y_{n+1} \in \hat{\mathcal{H}}(Y_{n+1}$ such that $i^* y_{n+1} = y_n$. This is straightforward using the axioms established for $\hat{\mathcal{H}}$, and we refer the reader to Adams [4] or Brown [15] for details. □

We are now ready to prove the following slight generalization of Brown's representability theorem (Theorem 10.3 above).

Theorem 10.12. *Let Y_0 be a CW-complex equipped with a class $y_0 \in \hat{\mathcal{H}}(Y_0)$. Then there exists a CW-complex Y together with an embedding $i : Y_0 \hookrightarrow Y$ and an element $y \in \hat{\mathcal{H}}(Y)$ such that $i^*(y) = y_0$, and such that the corresponding natural transformation*

$$T : [K, Y] \to \mathcal{H}(K)$$

is a bijection of sets for all finite complexes K.

Remarks. 1. Theorem 10.3 is a special case of Theorem 10.12 reflecting the case $Y_0 = point$.

2. Y is called a *representing complex* for the functor $\mathcal{H}$.

Proof. Let K run over a countable set of representatives of finite complexes as in the proof of Lemma 10.11. For each K let h run over $\mathcal{H}(K)$. Form

$$Y_1 = Y_0 \vee \bigvee_{K,h} K$$

Using the fact that $\hat{\mathcal{H}}$ satisfies the Wedge Axiom, let $y_1 \in \hat{\mathcal{H}}(Y_1)$ be the element that restricts to y_0 in $\hat{\mathcal{H}}(Y_0)$, and to h on the $(K, h)^{th}$ summand of the wedge. This then implies that the natural transformation

$$T_1 : [K, Y_1] \to \mathcal{H}(K)$$

corresponding to $y_1 \in \hat{\mathcal{H}}(Y_1)$ is surjective for every K.

Now construct complexes

$$Y_1 \hookrightarrow Y_2 \hookrightarrow \cdots Y_n \hookrightarrow \cdots$$

and elements $y_n \in \hat{\mathcal{H}}(Y_n)$, as in Lemma 10.11. Let $Y = \bigcup_n Y_n$, and let $y \in \hat{\mathcal{H}}(Y)$ be the element that restricts to $y_n \in \hat{\mathcal{H}}(Y_n)$ for every n. (This uses Lemma 10.8.) The corresponding natural transformation

$$T : [K, Y] \to \mathcal{H}(K)$$

is still surjective. But notice that it is also injective. To see this, let $f, g : K \to Y$ be any two maps such that $f^*(y) = g^*(y)$. Then since K is a finite complex f and g both must map into Y_n for some n. This means $f^*(y_n) = g^*(y_n)$ and f is homotopic to g in Y_{n+1} by Lemma 10.11. This completes the proof of Theorem 10.12. □

The following extension of Theorem 10.3 is straightforward, and we refer the reader to the paper by Adams [4] for details.

Theorem 10.13. *1. There is one and only one transformation*

$$\hat{T} : [X, Y]_w \to \hat{\mathcal{H}}(X)$$

defined and natural for all CW-complexes X, that reduces to T when X is finite.

2. The natural transformation $\hat{T}$ is a bijection of sets for any CW-complex X.

Lemma 10.10 together with the Brown Representability Theorem 10.12 allows us to prove the following quite easily. (See [4], Addendum 1.5.)

Proposition 10.14. *Let X be any CW complex (not necessarily finite), and let Y be a representing complex for the contravariant functor $\mathcal{H} : \mathcal{CW} \to \mathcal{G}$ satisfying the Homotopy, Wedge, and Mayer-Vietoris axioms. Let*

$$U : [K, X] \to [K, Y]$$

be a natural transformation of sets defined for finite CW complexes K. Then there is a map $f : X \to Y$ inducing U, and is unique up to weak homotopy.

Proof. We first show that such a map $f : X \to Y$ exists. Consider the composition

$$[K, X] \xrightarrow{U} [K, Y] \xrightarrow{T} \mathcal{H}(K),$$

where T is the natural "representing transformation" defined in Theorem 10.12. By Lemma 10.10 this composition corresponds to an element $\alpha \in \hat{\mathcal{H}}(X)$, while T itself corresponds to an element $\beta \in \hat{\mathcal{H}}(Y)$. By Theorem 10.13 there is a map $f : X \to Y$, which is well-defined up to weak homotopy, such that $f^*(\beta) = \alpha$. By the naturality statement in Lemma 10.10 this means that $Tf_* : [K, X] \to \mathcal{H}(K)$ is equal to $T \circ U$. Since T is a bijection, $f_* = U$.

To check the uniqueness statement, suppose $f_* = g_* : [K, X] \to [K, Y]$ for every finite complex K. By definition this means that f is weakly homotopic to g. □

We next show that the natural transformation $\hat{T} : [X, Y] \to \hat{\mathcal{H}}(Y)$ defined in Theorem 10.13 is an isomorphism of groups. In order to show this we need to show that the representing space Y has a multiplicative structure that endows $[X, Y]$ with a group structure with respect to which T is a homomorphism of groups.

Consider the product structure

$$\hat{\mathcal{H}}(X) \times \hat{\mathcal{H}}(X) \xrightarrow{\mu} \hat{\mathcal{H}}(X)$$

given by the group structure of $\mathcal{H}(X)$. Here X can be any CW complex. By Theorem 10.13 this defines a product

$$[X, Y \times Y]_w \xrightarrow{\cong} [X, Y]_w \times [X, Y]_w \xrightarrow{\mu} [X, Y]_w$$

for any CW-complex. By letting $X = Y \times Y$, we can let $\nu : Y \times Y \to Y$ represent the image of the identity in $[Y \times Y, Y \times Y]_w$ under this composition. $\nu : Y \times Y \to Y$ is well-defined up to weak homotopy. By construction, this product induces the product structure on $[X, Y]_w$ for any CW-complex and therefore on $[K, Y] \cong \mathcal{H}(K)$ for any finite CW-complex K. Also by construction, the natural transformation

$$T : [X, Y]_w \to \hat{\mathcal{H}}(X)$$

respects this product structure, and therefore by Theorem 10.13 is an isomorphism of groups.

10.2.2 Generalized (co)homology theories

We now apply the Brown representability theorem to classify *generalized cohomology theories*. We refer the reader to [15] for details.

Definition 10.8. *Let $\mathcal{CW}_2$ be the category of pairs (X, A) of finite CW-complexes. A (generalized) cohomology theory E is a collection of contravariant functors $E^q : \mathcal{CW}_2 \to \mathcal{G}$ and a collection of natural homomorphisms $\delta^q : E^q(A) \to E^{q+1}(X, A)$ defined for each pair $(X, A) \in \mathcal{CW}_2$, satisfying the following Eilenberg-Steenrod axioms:*

- **Homotopy:** If $f, g : (X, A) \to (Y, B)$ are homotopic, $f^* = g^* : E^q(Y, B) \to E^q(X, A)$. (Here, as above, we are using the superscript $*$ to denote the homomorphism of groups induced by the contravariant functor E applied to the map (morphism) in $\mathcal{CW}_2$.)
- **Exactness:** Let $\iota : (X, \emptyset) \to (X, A)$ and $j : A \hookrightarrow X$ be the natural inclusion maps. Then the following sequence is exact.

$$\cdots \to E^{q-1}(A) \xrightarrow{\delta^{q-1}} E^q(X, A) \xrightarrow{\iota^*} E^q(X) \xrightarrow{j^*} E^q(A) \to \cdots$$

- **Excision:** If $(X_1, X_1 \cap X_2)$ and $(X_2, X_1 \cap X_2)$ are CW-pairs in $\mathcal{CW}_2$, the map
$$E^q(X_1 \cup X_2, X_2) \to E^q(X_1, X_1 \cap X_2)$$
induced by the inclusion map is an isomorphism for all q.

We now describe Brown's theorem stating that all generalized cohomology theories determine, and are determined by a spectrum.

Let $\mathbb{E} = \{(\mathrm{E_q}, \epsilon_\mathrm{q} : \Sigma\mathrm{E_q} \to \mathrm{E_{q+1}})\}$ be an ω-spectrum. (Recall that this means that the adjoint mappings $\bar{\epsilon}_q : E_q \to \Omega E_{q+1}$ are homeomorphisms.). For a finite CW-complex X, with subcomplex $A \subset X$, let $E^q(X, A) = [X/A, E_q]$. Here, as above, we mean based homotopy classes of basepoint preserving maps.

Note. If A is not a subcomplex of X, but one rather simply has a map $\iota : A \to X$, one can replace the quotient X/A by the mapping cone $X \cup_\iota c(A)$. When A is the emptyset, we use the notation X/A to mean the space $X \sqcup pt$, where the basepoint is the disjoint point. In this case the set

$$[X/\emptyset, E_q] = [X \sqcup pt, E_q]$$

where the last set can simply be viewed as the set of homotopy classes of *unbased* maps from X to E_q. Let $S : X/A \simeq X \cup_\iota c(A) \to \Sigma A$ be the map that collapses $X \subset X \cup_\iota c(A)$ to a point. Let $\sigma : [A \sqcup pt, \Omega E_{q+1}] \xrightarrow{\simeq} [\Sigma(A \sqcup pt), E_{q+1}]$ be the usual adjunction isomorphism. We can then define

$$\delta^q : [A \sqcup pt, E_q] \to [X/A, E_{q+1}]$$

by letting $\delta(f) : X/A \to E_{q+1}$ be the composition

$$X/A \xrightarrow{S} \Sigma(A \sqcup pt) \xrightarrow{\Sigma f_+} \Sigma E_q \xrightarrow{\epsilon_q} E_{q+1}.$$

In this composition $f_+ : A \sqcup pt \to E_q$ is equal to f on A and sends the disjoint basepoint to the basepoint of E_q.

Notice that the set $[X, E_q]$ has a natural group structure since this set is equal to $[X, \Omega E_{q+1}]$. Indeed it is naturally an abelian group. (Consider the following exercise.)

Exercise. 1. Show that if X and Y are any based spaces, $[X, \Omega Y]$ has a natural group structure.

Hint. Recall how one defines the group structure on the fundamental group $\pi_1(Y)$, to show that ΩY has a product that defines a group structure "up to homotopy".

2. Show that the set $[X, \Omega^2 Z]$ is an abelian group. Here $\Omega^2(Z) = \Omega(\Omega(Z))$.

Hint. Recall how one shows that the second homotopy group $\pi_2(Z)$ is abelian.

The following is now a straightforward exercise.

Theorem 10.15. *For $\mathbb{E} = \{(\mathrm{E_q}, \epsilon_\mathrm{q} : \Sigma \mathrm{E_q} \to \mathrm{E_{q+1}})\}$ an ω-spectrum, the contravariant functor $(E^q(X, A), \delta^q)$ as defined above, forms a generalized cohomology theory. That is, it satisfies Definition 10.8 above.*

Exercise. Prove Theorem 10.15.

We now consider the converse.

Theorem 10.16. *(Brown [15]) Let $E = \{(E^q, \delta^q)\}$ be a generalized cohomology theory as defined above. Then there is an ω-spectrum $\mathbb{E} = \{(\mathrm{E_q}, \epsilon_\mathrm{q})\}$ and natural equivalences $\tau_q : [(X/A, E_q] \to E^q(X, A)$ defined for all pairs of finite CW-complexes (X, A). Furthermore we have the relation $\delta^q \tau_q = \tau_{q+1} \delta^q$ for each $q \in \mathbb{Z}$.*

Proof. Consider the contravariant functors

$$\begin{aligned} \tilde{E}^q : \mathcal{CW} &\to \mathcal{G} \\ X &\to E^q(X, x_0) \end{aligned}$$

where $x_0 \in X$ is the basepoint. Since E satisfies the Eilenberg-MacLane axioms as given in Definition 10.8, clearly $\tilde{E}^q$ satisfies the Homotopy Axiom. Furthermore, standard arguments show that since E satisfies exactness and excision, each $\tilde{E}^q$ satisfies the Wedge and Mayer-Vietoris axioms. Therefore by the Brown Representability Theorem 10.3, there is a representing space E_q for the functor $\tilde{E}^q$. That is, there is a natural equivalence

$$\tau_q : [X, E_q] \xrightarrow{\cong} \tilde{E}^q(X) = E^q(X, x_0).$$

We now show that the collection $\{E_q\}$ fit together to define a spectrum. We first prove a suspension isomorphism in the following lemma.

Lemma 10.17. *There is a natural isomorphism*

$$\delta^q : \tilde{E}^q(X) \xrightarrow{\cong} \tilde{E}^{q+1}(\Sigma X).$$

Proof. . Consider the triple $(c(X), X, x_0)$. Here $c(X)$ is the cone, $c(X) = X \times I/X \times \{1\} \cup x_0 \times I$ where $I = [0, 1]$ is the unit interval. X is viewed as a subspace of $c(X)$ as $X \times \{0\}$. By excision and exactness one has that

$$E^q(c(X), X) \cong E^q(\Sigma X, x_0) = \tilde{E}^q(\Sigma X)$$

By substituting this in to the exact sequence of a triple we have an exact sequence

$$\cdots \to E^q(c(X), x_0) \to E^q(X, x_0) \xrightarrow{\delta^q} E^{q+1}(c(X), X) \to \cdots .$$

Since the cone $c(X)$ is contractible and the cohomology theory E satisfies the Homotopy Axiom, $E^r(c(X), x_0) = 0$ for every r. Therefore this sequence becomes

$$\cdots \to 0 \to \tilde{E}^q(X) \xrightarrow{\delta^q} \tilde{E}^{q+1}(\Sigma X) \to 0 \to \cdots$$

Thus the connecting homomorphisms $\delta^q : \tilde{E}^q(X) \to \tilde{E}^{q+1}(\Sigma X)$ is an isomorphism. □

We now continue our proof of Theorem 10.16.

The natural suspension isomorphism given by the above lemma can be interpreted as a natural isomorphism

$$\delta^q : [X, E_q] \xrightarrow{\cong} [\Sigma X, E_{q+1}] \xrightarrow{\cong} [X, \Omega E_{q+1}].$$

By Proposition 10.14 above, the natural transformation δ^q is realized by a map we call $\epsilon_q : E_q \to \Omega E_{q+1}$ which is uniquely defined up to weak homotopy. Notice furthermore that these maps are weak homotopy equivalences, since for any finite complex K the map

$$[K, E_q] \xrightarrow{\delta^q = (\epsilon_q)_*} [K, \Omega E_{q+1}]$$

is an isomorphism. So the collection $\{(E_q, \epsilon_q)\}$ defines a spectrum which represents the cohomology theory E. Notice that we might think of this spectrum as a "weak homotopy ω-spectrum", since the adjoints of the structure maps $\bar{\epsilon}_q : E_q \to \Omega E_{q+1}$ are weak homotopy equivalences, where in the definition of an actual ω-spectrum, these maps are required to be homeomorphisms. In any case we can now replace this spectrum by its associated ω-spectrum as in Definition 10.5 which we call $\mathbb{E}$. This completes the proof. □

An analogous statement for *"generalized homology theories"*, which is to say that *covariant functors* $\{E_q, \delta_q\} : \mathcal{CW}_2 \to \mathcal{G}$ that satisfy the covariant analogues of the Eilenberg-Steenrod axioms 10.8, was proven by G. Whitehead [100]. In order to state his theorem we first introduce the following definition.

Definition 10.9. *Let X be a space with basepoint, and $\mathbb{E} = \{E_k, \epsilon_k\}$ be a spectrum. The smash product spectrum $X \wedge \mathbb{E}$ has as its k^{th} space,*

$$(X \wedge \mathbb{E})_k = X \wedge E_k$$

with structure maps $1 \wedge \epsilon_k : \Sigma(X \wedge E_k) = X \wedge \Sigma E_k \xrightarrow{1\wedge\epsilon_k} X \wedge E_{k+1}$.

Exercise. Show that for any based space X, the spectrum $\mathbb{S} \wedge X$ is weakly homotopy equivalent to the suspension spectrum, $\Sigma^\infty X$. That is to say, there is a morphism of spectra, $\phi : \mathbb{S} \wedge X \to \Sigma^\infty X$ that induces an isomorphism on homotopy groups.

G. Whitehead [100] proved the following analogue of Theorem 10.16 about about the representability of *generalized homology* theories.

Theorem 10.18. *(Whitehead [100]) Let $E_* = \{(E_q, \delta_q)\}$ be a generalized homology theory. That is, it is a collection of covariant functors $E_q : \mathcal{CW}_2 \to \mathcal{G}$ and a collection of natural homomorphisms $\delta_q : E_q(X, A) \to E_{q-1}(A)$ defined for each pair $(X, A) \in \mathcal{CW}_2$, satisfying the covariant analogues of the Eilenberg-Steenrod axioms: Homotopy, Exactness, and Excision (see Definition 10.8). Then there is an ω-spectrum $\mathbb{E} = \{(\mathrm{E_q}, \epsilon_\mathrm{q})\}$ and natural equivalences*

$$\tau_q : \pi_q((X/A) \wedge \mathbb{E}) \to \mathrm{E_q(X, A)}$$

defined for all pairs of finite CW-complexes (X, A). Furthermore we have the relation $\delta_q \tau_q = \tau_{q+1} \delta_q$ for each $q \in \mathbb{Z}$.

Examples.

1. Let $\mathbb{S}$ be the sphere spectrum. Since, by the above exercise, for any space X, $X \wedge \mathbb{S} \simeq \Sigma^\infty X$, we have that the generalized homology theory represented by $\mathbb{S}$ is *stable homotopy groups*. The generalized cohomology theory represented by $\mathbb{S}$ is known as *stable cohomotopy*. Notice that

$$\mathbb{S}^k(X) = \varinjlim_n [\Sigma^n X, S^{n+k}],$$

and may in particular be nonzero for $k < 0$.

2. Let G be an abelian group and $\mathbb{H}G$ the corresponding Eilenberg-MacLane spectrum. The cohomology theory this spectrum represents is ordinary cohomology with coefficients in G. It gives the well-known result of Hopf stating that

$$[X, K(G, m)] \cong H^m(X; G).$$

In homology, Whitehead's theorem gives that $\mathbb{H}G_*(X) \cong H_*(X; G)$ which is the very non-intuitive result that

$$H_k(X; G) \cong \pi_k(X_+ \wedge \mathbb{H}G) = \varinjlim_n \pi_{k+n}(X \wedge K(G, k+n)),$$

where $X_+ = X/\emptyset$ is X with a disjoint basepoint.

3. Let $\mathbb{KU}$ be the complex K-theory spectrum as defined above. Recall that $\mathbb{KU}_q = \mathbb{Z} \times BU$ if q is even, and $\mathbb{KU}_q = U$ if q is odd. As mentioned above, Bott periodicity implies that $\mathbb{KU}$ is an ω-spectrum. The associated cohomology theory it represents is referred to as *complex* K-theory, denoted by $K^*(X)$. Notice that

$$K^0(X) \cong [X, \mathbb{Z} \times BU]$$

and as studied above, is the Grothendieck group completion of the abelian monoid $Vect^{\mathbb{C}}(X)$ of complex vector bundles over X. The periodic nature of this spectrum tells us that

$$K^q(X) \cong \begin{cases} K^0(X) \cong [X_+, \mathbb{Z} \times BU] & \text{if } q \text{ is even, and} \\ K^1(X) \cong [X_+, U] & \text{if } q \text{ is odd.} \end{cases}$$

Generalized (co)homology theories are required to satisfy all the Eilenberg-Steenrod axioms, except the Dimension Axiom. Recall that the Dimension Axiom says that the (co)homology of a point is zero except in dimension zero. For a generalized theory, this need not be the case. As we see above,

$$E_*(point) = \pi_*(point_+ \wedge \mathbb{E}) = \pi_*(S^0 \wedge \mathbb{E}) = \pi_*(\mathbb{E}),$$

and this group is often nonzero in many dimensions.

Exercise. Show that $\pi_q(\mathbb{E}) = 0$ for all $q \neq 0$ if and only if $\mathbb{E}$ is an Eilenberg-MacLane spectrum.

With the classification of generalized (co)homology theories by spectra, we can now understand the notions of (co)homology of a spectrum, as well as of a space.

Definition 10.10. *Let $\mathbb{E}$ be an ω-spectrum representing a generalized cohomology theory E^* and generalized homology theory E_*. Let $\mathbb{X}$ be any connective spectrum (i.e a spectrum with $\pi_q(\mathbb{X}) = 0$ for $q < 0$). We defined the generalized homology groups*

$$E_q(\mathbb{X}) = \pi_q(\mathbb{E} \wedge \mathbb{X}).$$

We similarly define the generalized cohomology groups by

$$E^q(\mathbb{X}) = [\mathbb{X}, \mathbb{E}]_q$$

where by this notation we mean weak homotopy classes of maps from $\mathbb{X}$ to $\Sigma^q\mathbb{E}$.

Exercise. Show that if X is a finite CW-complex and $\mathbb{E}$ is a spectrum representing (co)homology theories E_* and E^*, then

$$E_*(X) \cong E_*(\Sigma^\infty(X_+)) \quad \text{and} \quad E^*(X) \cong E^*(\Sigma^\infty(X_+)).$$

10.2.3 Application: The finiteness of the positive dimensional stable homotopy groups of spheres.

A famous theorem of Serre [84] is that the stable homotopy groups of spheres are finite in positive dimensions:

Theorem 10.19. *(Serre) [84]*

$$\varinjlim_{k} \pi_{q+k}(S^k) = \pi_q(\mathbb{S})$$

are finite abelian groups for $q > 0$.

As an application of the theory of spectra and generalized homology theories, we sketch a proof of Serre's theorem, modulo one result that Serre proved along the way.

Lemma 10.20. *(Serre)[84] $\pi_k(\mathbb{S})$ is a finitely generated abelian group for every k.*

Let $\mathbb{S}_{\mathbb{Q}}$ be the spectrum that represents the generalized homology theory, given by "rational stable homotopy", $(X, A) \to \pi_*^s(X, A) \otimes \mathbb{Q}$.

Exercise. Show that the homotopy groups of this representing spectrum are the rational stable homotopy groups of spheres.

$$\pi_s(\mathbb{S}_{\mathbb{Q}}) \cong \pi_*(\mathbb{S}) \otimes \mathbb{Q}.$$

We now observe that the homology of $\mathbb{S}_{\mathbb{Q}}$ is quite simple.

Lemma 10.21.

$$H_*(\mathbb{S}_{\mathbb{Q}}; \mathbb{Z}) = \begin{cases} \mathbb{Q}, & \textit{for } q = 0 \\ 0 & \textit{otherwise.} \end{cases}$$

Proof.

$$\begin{aligned} H_*(\mathbb{S}_{\mathbb{Q}}; \mathbb{Z}) &= \pi_*(\mathbb{S}_{\mathbb{Q}} \wedge H\mathbb{Z}) \\ &\cong (\mathbb{S}_{\mathbb{Q}})_*(H\mathbb{Z}) \\ &= \pi_*(H\mathbb{Z}) \otimes \mathbb{Q} \end{aligned}$$

So the result follows. □

Now consider the rational Hurewicz map, viewed as a map of generalized homology theories:

$$h : \pi_*(-) \otimes \mathbb{Q} \to H_*(-; \mathbb{Q}).$$

This is induced by a map of representing spectra,

$$h : \mathbb{S}_{\mathbb{Q}} \to H\mathbb{Q}.$$

The following is immediate from the lemma.

Proposition 10.22. *The map of spectra $h : \mathbb{S}_{\mathbb{Q}} \to H\mathbb{Q}$ is an equivalence. Therefore there is an isomorphism*

$$h_*; \pi_*(\mathbb{E}) \otimes \mathbb{Q} \xrightarrow{\cong} H_*(\mathbb{E}; \mathbb{Q})$$

for any spectrum $\mathbb{E}$.

Applying this proposition to the sphere spectrum $\mathbb{S}$, we have that

$$\pi_*(\mathbb{S}) \otimes \mathbb{Q} \cong H_*(\mathbb{S}; \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } q = 0 \\ 0 & \text{otherwise} \end{cases}$$

In particular this means that $\pi_q(\mathbb{S}) \otimes \mathbb{Q} = 0$ for $q > 0$. By Lemma 10.20, this implies that $\pi_q(\mathbb{S})$ is a finite abelian group for $q > 0$.

10.3 The Atiyah-Hirzebruch spectral sequence

The Atiyah - Hirzebruch spectral sequence provides one with a computational technique for computing the generalized (co)homology of a CW complex X in terms of its ordinary (co)homology and the homotopy groups of the spectrum representing the generalized theory. It is based on filtering X by its skeletal filtration.

10.3.1 The spectral sequence

Let X be a finite, n-dimensional based CW complex, with basepoint $x_0 \in X$. Let h^* be a generalized cohomology theory represented by a spectrum $\mathbb{E}$. Let $\tilde{h}^*$ be the reduced theory,

$$\tilde{h}^*(X) = h^*(X, x_0) = ker\,(h^*(X) \to h^*(x_0)).$$

Consider the skeletal filtration of X:

$$x_0 = X^{-1} \subset X^0 \subset X^1 \subset \cdots \subset X^n = X. \tag{10.5}$$

Let $F_m\tilde{h}^p(X) = ker\,(\tilde{h}^p(X) \to \tilde{h}^p(X^m))$. We then have a filtration on $\tilde{h}^p(X)$:

$$0 = F_n\tilde{h}^p(X) \subset F_{n-1}\tilde{h}^p(X) \subset \cdots \subset F_0\tilde{h}^p(X) \subset F_{-1}\tilde{h}^p(X) = \tilde{h}^p(X). \tag{10.6}$$

The Atiyah Hirzebruch spectral sequence (AHSS) will have as its E_1-term the homology of the subquotients of the skeletal filtration

$$E_1^{p,n-p} = h^n(X^p, X^{p-1}) \cong \tilde{h}^n(X^p, X^{p-1}).$$

It will converge to $\tilde{h}^*(X)$ in the sense that the E_∞-term is given by the subquotients of the above filtration,

$$E_\infty^{p,n-p} = F_p\tilde{h}^n(X)/F_{p-1}\tilde{h}^n(X)).$$

To understand the basic idea, first notice that the subquotient of the skeletal filtration X^p/X^{p-1} is a wedge of $(p-1)$-dimensional spheres, and these spheres form a basis of the cellular chain group $C_p(X)$. We call that basis β_p. That is,

$$X^p/X^{p-1} \simeq \bigvee_{\beta_p} S^p.$$

We therefore have

$$\begin{aligned} \tilde{h}^n(X^p/X^{p-1}) = \bigoplus_{\beta_p} \tilde{h}^n(S^p) &\cong \bigoplus_{\beta_p} \tilde{h}^{n-p}(S^0) \\ &\cong \bigoplus_{\beta_p} h^{n-p}(pt) = Hom(C_p(X), h^{n-p}(pt)) \\ &= C^p(X; h^{n-p}(pt)) \cong C^p(X; \pi_{n-p}(\mathbb{E})). \end{aligned} \tag{10.7}$$

These cochain groups form the E_1-term of the *Atiyah - Hirzebruch spectral sequence*. Here is the statement of the theorem asserting the existence of this spectral sequence.

Theorem 10.23. *(Atiyah and Hirzebruch [8]). Let X be a finite CW-complex and h^* a generalized cohomology theory represented by a spectrum $\mathbb{E}$. Then there is a spectral sequence converging to $\tilde{h}^*(X)$, satisfying the following properties:*

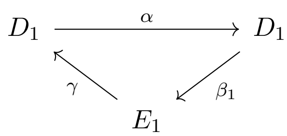
1. $E_1^{p,q} = C^p(X; h^q(pt)) = C^p(X; \pi_q(\mathbb{E}))$. *These are the cellular cochains of X.*
2. $E_2^{p,q} = \tilde{H}^p(X; h^q(pt))$.
3. $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$
4. $E_\infty^{p,q} = F_{p-1}\tilde{h}^{p+q}(X)/F_p\tilde{h}^{p+q}(X))$.

We remark that there is a similar spectral sequence converging to the generalized homology.

10.3.2 The spectral sequence of an exact couple and the construction of the AHSS

The construction of the Atiyah-Hirzebruch spectral sequence (AHSS) is explained well in Adam's well-known book [5]. Here we indicate its construction as an example of a spectral sequence arising from an *exact couple*. We begin by describing this general construction, and then show how it can be used to construct the AHSS with the properties described in Theorem 10.23.

Definition 10.11. *An exact couple is a triangle of abelian groups or chain complexes and homomorphisms between them that are exact.*

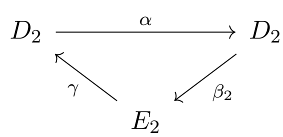


In other words, $ker\,\alpha = Image\,\gamma$, $ker\,\gamma = Image\,\beta_1$, and $ker\,\beta_1 = Image\,\alpha$.

One might think of E_1 as the first term of a spectral sequence. If one lets

$$d_1 = \beta_1 \circ \gamma : E_1 \to E_1$$

then one can define $E_2 = H_*(E_1, d_1)$, $D_2 = Image\,\alpha$, and $\beta_2 = \beta_1 \circ \alpha^{-1} : D_2 \to E_2$, and then one can check that



is an exact couple as well (called the *derived* exact couple). We then think of E_2 as the second term of the spectral sequence. Continuing in this fashion produces all the terms in a spectral sequence.

Applying the reduced generalized cohomology to the skeletal filtration of a finite CW complex $\tilde{h}^*(X)$ leads to an exact couple in the following way.

We let $D_1 = \bigoplus_{p,q} \tilde{h}^{p+q}(X^p)$ and $E_1 = \bigoplus_{p,q} \tilde{h}^{p+q}(X^p/X^{p-1}) \cong C^p(X; h^q(pt))$. For each p one has a long exact sequence in generalized cohomology,

$$\cdots \xrightarrow{\beta_1} h^{p+q}(X^p, X^{p-1}) \xrightarrow{\gamma} h^{p+q}(X^p) \xrightarrow{\alpha} h^{p+q}(X^{p-1}) \xrightarrow{\beta_1} h^{p+q+1}(X^p, X^{p-1}) \xrightarrow{\gamma} \cdots$$

One can easily check that this defines an exact couple, with the spaces and maps being the direct sum of all long exact sequences associated to the various skeletal pairs (X^p, X^{p-1}). The resulting spectral sequence is the *Atiyah-Hirzebruch spectral sequence.*

To compute the E_2-term of this spectral sequence we need to compute the homology $H_*(E_1, d_1)$ where

$$d_1 = \beta_1 \circ \gamma : \bigoplus_{p,q} \tilde{h}^{p+q}(X^p/X^{p-1}) \to \bigoplus_{p,q} h^{p+q+1}(X^{p+1}, X^p) \tag{10.8}$$
$$\bigoplus_{p,q} C^p(X; h^q(pt)) \to \bigoplus_{p,q} C^{p+1}(X; h^q(pt)).$$

Exercise. Prove that $d_1 : \bigoplus_{p,q} C^p(X; h^q(pt)) \to \bigoplus_{p,q} C^{p+1}(X; h^q(pt))$ as defined above is the coboundary map in the cellular cochain complex and therefore the E_2-term in the Atiyah-Hirzebruch spectral sequence is

$$E_2^{p,q} = \tilde{H}^p(X; h^q(pt)).$$

The remaining properties of the Atiyah-Hirzebruch spectral sequence as described in Theorem 10.23 are proved in a rather straightforward way. See [5] for a clear treatment. We now apply the AHSS to compute the K-theory of some important, familiar spaces.

10.3.3 Some K-theory calculations with the AHSS

In this subsection we will use the Atiyah-Hirzebruch spectral sequence to calculate the (complex) K-theory of certain important manifolds. We begin with the calculation of the K-theory of closed orientable surfaces.

Proposition 10.24. *Let Σ_g be a closed, orientable surface of genus g. Then*

$$K^0(\Sigma_g) \cong \mathbb{Z}^2 \quad \text{and} \quad K^1(\Sigma_g) \cong \mathbb{Z}^{2g}.$$

Note. By Bott periodicity this result determines the K-cohomology of Σ_g in all dimensions. Namely,

$$K^q(\Sigma_g) = \begin{cases} \mathbb{Z}^2 & \text{if } q \text{ is even} \\ \mathbb{Z}^{2g} & \text{if } q \text{ is odd.} \end{cases}$$

Proof. The cohomology of Σ_g is nonzero in only three dimensions: $H^0(\Sigma_g;\mathbb{Z}) \cong \mathbb{Z}$, $H^1(\Sigma_g;\mathbb{Z}) \cong \mathbb{Z}^{2g}$, and $H^2(\Sigma_g;\mathbb{Z}) \cong \mathbb{Z}$. Therefore the E_2-term of the Atiyah-Hirzebruch spectral sequence has only the following nonzero groups:

$$E_2^{0,2m} \cong \mathbb{Z}, \quad E_2^{1,2m} \cong \mathbb{Z}^{2g}, \quad E_2^{2,2m} \cong \mathbb{Z}$$

for each m. Again, all other groups in the E_2-term are zero. Since $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$, one immediately sees that all differentials must be zero. Therefore the spectral sequence "collapses", i.e $E_2^{p,q} = E_\infty^{p,q}$ and the result follows. □

Proposition 10.25. *The K-theory of $\mathbb{CP}^n$ is given by*

$$K^0(\mathbb{CP}^n) \cong \mathbb{Z}^{n+1} \quad K^1(\mathbb{CP}^n) = 0,$$

Proof. The E_2- term of the AHSS is given by

$$E_2^{p,q} = H^p(\mathbb{CP}^n; K^q(pt)) = \begin{cases} \mathbb{Z} & \text{if } p \text{ and } q \text{ are both even and } 0 \le p \le n \\ 0 & \text{otherwise.} \end{cases}$$

If we call the total degree of an element of $E_r^{p,q}$ $p+q$, then we see that the only nonzero terms in this spectral sequence have even total degree. Yet the differentials change the total degree of an element by $+1$. Therefore the differentials are all zero and the spectral sequence collapses at the E_2-term. Now

$$\bigoplus_{p+q=0} E_2^{p,q} = \mathbb{Z}^{n+1}$$

so the result follows. □

Exercises. .

1. What is $K^q(\mathbb{CP}^n)$ for all q? *Hint.* Use Bott periodicity.

2. Show that if X is any space with $H^q(X;\mathbb{Z}) = 0$ for q odd, then the AHSS collapses at the E_2-term, which is to say

$$\begin{aligned} E_\infty^{p,q} &= H^p(X; K^q(pt)) \\ &= \begin{cases} H^p(X;\mathbb{Z}) & \text{if } p \text{ and } q \text{ are both even} \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{10.9}$$

10.4 Symmetric spectra, ring spectra and module spectra

The graded abelian group $E_*(point) \cong \pi_*(\mathbb{E})$ is called the *coefficients* of the generalized homology theory E. Now motivated by structures in ordinary (co)homology, one might expect to find structures such as a evaluation map of a generalized cohomology theory on its corresponding generalized homology theory, taking values in the coefficients, or perhaps a cup product in the generalized cohomology. Notice that even in ordinary (Eilenberg-MacLane) (co)homology, $H^*(X;G)$ has these structures only if G is a ring. In a generalized theory we will need the representing spectrum $\mathbb{E}$ to be a "ring spectrum", which means there is a monoid structure

$$\mathbb{E} \wedge \mathbb{E} \to \mathbb{E}.$$

Of course we don't have a definition of the smash product of spectra yet. So far we only know how to take the smash product of a space with a spectrum. Defining an associative smash product has the effect of giving the category of spectra a "monoidal structure". For the purposes of defining structures at the level of generalized (co)homology such as cup product, having a ring structure "up to homotopy" suffices, and that was all that existed from the time of Whitehead's seminal paper [100] until the 1990's. Defining such a structure that is actually associative, instead of just associative up to homotopy, is quite a technical challenge. It was first accomplished by Hovey, Shipley, and Smith in [48]. Such a structure allows one to talk about “ring spectra”, “module spectra”, and roughly speaking, to do homological algebra in the category of spectra. In this section we describe a monoidal structure on the category of spectra, define the notion of a “ring spectrum”, and discuss several applications. In the next chapter this structure will prove quite useful in studying cobordisms of manifolds in the setting of a generalized (co)homology theories.

Our goal in this subsection is to show that a category of spectra exists which is in some sense equivalent to the one described above, and that has a monoidal structure defined by smash product of spectra. We begin with a definition of type of categorical monoidal structure we are looking for.

Definition 10.12. *A monoidal category is a category $\mathcal{C}$ equipped with a monoidal* structure. A monoidal structure consists of the following:

- a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called the tensor product or monoidal product,
- an object I called the unit object or identity object,
- three natural isomorphisms subject to certain coherence conditions expressing the fact that the tensor operation
  - is associative: there is a natural (in each of three arguments A, B, C) isomorphism α called the *associator*, with components

$$\alpha_{A,B,C} \colon A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$$

  - has I as left and right identity: there are two natural isomorphisms λ and ρ called the left and right unitor respectively, with components $\lambda_A : I \otimes A \cong A$ and $\rho_A : A \otimes I \cong A$.
- The coherence conditions for these natural transformations are:
  - for all A, B, C, and D in $\mathcal{C}$, the following diagram commutes

$$\begin{array}{ccccc} (A \otimes (B \otimes (C \otimes D)) & \xrightarrow{\alpha_{A,B,C\otimes D}} & (A \otimes B) \otimes (C \otimes D) & \xrightarrow{\alpha_{A\otimes B,C,D}} & ((A \otimes B) \otimes C) \otimes D \\ {\scriptstyle I_A \otimes \alpha_{B,C,D}} \downarrow & & & & \uparrow {\scriptstyle \alpha_{A,B,C} \otimes I_D} \\ A \otimes ((B \otimes C) \otimes D) & & \xrightarrow[\alpha_{A,B\otimes C,D}]{} & & (A \otimes (B \otimes C)) \otimes D \end{array}$$

  - for all A and B in $\mathcal{C}$, the following diagram commutes

$$\begin{array}{ccc} A \otimes (I \otimes B) & \xrightarrow{\alpha_{A,I,B}} & (A \otimes I) \otimes B \\ {\scriptstyle I_A \otimes \lambda_B} \downarrow & & \downarrow {\scriptstyle \rho_A \otimes I_B} \\ A \otimes B & \xrightarrow[=]{} & A \otimes B \end{array}$$

A **strict monoidal category** is one for which the natural isomorphisms α, λ, and ρ are identities. It turns out that every monoidal category is monoidally equivalent to a strict monoidal category.

Examples:

1. $Vect_k$, the category of finite dimensional vector spaces over a field k, with morphisms being k-linear transformations. The monoidal structure is tensor product (over k) of vector spaces. The unit is the one-dimensional vector space k.
2. $\mathcal{G}_{ab}$, category of Abelian groups and group homomorphisms. The monoidal structure is tensor product of abelian groups, and the unit is the group of integers $\mathbb{Z}$. More generally, the category $R - mod$ of modules over a commutative ring R is a monoidal category, with tensor product (over R) the monoidal structure, and with the unit being R itself.
3. Set, the category of finite sets and set maps. The monoidal structure is cartesian product, and the one-element set is the unit.
4. Cat, the category of small categories (i.e categories where the objects and morphisms both form sets) is a monoidal category, where the monoidal structure is the cartesian product of categories. The category with one object and whose only morphism is the identity morphism is the unit.

A *symmetric monoidal category* is a monoidal category where the monoidal structure $\otimes$ is commutative up to coherent isomorphism. Here is a strict definition.

Definition 10.13. *A symmetric monoidal category is a monoidal category $(\mathcal{C}, \otimes, I)$ such that, for every pair A, B of objects in $\mathcal{C}$, there is an isomorphism $s_{A,B} : A \otimes B \to B \otimes A$ that is natural in both A and B and such that the following coherence diagrams commute:*

- *The unit coherence:*

$$\begin{array}{ccc} A \otimes I & \xrightarrow{s_{A,I}} & I \otimes A \\ {\scriptstyle \rho_A}\downarrow & & \downarrow{\scriptstyle \lambda_A} \\ A & \xrightarrow[=]{} & A \end{array}$$

- *The associativity coherence:*

$$\begin{array}{ccc} (A \otimes B) \otimes C & \xrightarrow{s_{A,B} \otimes 1_C} & (B \otimes A) \otimes C \\ {\scriptstyle \alpha_{A,B,C}}\downarrow & & \downarrow{\scriptstyle \alpha_{B,A,C}} \\ A \otimes (B \otimes C) & & B \otimes (A \otimes C) \\ {\scriptstyle s_{A,B\otimes C}}\downarrow & & \downarrow{\scriptstyle 1_B \otimes s_{A,C}} \\ (B \otimes C) \otimes A & \xrightarrow[\alpha_{B,C,A}]{} & B \otimes (C \otimes A) \end{array}$$

- *The inverse law:*

$$\begin{array}{ccc} B \otimes A & \xrightarrow{=} & B \otimes A \\ {\scriptstyle s_{A,B}} \uparrow & & \downarrow {\scriptstyle s_{B,A}} \\ A \otimes B & \xrightarrow[1_{A\otimes B}]{} & A \otimes B \end{array}$$

Exercises.

1. Show that all of the categories in the above examples of monoidal categories in fact are symmetric monoidal.
2. Find an example of a monoidal category that is not symmetric monoidal.

Our goal in this subsection is to show that there is a category of spectra that has a symmetric monoidal structure, where the monoidal structure is a representation of smash product of spectra. Such symmetric monoidal categories of spectra were found in the late 1990's and early 2000's (see, for example [48], [31], [59]), and thankfully, they were al eventually shown to be equivalent in an appropriate sense. Here we describe the notion of *symmetric spectra* of [48]. Actually in [48] the authors work in the setting of simplicial sets, but here we work in the setting of topological spaces.

Definition 10.14. *. A symmetric spectrum $\mathbb{X}$ is a sequence of spaces $\{X_n, n \geq 0\}$ together with structure maps $\epsilon_n : \Sigma X_n \to X_{n+1}$, and actions of the symmetric groups $\Sigma_n \times X_n \to X_n$ so that if we think of S^p as the smash product*

$$S^p = S^1 \wedge \cdots \wedge S^1$$

with the action of Σ_p given by permutation of coordinates, then the composition

$$S^p \wedge X_q \xrightarrow{1\wedge\epsilon_q} S^{p-1} \wedge X_{1+q} \xrightarrow{1\wedge\epsilon_{1+q}} \cdots \xrightarrow{1\wedge\epsilon_{p-1+q}} S^1 \wedge X_{p-1+q} \xrightarrow{\epsilon_{p+q}} X_{p+q}$$

is $(\Sigma_p \times \Sigma_q)$-equivariant. Here $(\Sigma_p \times \Sigma_q)$ acts on X_{p+q} as it is naturally a subgroup of Σ_{p+q} consisting of those permutations of $p+q$ letters that fix the first p letters and the last q letters as sets.

A map (morphism) of symmetric spectra $f : \mathbb{X} \to \mathbb{Y}$ is a sequence of maps $f_n : X_n \to Y_n$ that is Σ_n equivariant, and which respect the structure maps. That is the following diagrams commute:

$$\begin{array}{ccc} \Sigma\mathbb{X}_n & \xrightarrow{\epsilon_n} & \mathbb{X}_{n+1} \\ {\scriptstyle \Sigma f_n} \downarrow & & \downarrow {\scriptstyle f_{n+1}} \\ \Sigma\mathbb{Y}_n & \xrightarrow[\epsilon_n]{} & \mathbb{Y}_{n+1} \end{array}$$

In order to complete the definition of the category of symmetric spectra, which we call $\mathbf{Sp}^{\boldsymbol{\Sigma}}$, we observe that the collection of morphisms between two symmetric spectra, $Map^{\mathbf{Sp}^{\mathbf{\Sigma}}}(\mathbb{X}, \mathbb{Y})$ is itself a symmetric spectrum. For this we define

$$Map^{\mathbf{Sp}^{\mathbf{\Sigma}}}(\mathbb{X}, \mathbb{Y})_n \subset \prod_i Map(\mathbb{X}_i, \mathbb{Y}_{i+n})$$

to be the subspace of all collections of Σ_i-equivariant maps $\{\phi_i : \mathbb{X}_i \to \mathbb{Y}_{i+n}\}$, as i varies, that respect the structure maps. That is, the following diagrams commute:

$$\begin{array}{ccc} \Sigma\mathbb{X}_i & \xrightarrow{\Sigma f_i} & \Sigma\mathbb{Y}_{i+n} \\ {\scriptstyle \epsilon_i}\downarrow & & \downarrow{\scriptstyle \epsilon_i} \\ \mathbb{X}_{i+1} & \xrightarrow[f_{i+1}]{} & \mathbb{Y}_{i+1+n} \end{array}$$

Notice that $Map^{\mathbf{Sp}^{\mathbf{\Sigma}}}(\mathbb{X}, \mathbb{Y})_0$ is just the space of all maps of symmetric spectra.

We leave it to the reader to check that the collection of spaces $\{Map^{\mathbf{Sp}^{\mathbf{\Sigma}}}(\mathbb{X}, \mathbb{Y})_n\}$ support natural symmetric group actions and structure maps to define a symmetric spectrum structure.

This definition defines a category of symmetric spectra that we call $\mathbf{Sp}^{\mathbf{\Sigma}}$.

Note: By replacing the symmetric groups Σ_n by the orthogonal groups $O(n)$ in the above definition one gets the notion of an *orthogonal spectrum*, and the category of such, $\mathbf{Sp}^{\mathbf{O}}$. Like symmetric spectra, orthogonal spectra have been very useful in homotopy theory. However for the purposes of these notes we emphasize symmetric spectra.

As mentioned above, one of the main reasons to consider symmetric or orthogonal spectra, is that the categories of such are symmetric monoidal, where the monoidal structure is an operation that defines smash product of such spectra. We now define the smash product of two symmetric spectra, $\mathbb{X}$ and $\mathbb{Y}$.

Definition 10.15. *The smash product of two symmetric spectra $\mathbb{X}$ and $\mathbb{Y}$ is the symmetric spectrum $\mathbb{X} \wedge \mathbb{Y}$ defined by*

$$(\mathbb{X} \wedge \mathbb{Y})_n = \bigvee_{p+q=n} \Sigma_{n_+} \wedge_{\Sigma_p \times \Sigma_q} (\mathbb{X}_p \wedge \mathbb{Y}_q)/\sim$$

where Σ_{n_+} denotes the symmetric group Σ_n with an additional disjoint basepoint, and the quotient relation identifies the images, for every r, of the two maps

$$\alpha : \Sigma_{(p+q+r)_+} \wedge (S^p \wedge \mathbb{X}_q \wedge \mathbb{Y}_r) \longrightarrow \Sigma_{(p+q+r)_+} \wedge_{\Sigma_q \times \Sigma_{p+r}} (\mathbb{X}_q \wedge \mathbb{Y}_{p+r})$$

and

$$\beta : \Sigma_{(p+q+r)_+} \wedge (S^p \wedge \mathbb{X}_q \wedge \mathbb{Y}_r) \longrightarrow \Sigma_{(p+q+r)_+} \wedge_{\Sigma_{p+q} \times \Sigma_r} (\mathbb{X}_{p+q} \wedge \mathbb{Y}_r)$$

where $\alpha(\sigma, t, x, y) = (\sigma \circ \tau_{q,p}, x, ty)$ *and* $\beta(\sigma, t, x, y) = (\sigma, tx, y)$. *Here* $(t, x) \to tx$ *is shorthand for the structure map*

$$S^p \wedge \mathbb{X}_q \to \mathbb{X}_{p+q}$$

and $\tau_{q,p} \in \Sigma_{p+q+r}$ *is the permutation that moves the first block of* q *letters past the second block of* p *letters and leaves the last block of* r *letters alone.*

The action of the symmetric group Σ_n *on* $(\mathbb{X} \wedge \mathbb{Y})_n$ *is induced by the action on the left hand coordinate of* $\Sigma_{n_+} \wedge_{\Sigma_p \times \Sigma_q} (\mathbb{X}_p \wedge \mathbb{Y}_q)$.

Exercise.
Define the structure maps $\epsilon_n : \Sigma(\mathbb{X} \wedge \mathbb{Y})_n \to (\mathbb{X} \wedge \mathbb{Y})_{n+1}$ and verify that $\mathbb{X} \wedge \mathbb{Y}$ is indeed a symmetric spectrum.

Note. The construction in this definition of identifying the images of the two maps α and β is known in category theory as the *"coequalizer"* of α and β.

The following, proved in [48] is not too difficult to prove, but is extremely important.

Theorem 10.26. *(Hovey, Shipley, and Smith [48], corollary 2.2.4) The smash product* $\mathbb{X} \wedge \mathbb{Y}$ *is a symmetric monoidal structure on the category of symmetric spectra,* $\mathbf{Sp}^{\mathbf{\Sigma}}$.

Exercise. . Verify that the unit in this symmetric monoidal structure is the sphere spectrum $\mathbb{S}$, where $\mathbb{S}_k$ is the k-fold smash product

$$\mathbb{S}_k = S^k = S^1 \wedge \cdots \wedge S^1$$

and the action of the symmetric group Σ_k is given by permuting the coordinates.

It is important to understand when a morphism of symmetric spectra $f : \mathbb{X} \to \mathbb{Y}$, is in an appropriate sense, an (homotopy) equivalence. The appropriate notion of equivalence is important because one would like to consider the associated "homotopy category", where one takes the same objects (symmetric spectra) and one "inverts" the equivalences. That is, in the homotopy category one formally adds inverse morphisms to every equivalence. This is a construction due to Quillen [80] and can be done whenever one has what is called a "model" structure on a category. A model category is one that has three distinguished types of morphisms, called "fibrations", "cofibrations", and "weak equivalences", satisfying several axioms. The associated homotopy category is defined by "localizing" with respect to the weak equivalences. This is a fascinating and important area of study, and there are several good texts on the subject. We refer the reader to [80], [47], and [66]. They are excellent references for this topic.

A model structure for the category of symmetric spectra, $\mathbf{Sp}^{\mathbf{\Sigma}}$, was described and studied in detail in [48]. It turns out that there is a subtlety when defining the notion of (stable) equivalence in the category $\mathbf{Sp^\Sigma}$. Following the notion of weak equivalence of spaces or of (ordinary) spectra, one might be inclined to declare that a morphism of symmetric spectra $f : \mathbb{X} \to \mathbb{Y}$ is a stable equivalence if it is a weak homotopy equivalence as a map of ordinary spectra; that is, if it induces an isomorphism of homotopy groups. However as is pointed out clearly in [48], this will not work. Namely our goal is to find a good notion of stable equivalence of symmetric spectra that has the property that when one takes the associated homotopy category, one obtains a category equivalent to taking the homotopy category of the category of ordinary spectra. So in particular, when one wants to do calculations depending only on the homotopy type of spectra and maps between them, it would not matter if one was using symmetric spectra or ordinary spectra. The above naive notion of weak equivalence simply won't satisfy this property, as pointed out in [48]. Essentially, the way the authors of [48] found to deal with this issue was to declare that a map of symmetric spectra $f : \mathbb{X} \to \mathbb{Y}$ is a stable equivalence if the induced map E^*f of cohomology groups is an isomorphism for every generalized cohomology theory E^*. We refer the reader to [48] for details of these issues. The main upshot for our purposes is that there is now a symmetric monoidal category of spectra, with unit the sphere spectrum, whose associated homotopy theory is equivalent to what one would expect from the naive notions of spectra that go all the way back to Lima and Whitehead in the 1950's and early 1960's.

With the existence of a symmetric monoidal structure, one can begin doing "algebra" in our category of spectra. For example, a ring spectrum (with unit) $\mathbb{X}$ is one that is equipped with a pairing

$$\mu : \mathbb{X} \wedge \mathbb{X} \to \mathbb{X}$$

together with a unit maps $\eta : \mathbb{S} \to \mathbb{X}$ that satisfy the usual associativity conditions. In other words, a "ring spectrum" is a monoid in the category of spectra. For example, the sphere spectrum $\mathbb{S}$ has the pairing given by the equality

$$\mathbb{S} \wedge \mathbb{S} = \mathbb{S}$$

which makes $\mathbb{S}$ a commutative ring spectrum. Another important class of examples comes from the suspension spectrum of a group (with a disjoint basepoint), $\Sigma^\infty(G_+)$. This is because

$$\Sigma^\infty(G_+) \wedge \Sigma^\infty(G_+) = \Sigma^\infty(G \times G_+).$$

The group multiplication defines the ring structure.

If X is a space with a group action $G \times X \to X$, then its suspension spectrum $\Sigma^\infty(X_+)$ becomes a *module spectrum* over the ring spectrum $\Sigma^\infty(G_+)$. Here a (right) module spectrum $\mathcal{M}$ over a ring spectrum $\mathbb{X}$ is a symmetric spectrum that is equipped with a pairing map

$$\mathcal{M} \wedge \mathbb{X} \to \mathcal{M}$$

that satisfies the usual associativity conditions. As one can see, once one has the structure of a symmetric monoidal category, one can start doing algebra in the category!

Now observe that one does not really need a group structure on G for $\Sigma^\infty(G_+)$ to have a ring structure. Indeed G just needs to be a monoid. An important class of such examples comes from the based loop space ΩX where X is any based space. In order that ΩX be a (strict) monoid, we take ΩX to refer to the space of "Moore loops". This is the space of pairs (r, α), where $r \geq 0$ and $\alpha : [0, r] \to X$ is a map that sends the endpoints 0 and r to the basepoint $x_0 \in X$. The multiplication in ΩX is given by juxtaposition:

$$(r, \alpha) \cdot (s, \beta) = (r + s, \alpha \cdot \beta) : [0, r + s] \to X$$

where

$$\alpha \cdot \beta(t) = \begin{cases} \alpha(t) & \text{if} \quad 0 \leq t \leq r \\ \beta(t - r) & \text{if} \quad r \leq t \leq r + s \end{cases}$$

The study of the spectrum $\Sigma^\infty(\Omega X_+)$ as a ring spectrum was initiated by Waldhausen [95]. It was shown how the study of the category of modules over $\Sigma^\infty(\Omega X_+)$ leads to an understanding of various automorphism groups of X if X is a manifold (eg diffeomorphism groups, homeomorphism groups, PL homeomorphism groups, etc.). It has lead to the study of what is now known as "Waldhausen K-theory" which has been a major area of research in algebraic and differential topology since the 1970's. The reader is referred to [95], [96] to learn more.

The Eilenberg MacLane spectrum $\mathbb{H}R$ where R is any ring, is also a ring spectrum. The ring structure is induced up to homotopy by the pairings

$$K(R, q) \times K(R, s) \to K(R, q + s)$$

which represents the cohomology class given by the cross product $\iota_q \times \iota_s \in H^{q+s}(K(R, q) \times K(R, s); R)$ given by the cross product of the fundamental classes $\iota_q \in H^q(K(R, q); R)$ and $\iota_s \in H^s(K(R, s); R)$.

In a similar fashion, if P is a right module over a ring R, $\mathbb{H}P$ has the structure of a right module spectrum over the ring spectrum $\mathbb{H}R$.

In these notes we will not further pursue the homological algebra that is possible in the category of spectra. But understanding this structure is a very active area of research and it has had many applcations.

From here on out, when we refer to "spectra", we will mean symmetric spectra, and we will most often leave out the reference to the category $\mathbf{Sp}^{\mathbf{\Sigma}}$. So for example when we write $Map(\mathbb{X}, \mathbb{Y})$ we will mean the mapping spectrum $Map^{\mathbf{Sp}^{\mathbf{\Sigma}}}(\mathbb{X}, \mathbb{Y})$.

10.5 Generalized cup and cap products

In this section we will study product structures on generalized cohomology theories represented by ring spectra. Our goal will be to apply them to the study of generalized orientations and duality structures on manifolds.

Let R be a ring, then a basic construction in algebraic topology is the cup product in the cohomology $H^*(X; R)$. If R is a commutative ring, then this product inherits a graded-commutative structure. Recall that the ingredients involved in this construction are the diagonal map

$$\Delta : X \to X \times X$$

and the ring multiplication $\mu : R \times R \to R$. More specifically the cup product is defined by

$$\cup : H^q(X; R) \times H^s(X; R) \xrightarrow{\times} H^{q+s}(X \times X; R) \xrightarrow{\Delta^*} H^{q+s}(X; R)$$

where the first map in this composition is the "cross product". This cross product is induced on the level of the representing Elienberg-MacLane spaces via the map $K(R, q) \times K(R, s) \to K(R, q + s)$ which represents the cross product class $\iota_q \times \iota_s \in H^{q+s}(K(R, q) \times K(R, s); R)$. Furthermore these classes define (up to homotopy) the ring spectrum structure on the representing Eilenberg-MacLane spectrum $\mathbb{H}R$.

This suggests that whenever we have a generalized cohomology theory E^* represented by a ring spectrum $\mathbb{E}$, then one can use that ring structure to define a "cross product" map, and a "cup product" structure in the generalized cohomology theory. This is indeed the case.

Definition 10.16. *Let E^* be a generalized cohomology theory represented by a ring spectrum $\mathbb{E}$. Let X and Y be spaces, and consider generalized cohomology classes $\alpha \in E^q(X)$, and $\beta \in E^s(Y)$. Let*

$$\phi_\alpha : \Sigma^\infty(X_+) \to \Sigma^q\mathbb{E} \quad \text{and} \quad \phi_\beta : \Sigma^\infty(Y_+) \to \Sigma^s\mathbb{E}$$

be maps of spectra representing α and β respectively. The "cross product" $\alpha \times \beta \in E^{q+s}(X \times Y)$ is defined to be the cohomology class represented by the composition

$$\begin{aligned}\phi_{\alpha\times\beta} : \Sigma^\infty((X \times Y)_+) = \Sigma^\infty(X_+) \wedge \Sigma^\infty(Y_+) &\xrightarrow{\phi_\alpha \times \phi_\beta} \Sigma^q\mathbb{E} \wedge \Sigma^s\mathbb{E} \\ = \Sigma^{q+s}(\mathbb{E} \wedge \mathbb{E}) &\xrightarrow{\mu} \Sigma^{q+s}\mathbb{E}\end{aligned}$$

Here $\mu : \mathbb{E} \wedge \mathbb{E} \to \mathbb{E}$ is the ring multiplication.

Definition 10.17. *Let E^* be a generalized cohomology theory represented by a ring spectrum $\mathbb{E}$. Let X be a space and $\alpha \in E^q(X)$ and $\beta \in E^s(X)$ generalized cohomology classes. The cup product $\alpha \cup \beta \in E^{q+s}(X)$ is defined to be the class represented by the composition*

$$\phi_{\alpha\cup\beta} : \Sigma^\infty(X_+) \xrightarrow{\Delta} \Sigma^\infty((X \times X)_+) \xrightarrow{\phi_{\alpha\times\beta}} \Sigma^{q+s}\mathbb{E}$$

where $\phi_{\alpha\times\beta}$ is the map representing the cross product as above.

Exercise. Verify that with the above definition, the generalized cohomology $E^*(X)$ has the structure of a graded ring. If $\mathbb{E}$ is a commutative ring spectrum, then verify that this ring structure on $E^*(X)$ is graded commutative, like it is for ordinary cohomology with coefficients in a commutative ring.

Notice that a key ingredient in the construction of these generalized cup products is the diagonal map $\Delta : X \to X \times X$ which induces a map on the level of spectra $\Delta : \Sigma^\infty(X_+) \to \Sigma^\infty(X_+) \wedge \Sigma^\infty(X_+)$. This map is called a coproduct, and this structure is often referred to as a *"coalgebra"* structure on the suspension spectrum $\Sigma^\infty(X_+)$. In general not all connective spectra $\mathbb{X}$ have this structure, and so their generalized cohomologies, $E^*(\mathbb{X})$ do not have cup products.

One might form a more general, "twisted" form of this construction in the following way. Suppose $p : \zeta \to X$ is a vector bundle over a finite CW complex. The diagonal map on $\Delta : X \to X \times X$ defines maps on the level of vector bundles

$$\begin{array}{ccc} \zeta & \xrightarrow{\Delta_R^\zeta} & \zeta \times X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p\times 1} \\ X & \xrightarrow[\Delta]{} & X \times X, \end{array}$$

and similarly $\Delta_L^\zeta : \zeta \to X \times \zeta$.

Notice that the Thom space of $\zeta \times X$ is given by

$$T(\zeta \times X) = T\zeta \wedge X_+$$

and similarly $T(X \times \zeta) = X_+ \wedge T\zeta$.

The diagonal maps then induce maps on the Thom spaces for which, by abuse of notation, we use the same notation,

$$\Delta_R^\zeta : T\zeta \to T\zeta \wedge X_+ \quad \text{and} \quad \Delta_L^\zeta : T\zeta \to X_+ \wedge T\zeta.$$

Now let $\mathbb{E}$ be a commutative ring spectrum representing the generalized cohomology theory E^*.

Exercises.

1. Define a cross product map

$$E^q(\Sigma^\infty(T\zeta)) \times E^s(X) \to E^{q+s}(\Sigma^\infty(T\zeta \wedge X_+)).$$

2. Using the map of Thom spaces Δ^ζ_R, show that there is an induced pairing

$$E^q(\Sigma^\infty(T\zeta)) \times E^s(X) \xrightarrow{\mu} E^{q+s}(\Sigma^\infty(T\zeta))$$

that gives $E^*(\Sigma^\infty(T\zeta))$ the structure of right module over the graded ring $E^*(X)$. Similarly $E^*(\Sigma^\infty(T\zeta))$ is a left module over $E^*(X)$ using the map Δ^ζ_L.

3. Show that if ζ is the trivial zero dimensional bundle over X, i.e $\zeta = X \xrightarrow{p=id} X$, then $T\zeta = X_+$ and the above module structures are the cup product structure in $E^*(X)$.

There are other constructions from ordinary cohomology theory that also have analogues in generalized cohomology. The evaluation map of cohomology on homology, and more generally, the cap product maps are important examples.

As above let $\mathbb{E}$ be a ring spectrum representing the generalized cohomology theory E^* and the generalized homology theory E_*. Let X be a space (of the homotopy type of a CW-complex) and consider classes $\theta \in E_q(X) = \pi_q(X_+ \wedge \mathbb{E})$ and $\alpha \in E^q(X) = [X, \Sigma^q\mathbb{E}]$, which we take to mean weak homotopy classes of maps.

Definition 10.18. *Let $\psi_\theta : S^q \to X_+ \wedge \mathbb{E}$ and $\phi_\alpha : \Sigma^\infty(X_+) \to \Sigma^q\mathbb{E}$ be maps representing the classes $\theta \in E_q(X)$ and $\alpha \in E^q(X)$, respectively, The evaluation class $\langle \alpha, \theta \rangle \in \pi_0\mathbb{E}$ is defined to be the class represented by the composition*

$$S^q \xrightarrow{\psi_\theta} X_+ \wedge \mathbb{E} \xrightarrow{\phi_\alpha \wedge 1} \Sigma^q\mathbb{E} \wedge \mathbb{E}. \xrightarrow{\mu} \Sigma^q\mathbb{E}$$

where $\mu : \mathbb{E} \wedge \mathbb{E} \to \mathbb{E}$ is the ring multiplication.

Exercise. Show that the evaluation pairing defines a ring homomorphism

$$E^*(X) \to Hom(E_*(X), \pi_0(\mathbb{E})).$$

Like in ordinary (co)homology theory, this evaluation pairing extends to define a "cap product" in generalized cohomology theories represented by ring spectra.

Definition 10.19. *Let $\mathbb{E}$ and X be as above. Suppose $\theta \in E_q(X)$ is represented by $\psi_\theta : S^q \to X_+ \wedge \mathbb{E}$, and $\beta \in E^r(X)$ is represented by $\phi_\beta : \Sigma^\infty(X_+) \to \Sigma^r\mathbb{E}$. We define the cap product $\theta \cap \beta \in E_{q-r}(X) = \pi_{q-r}(X_+ \wedge \mathbb{E})$ to be the class represented by the composition*

$$\psi_{\theta\cap\beta} : S^q \xrightarrow{\psi_\theta} X_+ \wedge \mathbb{E} \xrightarrow{\Delta\wedge 1} X_+ \wedge X_+ \wedge \mathbb{E} \xrightarrow{1\wedge\phi_\beta\wedge 1} X_+ \wedge \Sigma^r\mathbb{E} \wedge \mathbb{E} \xrightarrow{1\wedge\mu} X_+ \wedge \Sigma^r\mathbb{E}.$$

Note. This cap product pairing is a map $\cap : E_q(X) \times E^r(X) \to E_{q-r}(X)$. We leave it to the reader that this definition also applies to give slightly more general pairings (compare 1.6)

$$E_q(X,A) \times E^r(X) \xrightarrow{\cap} E_{q-r}(X,A) \quad \text{and} \quad E_q(X,A) \times E^r(X,A) \xrightarrow{\cap} E_{q-r}(X).$$

10.6 Thom spectra

Our next goal is to apply generalized cohomology theory to the study of manifolds, and in particular to prove a generalized form of Poincaré duality with respect to a generalized cohomology theory E^*. As we recall, the notion of orientations played a crucial role in Poincaré for usual (co)homology. So we need to study the notion of orientations with respect to generalized cohomology theories. For this we will begin by generalizing the notion of Thom spaces, to *"Thom spectra"*.

Let $\zeta \to X$ be a k-dimensional vector bundle over a finite CW-complex. As before, let $T\zeta$ be it's Thom space. Let $\epsilon^m \to X$ be an m-dimensional trivial bundle, $\epsilon^m = X \times \mathbb{R}^m$. Observe, as we have earlier, that

$$T\epsilon^m = (X \times D^k)/(X \times S^{k-1}) \cong \Sigma^m(X_+).$$

Consider the Whitney sum bundle $\zeta \oplus \epsilon^m \to X$.

Exercise. Prove that there is a natural homeomorphism,

$$T(\zeta \oplus \epsilon^m) \cong \Sigma^m T\zeta.$$

Given the result of this exercise we can think of the suspension spectrum $\Sigma^\infty(T\zeta)$ as having its m^{th}-space equal to $T(\zeta \oplus \epsilon^m) = T(\zeta \times \mathbb{R}^m)$.

Exercise. Show that the natural symmetric spectrum structure on the suspension spectrum can be described in terms of the symmetric groups Σ_m acting on $\mathbb{R}^m$ by permuting the coordinates.

The above observation tells us that we have a natural equivalence of spectra,

$$\Sigma^m \Sigma^\infty(T\zeta) \simeq \Sigma^\infty(T(\zeta \oplus \epsilon^m)).$$

Suppose the k-dimensional bundle ζ is classified by a map

$$f_\zeta : X \to BO(k).$$

Then the above observations say that $\Sigma^m \Sigma^\infty(T\zeta)$ is the suspension spectrum of the Thom space of the $(k+m)$-dimensional vector bundle $\zeta \oplus \epsilon^m$ which is represented by the composition

$$f_{\zeta \oplus \epsilon^m} : X \xrightarrow{f_\zeta} BO(k) \to BO(k+m).$$

By allowing m to be negative in the above discussion, we are motivated to define the following.

Definition 10.20. *As above, let $\zeta \to X$ be a k dimensional vector bundle over a finite CW-complex X, classified by a map*

$$f_\zeta : X \to BO(k).$$

The Thom spectrum, which we denote using the exponential notation X^ζ, is defined to be the k-fold desuspension of the suspension spectrum of the Thom space,

$$X^\zeta = \Sigma^{-k}\Sigma^\infty(T\zeta).$$

We say that X^ζ is the Thom spectrum of the virtual zero-dimensional bundle $\zeta - \epsilon^k$ classified by the map

$$\phi_\zeta : X \to BO$$

defined to be the composition $\phi_\zeta : X \xrightarrow{f_\zeta} BO(k) \to BO$.

Let us consider Thom spectra from a different perspective. Suppose X is a finite CW-complex, and we are given a map

$$f : X \to BO.$$

The question we would like to now address is the following:

Question. . Can we define a Thom spectrum X^f associated to the map $f : X \to BO$?

Notice that if we were given a factorization of f through a finite $BO(k)$, i.e a map $f_k : X \to BO(k)$ such that the composition $X \xrightarrow{f_k} BO(k) \to BO$ is homotopic to f, then, of course, f_k classifies a k-dimensional vector bundle ζ_{f_k}, and we can define the Thom spectrum X^f to be the Thom spectrum of this factorization,

$$X^f = X^{\zeta_{f_k}} = \Sigma^{-k}\Sigma^\infty(T(\zeta_{f_k})). \tag{10.10}$$

But is this spectrum, or at least its homotopy type, independent of the choice of factorization?

To address this, suppose $\tilde{f}_q : X \to BO(q)$ is another factorization of f. (The integer q may or not be the same as k.)

Proposition 10.27. *The spectra* $X^{\zeta_{f_k}} = \Sigma^{-k}\Sigma^\infty(T(\zeta_{f_k}))$ *and* $X^{\zeta_{\tilde{f}_q}} = \Sigma^{-q}\Sigma^\infty(T(\zeta_{\tilde{f}_q}))$ *are (weakly) homotopy equivalent.*

Proof. Since the two compositions

$$X \xrightarrow{f_k} BO(k) \to BO \quad \text{and} \quad X \xrightarrow{\tilde{f}_q} BO(q) \to BO$$

are both homotopic to $f : X \to BO$, they are therefore homotopic to each other. Let $X^{(m)}$ be any finite subcomplex of X. Then there must be a finite N larger than both k and q such that the compositions

$$f_k^N : X^{(m)} \xrightarrow{f_k} BO(k) \to BO(N) \quad \text{and} \quad \tilde{f}_q^N : X^{(m)} \xrightarrow{\tilde{f}_q} BO(q) \to BO(N)$$

are homotopic. Therefore they classify isomorphic N-dimensional vector bundles over $X^{(m)}$, and so have homotopy equivalent Thom spaces. Notice that the bundle classified by f_k^N is equal to $\zeta_{f_k} \oplus \epsilon^{N-k}$, whereas the bundle classified by $\tilde{f}_q^N$ is equal to $\zeta_{\tilde{f}_q} \oplus \epsilon^{N-q}$. We therefore have a bundle isomorphism over $X^{(m)}$

$$\zeta_{f_k} \oplus \epsilon^{N-k} \cong \zeta_{\tilde{f}_q} \oplus \epsilon^{N-q}.$$

On the level of Thom spaces we have a homotopy equivalence

$$\begin{aligned} T(\zeta_{f_k} \oplus \epsilon^{N-k}) &\simeq T(\zeta_{\tilde{f}_q} \oplus \epsilon^{N-q}) \\ \Sigma^{N-k}T(\zeta_{f_k}) &\simeq \Sigma^{N-q}T(\zeta_{\tilde{f}_q}). \end{aligned}$$

We therefore have an equivalence of spectra,

$$\begin{aligned} (X^{(m)})^{\zeta_{f_k}} &= \Sigma^{-k}\Sigma^\infty(T(\zeta_{f_k})) = \Sigma^{-N}\Sigma^{N-k}\Sigma^\infty(T(\zeta_{f_k})) = \Sigma^{-N}\Sigma^\infty\Sigma^{N-k}T(\zeta_{f_k})) \\ &\simeq \Sigma^{-N}\Sigma^\infty\Sigma^{N-q}T(\zeta_{\tilde{f}_q}) \simeq \Sigma^{-N}\Sigma^{N-q}\Sigma^\infty T(\zeta_{\tilde{f}_q}) \\ &= \Sigma^{-q}\Sigma^\infty T(\zeta_{\tilde{f}_q}) = (X^{(m)})^{\zeta_{\tilde{f}_q}} \end{aligned}$$

□

Since this equivalence is true for any subcomplex $X^{(m)}$ of X, we can conclude that $X^{\zeta_{f_k}}$ and $X^{\zeta_{\tilde{f}_q}}$ have the same weak homotopy type.

The following exercise is proved in a similar manner.

Exercise. Let $f : X \to BO$ and $g : Y \to BO$ be maps where X and Y are finite CW complexes. Suppose there is a map $\phi : X \to Y$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\phi} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ BO & \xrightarrow[=]{} & BO \end{array}$$

Show that there is an induced map on the level of Thom spectra

$$T\phi : X^f \to Y^g$$

that is well defined up to weak homotopy.

At this point we have associated to every map $f : X \to BO$, where X is a finite CW-complex, a Thom spectrum X^f, which is well-defined up to homotopy. And to every "map over BO", that is a map $\phi : X \to Y$ respecting maps $f : X \to BO$ and $g : Y \to BO$ as in the exercise, we have associated a map of Thom spectra, $T\phi : X^f \to X^g$, which again, is well-defined up to homotopy. This suggests that there might be a functoriality result where we can remove the "up-to-homotopy" restriction. Indeed there is such a result, and it is fairly recent. In order to describe it, we first consider Thom spectra for maps $f : X \to BO$ where X is a CW-complex that is not necessarily finite. Consider the skeletal filtration of X:

$$X^0 \hookrightarrow X^{(1)} \hookrightarrow \cdots \hookrightarrow X^{(k-1)} \hookrightarrow X^{(k)} \hookrightarrow \cdots \hookrightarrow X.$$

We first observe the following:

Theorem 10.28. *Any map $f : X \to BO$ is homotopic to one which takes the k^{th}-skeleton $X^{(k)}$ to $BO(k)$. That is, there is a commutative diagram*

$$\begin{array}{ccccccccccc}
X^{(0)} & \xrightarrow{\hookrightarrow} & \cdots & \xrightarrow{\hookrightarrow} & X^{(k)} & \xrightarrow{\hookrightarrow} & X^{(k+1)} & \xrightarrow{\hookrightarrow} & \cdots & \xrightarrow{\hookrightarrow} & X \\
{\scriptstyle f^{(0)}}\downarrow & & & & {\scriptstyle f^{(k)}}\downarrow & & \downarrow{\scriptstyle f^{(k+1)}} & & & & \downarrow{\scriptstyle f} \\
BO(0) & \xrightarrow[\hookrightarrow]{} & \cdots & \xrightarrow[\hookrightarrow]{} & BO(k) & \xrightarrow[\hookrightarrow]{} & BO(k+1) & \xrightarrow[\hookrightarrow]{} & \cdots & \xrightarrow[\hookrightarrow]{} & BO
\end{array}$$

Proof. This follows from obstruction theory and in particular Theorem 7.11 and Proposition 7.14.

□

Consider a skeletal filtration preserving map $f : X \to BO$ as in the statement of this theorem. Let $\zeta^{(k)} \to X^{(k)}$ be the k-dimensional vector bundle classified by $f^{(k)} : X^{(k)} \to BO(k)$. Consider the composition

$$X^{(k)} \xrightarrow{f^{(k)}} BO(k) \hookrightarrow BO(k+1).$$

This classifies the $(k+1)$-dimensional bundle $\zeta^{(k)} \oplus \epsilon^1$ over $X^{(k)}$. The Thom space of this bundle is the suspension

$$T(\zeta^{(k)} \oplus \epsilon^1) = \Sigma T(\zeta^{(k)}).$$

Now by the commutativity of the diagram in this theorem we have maps of vector bundles,

$$\begin{array}{ccc} \zeta^{(k)} \oplus \epsilon^1 & \longrightarrow & \zeta^{(k+1)} \\ \downarrow & & \downarrow \\ X^{(k)} & \xrightarrow[\hookrightarrow]{} & X^{(k+1)} \end{array}$$

and hence we have a map of Thom spaces that we call

$$\epsilon_k : \Sigma T(\zeta^{(k)}) \to T(\zeta^{(k+1)}).$$

We can then make the following definition:

Definition 10.21. *. Given the above situation we define the spectrum*

$$X^f = \{T(\zeta^{(k)}); \quad \epsilon_k : \Sigma T(\zeta^{(k)}) \to T(\zeta^{(k+1)})\}.$$

Exercises.

1. Show that the weak homotopy type of X^f is well defined. That is, it does not depend on the choices of homotoping $f : X \to BO$ into a skeletal filtration preserving map, as in the statement of Theorem 10.28.

2. Give the Thom spectrum X^f the structure of a symmetric spectrum.

3. Suppose X is a finite CW-complex and $f : X \to BO$ is given by a factorization $X \xrightarrow{f_k} BO(k) \to BO$, and that the map f_k classifies the k-dimensional vector bundle

$$\zeta^k \to X.$$

Show that the definition of the Thom spectrum given above (10.10)

$$X^f = \Sigma^{-k}\Sigma^{\infty}(T(\zeta^k))$$

agrees, up to homotopy, with Definition 10.21.

4. Suppose $f : X \to BO$ and $g : Y \to BO$ are maps from (not-necessarily-finite) CW complexes. Suppose furthermore that

$$\phi : X \to Y$$

is a map making the following diagram commute:

$$\begin{array}{ccc} X & \xrightarrow{\phi} & Y \\ f\downarrow & & \downarrow g \\ BO & \xrightarrow[=]{} & BO. \end{array}$$

Define an induced map of Thom spectra $T\phi : X^f \to Y^g$ that extends the definition given in the previous exercise set when X and Y are assumed to be finite.

Examples.

1. Consider the identity map $id : BO \to BO$. Then the construction above defines a Thom spectrum BO^{id} for which we use the standard notation bmo. If $MO(n)$ denotes the Thom space of the universal bundle over $BO(n)$, then as a spectrum $\mathbb{MO}$ is made up of the spaces $MO(n)$ together with the structure maps $\epsilon_n : \Sigma MO(n) \to MO(n+1)$, defined as in Definition 10.21.

2. The inclusion of the unitary group into the orthogonal group $U(n) \hookrightarrow O(2n)$ defines a map on classifying spaces $\gamma_n : BU(n) \to BO(2n)$. On the level of bundles it takes an n-dimensional complex vector bundle, forgets its complex structure and views it as a $2n$-dimensional real vector bundle. The maps γ_n fit together to define a map

$$\gamma : BU \to BO.$$

   It has a corresponding Thom spectrum which we denote by $\mathbb{MU}$.

The Thom spectra $\mathbb{MO}$ and $\mathbb{MU}$ were originally introduced by R. Thom in [94] and play an essential role in cobordism theory which we will see in the next chapter.

We now consider certain multiplicative properties of Thom spectra. First suppose that $\zeta^k \to X$ and $\xi^q \to Y$ are k and q dimensional vector bundles, respectively, where the base spaces are CW complexes. Let $f_\zeta : X \to BO(k)$ and $g_\xi : Y \to BO(q)$ be classifying maps for these bundles. One can consider the external product bundle

$$\zeta^k \times \xi^q \to X \times Y.$$

As we've observed before, this $(k+q)$-dimensional vector bundle is classified by the composition map

$$X \times Y \xrightarrow{f_\zeta \times g_\xi} BO(k) \times BO(q) \xrightarrow{\mu_{k,q}} BO(k+q)$$

where $\mu_{k,q} : BO(k) \times BO(q) \to BO(k+q)$ is the "Whitney sum" pairing induced by the "block addition" homomorphism

$$O(k) \times O(q) \to O(k+q)$$

given by sending a $k \times k$ matrix A and a $q \times q$ matrix B to the $(k+q) \times (k+q)$ matrix that has A in the upper left $k \times k$ block, B in the lower right $q \times q$ block, and zero's elsewhere.

The following is simply a parameterized form of the fact that

$$(D^k \times D^q)/\partial(D^k \times D^q) \cong D^k/\partial D^k \wedge D^q/\partial D^q.$$

We leave its proof to the reader.

Proposition 10.29. *The Thom space of $\zeta^k \times \xi^q$ is given by*

$$T(\zeta^k \times \xi^q) \cong T(\zeta^k) \wedge T(\xi^q).$$

The pairing maps $\mu_{k,q} : BO(k) \times BO(q) \to BO(k+q)$ fit together to give a pairing

$$\mu : BO \times BO \to BO \tag{10.11}$$

This can be verified directly, which we encourage the reader to do. This pairing also follows as a consequence of (real) Bott periodicity. Recall that (complex) Bott periodicity says that

$$\mathbb{Z} \times BU \simeq \Omega U \quad \text{and, of course} \quad U \simeq \Omega BU = \Omega(\mathbb{Z} \times BU)$$

which implies the two-fold periodicity

$$\mathbb{Z} \times BU \simeq \Omega^2(\mathbb{Z} \times BU).$$

In the case of BO, in [11] Bott proved that there is an eight-fold periodicity

$$\mathbb{Z} \times BO \simeq \Omega^8(\mathbb{Z} \times BO). \tag{10.12}$$

Indeed Bott showed that

$$\mathbb{Z} \times BO \simeq \Omega(U/O) \tag{10.13}$$

where $U/O = \varinjlim_n U(n)/O(n)$. Here $O(n) \subset U(n)$ is the subspace of all unitary matrices with the property that all of their entries are real (i.e have zero imaginary parts).

Using Moore loops, the loop space $\Omega(U/O)$ has the structure of an associative monoid. This gives a homotopy theoretic model of $\mathbb{Z} \times BO$ with the structure of an associative monoid. We call this product

$$\mu : (\mathbb{Z} \times BO) \times (\mathbb{Z} \times BO) \to \mathbb{Z} \times BO.$$

It restricts on components to give pairings

$$\mu_{m,n} : (\{m\} \times BO) \times (\{n\} \times BO) \to \{m+n\} \times BO.$$

In particular it defines a monoid structure on $BO = \{0\} \times BO \hookrightarrow \mathbb{Z} \times BO$. This monoid structure corresponds, up to homotopy, with the Whitney sum pairings $\mu_{k,q} : BO(k) \times BO(q) \to BO(k+q)$ described above, and hence the (abuse of) notation. BU has a similar monoid structure. We refer the reader to [63] for a much more complete discussion of these structures.

Corollary 10.30. *. If $f : X \to BO$ and $g : Y \to BO$ are maps from CW-complexes to BO, consider the composition,*

$$f \cdot g : X \times Y \xrightarrow{f \times g} BO \times BO \xrightarrow{\mu} BO.$$

Then there is an equivalence of Thom spectra

$$(X \times Y)^{f \cdot g} \simeq X^f \wedge X^g.$$

We now define a category $\mathcal{C}_{BO}$ of "spaces over BO". The objects of $\mathcal{C}_{BO}$ are maps from CW-complexes, $f : X \to BO$, and a morphisms between objects $f : X \to BO$ and $g : Y \to BO$ are maps $\phi : X \to Y$ making the following diagram commute.

$$\begin{array}{ccc} X & \xrightarrow{\phi} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ BO & \xrightarrow[=]{} & BO. \end{array}$$

Since BO is a monoid, the category $\mathcal{C}_{BO}$ inherits a monoidal structure. In particular the product of two objects $f : X \to BO$ and $g : Y \to BO$ is the composition

$$f \times g : X \times Y \xrightarrow{f \times g} BO \times BO \xrightarrow{\mu} BO.$$

We leave it to the reader to check that $\mathcal{C}_{BO}$ satisfies the properties of being a monoidal category. (This is true of the category $\mathcal{C}_M$ of spaces over any monoid M.)

Notice that a monoid in the category $\mathcal{C}_{BO}$ is an object $f : X \to BO$ together with a monoid structure on X, $\nu : X \times X \to X$ that lives above BO. That is, the following diagram commutes:

$$\begin{array}{ccc} X \times X & \xrightarrow{\nu} & X \\ {\scriptstyle f \times f}\downarrow & & \downarrow{\scriptstyle f} \\ BO \times BO & \xrightarrow[\mu]{} & BO \end{array}$$

If $f : X \to BO$ is a monoid in $\mathcal{C}_{BO}$, which we refer to as a "monoid over BO", then the commutativity of this diagram and Corollary 10.30 implies the following.

Proposition 10.31. *If $f : X \to BO$ is a monoid over BO with monoid product $\nu : X \times X \to X$, then there is a map of spectra $T\nu$ which is well-defined up to homotopy,*

$$T\nu : X^f \wedge X^f \to X^f.$$

Furthermore this map is associative up to homotopy, and there exists a "unit map" $u : \mathbb{S} \to X^f$ so that the compositions

$$X^f = X^f \wedge \mathbb{S} \xrightarrow{1 \wedge u} X^f \wedge X^f \xrightarrow{T\nu}, \textit{ and}$$
$$X^f = \mathbb{S} \wedge X^f \xrightarrow{u \wedge 1} X^f \wedge X^f \xrightarrow{T\nu} \textit{ and}$$

are homotopic to the identity map. In other words, X^f is a "ring spectrum up to homotopy."

Note. The unit map $u : \mathbb{S} \to X^f$ is the map of Thom spectra induced by the inclusion of the basepoint (i.e the unit of the monoid), $x_0 \hookrightarrow X$.

This proposition suggests that there might be a functor $Th : \mathcal{C}_{BO} \to \mathbf{Sp^\Sigma}$ that assigns to a space over BO, $f : X \to BO$, its associated Thom spectrum, X^f.Furthermore this functor should preserves products. That is, it should send a monoid over BO to a ring spectrum. This proposition says that this can be done "up to homotopy". But in recent years it has been proven that one indeed can define a "Thom fumctor" that preserves this monoidal structure. Equivalent forms of the following result were proved in [56], [1], and [10].

Theorem 10.32. *[56], [1], [10] There is a monoidal functor*

$$Th : \mathcal{C}_{BO} \to \mathbf{Sp^\Sigma}$$

that takes an object $f : X \to BO$ to its Thom spectrum X^f.

The following is a more descriptive way of stating this result.

Corollary 10.33. *If $f : X \to BO$ is a monoid over BO, it's Thom spectrum X^f is a ring spectrum. If $Y \to BO$ is another monoid over BO and $\phi : X \to Y$ is a morphism in $\mathcal{C}_{BO}$ that preserves the monoid structures, then the induced map on the level of Thom spectra,*

$$T\phi : X^f \to Y^g$$

is a map of ring spectra.

Examples

$\mathbb{MO}$, the Thom spectrum of the identity map $id : BO \to BO$, is a ring spectrum, as is the Thom spectrum $\mathbb{MU}$ of the canonical map $BU \to BO$.

Note. These spectra are essential to the study of cobordisms of manifolds, as we will see in the next chapter. We will also see that their ring spectrum structures are crucial for being able to do cobordism calculations.

10.7 The ring structure of $H_*(BO;\mathbb{Z}/2)$, $H_*(BU;\mathbb{Z})$, $H_*(\mathbb{MO};\mathbb{Z}/2)$, and $H_*(\mathbb{MU};\mathbb{Z})$

In the previous section we observed that as a result of Bott periodicity, BO and BU are infinite loop spaces. This in particular means they have homotopy commutative product maps

$$BO \times BO \to BO \quad \text{and} \quad BU \times BU \to BU.$$

This implies that the homologies $H_*(BO;\mathbb{Z}/2)$ and $H_*(BU;\mathbb{Z})$ are graded commutative rings.

We also saw that the Thom spectra $\mathbb{MO}$ and $\mathbb{MO}$ have the induced structure of homotopy commutative ring spectra. This implies that their homologies $H_*(\mathbb{MO};\mathbb{Z}/2)$ and $H_*(\mathbb{MU};\mathbb{Z})$ also are graded commutative rings. The goal of this section is to compute these rings.

We begin with a calculation of $H_*(BO;\mathbb{Z}/2)$.

Theorem 10.34. *There is an isomorphism of graded algebras,*

$$H_*(BO;\mathbb{Z}/2) \cong \mathbb{Z}/2[\tilde{H}_*(\mathbb{RP}^\infty;\mathbb{Z}/2)] = \mathbb{Z}/2[a_1, a_2, \cdots, \textit{ where} |a_i| = i]$$

Proof. For ease of notation we leave off the coefficients in (co)homology. All coefficients will be $\mathbb{Z}/2$.

Recall that the "Splitting Principle" (Theorem 5.20) says that the product map

$$\mu : BO(1)^{\times m} \to BO(m)$$

induces a monomorphism in cohomology, $\mu^* : H^*(BO(m)) \to H^*(BO(1))^{\otimes m}$, or equivalently, the map in homology, $\mu_* : H_*(BO(1))^{\otimes m} \to H_*(BO(m))$ is surjective.

Using the facts that $BO(1) = \mathbb{RP}^\infty$ and that $\iota_* : H_q(BO(m)) \to H_q(BO)$ is an isomorphism through dimension m (see Theorem 5.15), we can conclude that

$$\mu_* : H_*(\mathbb{RP}^\infty)^{\otimes m} \to H_*(BO)$$

is surjective through dimension m. Since μ_* induces the product structure in $H_*(BO)$, this says that in the algebra structure, every element in $H_q(BO)$ can be written as a linear combination of monomials of length $\leq m$ for $q \leq m$. Since $H_*(BO)$ is a commutative algebra, this says that μ_* induces a surjective map of algebras,

$$\mu_* : \mathbb{Z}/2[\tilde{H}_*(\mathbb{RP}^\infty)] \to H_*(BO).$$

Now since $\mathbb{Z}/2[\tilde{H}_*(\mathbb{RP}^\infty)] = \mathbb{Z}/2[a_i,\, i \geq 1\, :\, |a_i| = i]$ and from Theorem 5.15 we know that the cohomology, $H^*(BO) \cong \mathbb{Z}/2[w_i,\, i \geq 1, :\, |w_i| = i]$, we can conclude that as $\mathbb{Z}/2$ vector spaces, $\mathbb{Z}/2[\tilde{H}_*(\mathbb{RP}^\infty)]$ and $H_*(BO)$ have the same rank in each dimension. Therefore μ_*, being a surjective map of algebras over $\mathbb{Z}/2$ that have the same rank in every dimension, must be an isomorphism. □

Exercise. Show, using an argument like above, that

$$H_*(BU;\mathbb{Z}) \cong \mathbb{Z}[\tilde{H}_*(\mathbb{CP}^\infty;\mathbb{Z})] = \mathbb{Z}[u_i,\, i \geq 1\, :\, |u_i| = 2i].$$

We now discuss the homology of the corresponding Thom spectra, $H_*(\mathbb{MO};\mathbb{Z}/2)$ and $H_*(\mathbb{MU};\mathbb{Z})$.

We continue with the notational convention that if we do not put coefficients in (co)homology, we mean coefficients in $\mathbb{Z}/2$. Recall the Thom isomorphism,

$$\cup u_k : H^*(BO(k)) \xrightarrow{\cong} H^{*+k}(MO(k)).$$

Now consider the image of the Thom class $u_k \in H^k(MO(k))$ in $H^{k-1}(MO(k-1))$ under the composition

$$H^k(MO(k)) \xrightarrow{\epsilon_{k-1}^*} H^k(\Sigma MO(k-1)) \xrightarrow{\cong} H^{k-1}(MO(k-1))$$

where $\epsilon_{k-1} : \Sigma MO(k-1) \to MO(k)$ is the structure map of the spectrum $\mathbb{MO}$, and the second map in this composition is the suspension isomorphism. Since the structure map ϵ_{k_1} is the map induced on Thom spaces by the map $BO(k-1) \to BO(k)$, then the Thom classes are preserved. That is, the image of u_k under this composition is u_{k-1}. Therefore the Thom classes fit together to define a zero dimensional cohomology class in the spectrum,

$$u \in H^0(\mathbb{MO}),$$

and so the Thom isomorphism can be viewed as an isomorphism between the cohomology of the base space BO and that of the spectrum $\mathbb{MO}$:

$$\cup u : H^*(BO) \xrightarrow{\cong} H^*(\mathbb{MO}). \tag{10.14}$$

Notice that when viewed in this way the Thom isomorphism does not shift degrees.

Now recall that the dual of the cup product with the Thom class in cohomology, is taking the cap product with the Thom class,

$$\cap u_k : H_*(MO(k)) \xrightarrow{\cong} H_{*-k}(BO(k)).$$

On the spectrum level the dual of the Thom isomorphism 10.14 can therefore be written

$$\cap u : H_*(\mathbb{MO}) \xrightarrow{\cong} H_*(BO). \tag{10.15}$$

Again, from this perspective there is no dimension shift.

Lemma 10.35. *Taking the cap product with the Thom class*

$$\cap u : H_*(\mathbb{MO}) \xrightarrow{\cong} H_*(BO)$$

is an isomorphism of graded rings.

Proof. Since we know that $\cap u$ is an isomorphism, we need only show that it preserves the product structure.

As discussed earlier, the product structure on $H_*(BO)$ is induced by the product maps $m_{k,r} : BO(k) \times BO(r) \to BO(k+r)$ and the product structure on $H_*(\mathbb{MO})$ is induced by the ring spectrum structure on $\mathbb{MO}$, which in turn is induced by the maps of Thom spaces $\mu_{k,r} : MO(k) \wedge MO(r) \to MO(k+r)$ induced by the maps $m : BO(k) \times BO(r) \to BO(k+r)$. In particular this means that the homomorphisms $\mu^*_{k,r}$ preserve Thom classes. That is,

$$\mu^*_{k,r}(u_{k+r}) = u_k \otimes u_r \in \tilde{H}^*(MO(k) \wedge MO(r)) = \tilde{H}^*(MO(k)) \otimes \tilde{H}^*(MO(r)).$$

Because of the relationship between cup and cap product, this means that for every $\alpha \in H_q(MO(k))$ and $\beta \in H_s(MO(r))$, then

$$u_{k+r} \cap (\mu_{k,r})_*(\alpha \otimes \beta) = m_*((u_k \cap \alpha) \otimes (u_r \otimes \beta).$$

Translated to the spectrum level, this is exactly the statement that $\cap u : H_*(\mathbb{MO}) \xrightarrow{\cong} H_*(BO)$ is a ring homomorphism. □

Let $\mathbb{MO}(k)$ be the spectrum $\Sigma^{-k}\Sigma^*(MO(k))$. Note that there are maps of spectra. $\mathbb{MO}(1) \to \mathbb{MO}(2) \to \cdots \mathbb{MO}(k) \to \cdots \mathbb{MO}$. Moreover, using these spectra, the Thom isomorphisms do not shift degrees:

$$\cup u_k : H^*(BO(k)) \xrightarrow{\cong} H^*(\mathbb{MO}(k)).$$

From this lemma and Theorem 10.34 we can conclude the following.

Theorem 10.36. *There is an isomorphism of graded algebras,*

$$H_*(\mathbb{MO}) \cong \mathbb{Z}/2[H_*(\mathbb{MO}(1))] = \mathbb{Z}/2[e_1, e_2, \cdots, \textit{where}\, |e_i| = i].$$

Exercise. Adapt the above arguments to prove the following:

Theorem 10.37. *There is an isomorphism of graded algebras,*

$$H_*(\mathbb{MU}; \mathbb{Z}) \cong \mathbb{Z}/2[H_*(\mathbb{MU}(1); \mathbb{Z})] = \mathbb{Z}[t_1, t_2, \cdots, \textit{where}\, |t_i| = 2i].$$

10.8 Generalized orientations, the generalized Thom isomorphism, and the generalized Poincaré and Alexander duality theorems

As we saw in Chapter 1, orientations are a crucial property for studying Poincaré duality for manifolds. For a commutative ring R we described the notion of R-orientability of a manifold (Definition 1.4), and in particular we proved that if a manifold is $\mathbb{Z}$-orientable, then it is R-orientable for any commutative ring R. We also proved that if a closed topological manifold is R-orientable, it satisfies Poincaré duality with respect to (co)homology with R-coefficients.

In this subsection we verify the analogues of these results with respect to *generalized* (co)homology theories. In particular we define the notion of orientability of a manifold with respect to a generalized (co)homology theory, when the representing spectrum for that theory is a ring spectrum. We prove that if a manifold is orientable with respect to stable (co)homotopy, the generalized theories represented by the sphere spectrum $\mathbb{S}$, then the manifold is orientable with respect to any generalized (co)homology theory $\mathbb{E}$ represented by a ring spectrum. We then prove the appropriate version of Poincaré duality for $\mathbb{E}$-oriented manifolds. All of these results were originally proved by G. Whitehead in [100].

10.8.1 Orientations

Recall from Chapter 1, Definition 1.2, that if M^n is a (topological) manifold, a local orientation of M^n at $x \in M^n$ is a choice of generator of

$$\begin{aligned} H_n(M^n, M^n - \{x\}) &\cong H_n(U_x, U_x - x) \\ &\cong H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}) \\ &\cong H_n(S^n, point) \\ &\cong \mathbb{Z} \end{aligned}$$

Here U_x is an open neighborhood (chart) of x, homeomorphic to $\mathbb{R}^n$.

Now let E_* be a generalized homology theory represented by a ring spectrum $\mathbb{E}$. To make an analogous definition of local E_*-orientation, we need to consider

$$\begin{aligned} E_n(S^n, point) &= \pi_n(S^n \wedge \mathbb{E}) \\ &\cong \pi_0(S^0 \wedge \mathbb{E}) = \pi_0(\mathbb{E}) \quad \text{by the suspension homomorphism.} \end{aligned}$$

Now since $\mathbb{E}$ is a ring spectrum, $\pi_*(\mathbb{E})$ is a graded ring, and $\pi_0(\mathbb{E})$ is a (nongraded) subring. This leads us to the following more general definition.

Definition 10.22. *Let E_* be a generalized homology theory represented by a ring spectrum $\mathbb{E}$, and let M^n be a topological manifold. Then an E_*-local orientation (equivalently referred to as an $\mathbb{E}$-local orientation) of M^n at $x \in M^n$ is a choice of unit (generator) in the ring $E_n(M^n, M^n - \{x\}) \cong E_n(S^n, point) \cong \pi_0(\mathbb{E})$.*

Now recall from the observation after the statement of Theorem 1.3 in Chapter 1 that if M^n is a closed, connected manifold, it has a global orientation if an only if there is a "fundamental class" $[M^n] \in H_n(M^n; \mathbb{Z})$ whose image in $H_n(M^n, M^n - \{x\})$ defines a local orientation for every $x \in M^n$ (i.e is a generator of $H_n(M^n, M^n - \{x\})$ for every $x \in M^n$). This leads to the following definition.

Definition 10.23. *Let E_* be a generalized homology theory represented by a connective ring spectrum $\mathbb{E}$ (remember that means that $\pi_q(\mathbb{E}) = 0$ for $q < 0$), and let M^n be a closed topological manifold. Then M^n is E_*-orientable (or equivalently, $\mathbb{E}$-orientable) if there is a class $[M^n]_E \in E_n(M^n)$ so that the restriction to $E_n(M^n, M^n - \{x\}) \cong \pi_0(\mathbb{E})$ defines a local E_*-orientation for every $x \in M^n$ (i.e is a unit of $E_n(M^n, M^n - \{x\}) \cong \pi_0(\mathbb{E})$).*

We note that this condition can be described more homotopy theoretically in the following way. A class $[M^n]_E \in E_n(M^n)$ is represented by a map

$$\zeta_M : S^n \to M^n_+ \wedge \mathbb{E}.$$

For $x \in M^n$, let U_x be an open neighborhood homeomorphic to $\mathbb{R}^n$ as above, and consider the projection map

$$p_x : M^n \to M^n/M^n - U_x \cong D^n/\partial D^n = S^n$$

where D^n is the closed n-dimensional disk. Now consider the composition

$$\rho^\zeta_x : S^n \xrightarrow{\zeta_M} M^n_+ \wedge \mathbb{E} \xrightarrow{p_x \wedge 1} (M^n/M^n - U_x) \wedge \mathbb{E} \cong S^n \wedge \mathbb{E}.$$

This composition represents a class in $\pi_n(S^n \wedge \mathbb{E}) \cong \pi_0(\mathbb{E})$. So the condition of $\mathbb{E}$-orientability is that there is a class $[M^n]_E \in E_n(M^n)$ represented by a map $\zeta_M : S^n \to M^n_+ \wedge \mathbb{E}$ so that the induced map $\rho^\zeta_x : S^n \to S^n \wedge \mathbb{E}$ represents a unit in the ring $\pi_0(\mathbb{E})$ for every $x \in M^n$.

Using the language we used for ordinary homology, a class $[M^n]_E \in E_n(M^n)$ satisfying the above property is called an E_*-fundamental class (or $\mathbb{E}$-fundamental class) of M^n. It is also often referred to as an E_*-orientation class of M^n.

Exercises.

1. Show that if a closed manifold M^n is $\mathbb{S}$-orientable, then it is $\mathbb{E}$-orientable for any ring spectrum $\mathbb{E}$. In particular it is orientable with respect to integral homology.

2. Show that the sphere S^n is $\mathbb{S}$-orientable for every n.

3. Let $H\mathbb{Z}$ be the integral Eilenberg-MacLane spectrum. Let $u : \mathbb{S} \to H\mathbb{Z}$ be the unit. Notice that u induces a map in generalized homology theories, called the "stable Hurewicz homomorphism",

$$u_* : \pi^s_*(X) \to H_*(X; \mathbb{Z}).$$

Now let M^n be a closed, connected, $\mathbb{S}$-oriented n-dimensional manifold. Show that the stable Hurewicz homomorphism in dimension n,

$$u_* : \pi^s_n(M^n) \to H_n(M^n; \mathbb{Z})$$

is surjective.

4. Continuing to assume that M^n is a closed, connected, $\mathbb{S}$-oriented n-dimensional manifold, let $[M^n]_{\mathbb{S}} \in \pi_n^s(M^n)$ be a $\mathbb{S}$-fundamental class, represented by a map of spectra

$$[M^n]_{\mathbb{S}} : \Sigma^\infty S^n \to \Sigma^\infty(M^n_+).$$

Let $x_0 \in M^n$ be a basepoint, with an open neighborhood U_{x_0} homeomorphic to $\mathbb{R}^n$. Let $\tilde{M}^n$ be M^n punctured at x_0. That is, $\tilde{M}^n$ is the complement

$$\tilde{M}^n = M^n - U_{x_0}.$$

Show that there is a weak homotopy equivalence of suspension spectra

$$\phi : \Sigma^\infty((S^n \vee \tilde{M}^n) \xrightarrow{\simeq} \Sigma^\infty(M^n).$$

Hint. Use the result of the previous exercise.

We observe that the notion of E_*-orientability can also be described via an orientation covering space as was done for orientation with respect to an ordinary homology theory in chapter one. Namely, one can construct E_*-orientation covering space over any connected (not necessarily compact) n-manifold M^n

$$p : Or_{E_*}(M^n) \to M^n$$

where the fiber over a point $x \in M^n$ is the set of units in the ring $E_n(M^n, M^n - \{x\}) \cong \pi_0(\mathbb{E})$. Details of the construction are left to the reader.

A global E_*-orientation is then a section of the covering space $Or_{E_*}(M^n)$. Because its proof only relied on the Eilenberg-Steenrod axioms, we immediately have the following generalization of Theorem 1.3.

Theorem 10.38. *Let M^n be an n-manifold and $A \subset M^n$ a compact subspace. Let E_* be a generalized homology theory represented by a connective ring spectrum $\mathbb{E}$. Then if $\alpha : A \to Or_{E_*}(M^n)$ is a section of the orientation covering space over A, then there exists a unique homology class $\alpha_A \in E_n(M, M - A)$ whose image in $E_n(M, M - x)$ is $\alpha(x)$ for every $x \in A$.*

In particular if M^n is closed, then by taking $A = \emptyset$, this gives the equivalence of having a section of $Or_{E_*}(M^n)$ and the existence of a fundamental class $\alpha_\emptyset = [M^n]_{\mathbb{E}} \in E_n(M^n)$.

10.8.2 Poincaré and Alexander duality, and the Thom isomorphism for generalized (co)homology

Our goal is to use the notion of E_*-orientation and derive, like we did in chapter 1 for ordinary (co)homology, Poincaré duality for generalized (co)homology theories. Throughout this subsection we continue to assume that E_* is a generalized homology theory represented by a ring spectrum, $\mathbb{E}$.

Our first step is to understand the notion of generalized cohomology with compact supports. When we defined ordinary cohomology with compact supports in Chapter 1, we used cochains. For generalized cohomology we make use of mapping spectra.

Recall from Definition 10.14 that for symmetric spectra $\mathbb{X}$ and $\mathbb{Y}$ we have an associated morphism spectra $Map(\mathbb{X}, \mathbb{Y})$. In the setting when $\mathbb{X}$ is the suspension spectrum of a space X, this has a particularly easy definition. Namely $Map(X, \mathbb{Y})_n = Map(X, \mathbb{Y}_n)$ with the obvious structure maps. (These mapping spaces consist of *basepoint preserving* maps.) Now given our ring spectrum $\mathbb{E}$, notice that the generalized cohomology group is given by

$$\begin{aligned} E^n(X) &= [X_+, \Sigma^n \mathbb{E}] \\ &= \pi_0(Map(X_+, \Sigma^n \mathbb{E})) \\ &= \pi_{-n}(Map(X_+, \mathbb{E})) \end{aligned} \tag{10.16}$$

For this reason we will choose to define generalized cohomology with compact supports using mapping spectra. In particular let M^n be an n-dimensional manifold, not necessarily compact. Let $K \subset M^n$ be a compact subspace. Notice that if Y is some other space, the mapping space $Map(M^n/M^n - K, Y)$ can be interpreted as the space of basepoint preserving maps from M^n to Y that map the complement of K to the basepoint of Y. Notice furthermore that if K_1 and K_2 are compact subspaces of M^n with $K_1 \subset K_2$, then $(M^n - K_2) \subset (M^n - K_1)$ we have an induced map, which is an inclusion,

$$Map(M^n/(M^n - K_1), Y) \to Map(M^n/(M^n - K_2), Y).$$

Definition 10.24. *We define the space of compactly supported maps, $Map^c(M^n, Y)$ to be the colimit, which can be viewed as the union,*

$$Map^c(M^n; Y) = \operatorname{colim}_{K \subset M^n} Map(M^n/(M^n - K), Y)$$

where the colimit (union) is taken over all compact subsets K of M^n.

Notice that $Map^c(M^n; Y) \subset Map(M^n, Y)$ consists of all maps that send the complement of some compact subspace $K \subset M^n$ to the basepoint of Y. This allows us to define the compactly supported mapping spectrum $Map^c(M^n_+, \mathbb{E})$ as follows:

Definition 10.25. *We define the spectrum*

$$Map^c(M^n_+, \mathbb{E})$$

by $Map^c(M^n_+, \mathbb{E})_k = Map^c(M^n_+; \mathbb{E}_k) \subset Map(M^n_+; \mathbb{E}_k)$ with the induced structure maps.

Motivated by observation (10.16), we make the following definition of *generalized cohomology with compact supports.*

Definition 10.26. *Let M^n be an n-dimensional manifold and E^* a generalized cohomology theory represented by a connective ring spectrum $\mathbb{E}$. We define the E^*-cohomology with compact supports to be,*

$$E^q_c(M^n) = \pi_{-q} Map^c(M^n_+, \mathbb{E})$$

Notice there is a natural map $E^*_c(M^n) \to E^*(M^n)$ which is an isomorphism if M^n is compact.

Exercise. (Compare with the exercise after the statement of the Poincaré Duality Theorem 1.5 in Chapter 1.) Show that

$$E^*_c(\mathbb{R}^n) \cong \tilde{E}^*(S^n)$$

and more generally that if X is a space whose one-point compactification $X \cup \infty$ has the property that the point at infinity in the one-point compactification has a contractible open neighborhood, as is the case if X is a manifold, then

$$E^*_c(X) \cong \tilde{E}^*(X \cup \infty).$$

We can now state the Poincaré Duality Theorem for generalized cohomology. First observe that if E^* is a generalized cohomology theory represented by a ring spectrum $\mathbb{E}$, and if M^n is a $\mathbb{E}$-oriented n-dimensional manifold, then if $K \subset M^n$ is any compact space, we have an orientation class $\alpha_K \in E_n(M^n, M^n - K)$, which induces a cap product operation (see Definition 10.19)

$$\cap\alpha_K : E^q(M^n, M^n - K) \to E_{n-q}(M^n).$$

As seen in ordinary (co)homology, these operations respect the inclusions of one compact subspace into another, and define a map

$$D_{M^n} : E^q_c(M^n) \to \operatorname{colim}_{K \subset M^n} E^q(M^n, M^n - K) \xrightarrow{\operatorname{colim}_K\{\cap\alpha_K\}} E_{n-q}(M^n).$$

Theorem 10.39. *Let E^* be a generalized cohomology theory represented by a connective ring spectrum $\mathbb{E}$. Let M^n be a $\mathbb{E}$-oriented manifold. Then the duality map*

$$D_{M^n} : E^k_c(M^n) \to E_{n-k}(M^n).$$

is an isomorphism for all k.

Proof. This theorem is a generalization of Theorem 1.6. As you will recall in the proof of that theorem, the argument just needed that the theorem is true for $M^n = \mathbb{R}^n$, which we know in the generalized setting by the exercises above, as well as the fact that (co)homology satisfies the homotopy, exactness, and excision Eilenberg-Steenrod axioms, so that, for example, we get Mayer-Vietoris sequences. Of course, generalized (co)homology theories also satisfy these axioms, and so the proof of Poincaré duality goes through for such generalized theories. We leave the exercise of going through that proof and showing that all the steps are satisfied by generalized (co)homology theories to the reader. We remark that this was first proved by Whitehead in [100]. □

Now recall in Chapter 2, the notion of orientability was generalized from manifolds to vector bundles. In particular a manifold is orientable if and only if its tangent bundle is orientable. (See Definition 2.10.) The idea was to assign to a vector bundle $\zeta \to X$ an "orientation double cover", Or_ζ. An orientation of ζ is a section of this covering space. If no such section exists, the bundle ζ is not orientable.

There is a similar notion of $\mathbb{E}$-orientability of a k-dimensional vector bundle $\zeta \to X$, where $\mathbb{E}$ is a ring spectrum representing a generalized homology theory E_*. To define this, we consider the covering space $Or^\zeta_{E_*} \to X$, where the fiber over $x \in X$ is the set of units of $E_k(\zeta_x, \zeta_x - \{0\}) \cong E_k(\mathbb{R}^k, \mathbb{R}^k - \{0\}) \cong \pi_0(\mathbb{E})$. We leave it to the reader to adapt the methods used in Chapters 1 and 2 to define the topology of the space $Or^\zeta_{E_*}$. With this orientation cover we can make the following definition:

Definition 10.27. *Let $\mathbb{E}$ be a connective ring spectrum representing the generalized cohomology theory E^*. Let $\zeta \to X$ be a k-dimensional vector bundle. A $\mathbb{E}$-orientation of ζ is a section of the orientation cover $Or^\zeta_{E_*}$.*

Exercises.

1. Show that a manifold M^n is $\mathbb{E}$-orientable if and only if its tangent bundle $TM^n \to M^n$ orientable.
2. Show that a closed manifold M^n equipped with an embedding or immersion into $\mathbb{R}^L$ for some L, is orientable if and only if the normal bundle to this immersion is orientable. Indeed show that an orientation of its tangent bundle induces an orientation of its normal bundle, and vice versa.

An important property of oriented vector bundles is the Thom isomorphism theorem (5.10). There is an analogous Thom isomorphism theorem for $\mathbb{E}$-oriented vector bundles $\zeta \to X$, which we now state. The proof follows the proof of Theorem 5.10 at every step.

Theorem 10.40. *Let ζ be a $\mathbb{E}$- oriented n - dimensional real vector bundle over a connected space X, where $\mathbb{E}$ is a connective ring spectrum representing the generalized cohomology theory E^*. The orientation gives generators (units)*

$u_x \in E^n(\zeta_x, \zeta_x - \{0\}) \cong \pi_0(\mathbb{E})$. *Then there is a unique class (called the* $\mathbb{E}$*-Thom class) in the cohomology of the Thom space*

$$u \in E^n(T(\zeta))$$

so that for every $x \in X$, *if*

$$j_x : \zeta_x/(\zeta_x - \{0\}) \hookrightarrow \zeta/(\zeta - zero(X)) \cong T(\zeta)$$

is the natural inclusion, where $zero(X)$ *is the image of the zero section, then under the induced homomorphism in* E^**- cohomology,*

$$j_x^* : E^n(T(\zeta)) \to E^n(\zeta_x, \zeta_x - \{0\}) \cong \pi_0(\mathbb{E}),$$

$j_x^*(u) = u_x$.

Furthermore the induced cup product map

$$\gamma : E^q(X) \xrightarrow{\cup u} \tilde{E}^{q+n}(T(\zeta))$$

is an isomorphism for every $q \in \mathbb{Z}$.

This generalized Thom isomorphism theorem has many applications, but a particularly interesting one that we will discuss is an analogue of Alexander duality for $\mathbb{E}$-oriented manifolds.

Theorem 10.41. *(Alexander Duality) Let* $e : M^n \subset \mathbb{R}^N$ *be a regular embedding of a closed,* $\mathbb{E}$*-oriented manifold into Euclidean space. Let* $\mathbb{E}$ *be a connective ring spectrum representing the generalized cohomology theory* E^* *and homology theory* E_*. *Then there is an isomorphism*

$$E^r(M^n) \cong \tilde{E}_{N-r-1}(\mathbb{R}^N - M^n).$$

Before we prove this theorem, we note that one of the most striking applications of this duality theorem is to knot theory. Recall that a "knot" is the image of a regular embedding $e : S^1 \hookrightarrow \mathbb{R}^3$. We call the image of this embedding $K \subset \mathbb{R}^3$. As above, assume $\mathbb{E}$ is a ring spectrum. Then the Alexander Duality theorem, combined with Poincaré duality calculates the E_*-homology of the complement of the knot in terms of the E_* homology of S^1:

Corollary 10.42.

$$\tilde{E}_q(\mathbb{R}^3 - K) \cong E_{q-1}(S^1).$$

Notice that in the case of ordinary integral homology this corollary says that $H_1(\mathbb{R}^3 - K) \cong \mathbb{Z}$. This in particular says that the fundamental group of the complement of the knot, which can be quite complicated, always has the integers $\mathbb{Z}$ as its abelianization.

We now proceed with the proof of the Alexander duality theorem.

Proof. . Let η_e be a tubular neighborhood of the embedding $e : M^n \hookrightarrow \mathbb{R}^N$, and let $\nu_e \to M^n$ be its normal bundle. Since e is a regular embedding, its complement of the tubular neighborhood is a deformation retract of the complement of the manifold,

$$\mathbb{R}^N - \eta_e \xrightarrow{\cong} \mathbb{R}^n - M^n.$$

Now that the quotient space $\mathbb{R}^N/(\mathbb{R}^N - \eta_e)$ is homeomorphic to the one-point compactification $\eta_e \cup \infty$. But by the Tubular Neighborhood Theorem, this is homeomorphic to the one-point compactification of the normal bundle, $\nu_e \cup \infty$. Now notice that since M^n is compact, the one-point compactification of the normal bundle is homeomorphic to its Thom space, $T(\nu_e)$. So we have

$$\mathbb{R}^N/(\mathbb{R}^N - \eta_e) \cong \eta_e \cup \infty \cong \nu_e \cup \infty \cong T(\nu_e).$$

We use this observation in the following way. Since $\mathbb{E}$ is assumed to be $\mathbb{E}$-orientable, then as vector bundles, its tangent bundle is $\mathbb{E}$-orientable. But by an exercise above this is equivalent to its normal bundle ν_e being orientable. Now the $\mathbb{E}$-Thom isomorphism theorem 10.40, interpreted for homology (rather than cohomology) says that taking the cap product with the Thom class $u_e \in E^{N-n}((T(\nu_e))$ gives an isomorphism

$$\cap u_e : \tilde{E}_{q+N-n}(T(\nu_e)) \xrightarrow{\cong} E_q(M^n). \tag{10.17}$$

Combining this with the above homeomorphisms, together with the fact that E_* satisfies the Excision Axiom, we get an isomorphism

$$E_{q+N-n}(\mathbb{R}^N, \mathbb{R}^N - \eta_e) \cong E_q(M^n). \tag{10.18}$$

Now using the long exact sequence in reduced homology for the pair $(\mathbb{R}^N, \mathbb{R}^N - \eta_e)$ together with the fact that by the Homotopy Axiom which implies that $\tilde{E}_*(\mathbb{R}^N) = 0$, we see that the connecting homomorphism is an isomorphism,

$$\partial : E_r(\mathbb{R}^N, \mathbb{R}^N - \eta_e) \xrightarrow{\cong} \tilde{E}_{r-1}(\mathbb{R}^N - \eta_e)$$

for all $r \in \mathbb{Z}$. Combining this with isomorphism (10.18) we get an isomorphism

$$E_q(M^n) \cong \tilde{E}_{q+N-n-1}(\mathbb{R}^N - \eta_e) \cong \tilde{E}_{q+N-n-1}(\mathbb{R}^N - M^n).$$

But Poincaré duality gives us an isomorphism

$$\cap[M^n]_{\mathbb{E}} : E^{n-q}(M^n) \xrightarrow{\cong} E_q(M^n).$$

The theorem now follows by combining these isomorphisms. □

10.8.3 Spanier-Whitehead duality and Atiyah duality

An important result in the study of closed differentiable manifolds says that if a manifold M^n is embedded in $\mathbb{R}^N$, then the Thom spectrum of the normal bundle and the manifold itself, are in a sense that can be made precise, dual to each other. This is a stable homotopy theoretic generalization of the Alexander Duality theorem, and was proved by Atiyah in [6]. The type of duality that is appropriate in this setting is known as "Spanier-Whitehead" duality (see [88]). In this subsection we introduce and explore these concepts.

The notion of Spanier-Whitehead duality is a direct analogue of the notion of duality in linear algebra. Recall that if V and W are finite dimensional vector spaces over a field k, then they are said to be *dual* to each other if there is a bilinear pairing

$$V \times W \to k$$

whose adjoints define isomorphisms

$$V \xrightarrow{\cong} Hom(W, k) \quad \text{and}$$
$$W \xrightarrow{\cong} Hom(V, k).$$

In the setting of spectra, the notion of a finite dimensional vector space is replaced by the notion of a *"finite spectrum"*. Such a spectrum $\mathbb{X}$ is one whose homology is finite in the sense that

1. $H_q(\mathbb{X})$ is nonzero for only finitely many $q \in \mathbb{Z}$, and

2. $H_q(\mathbb{X})$ is a finitely generated abelian group for every $q \in \mathbb{Z}$.

The archetypical example of a finite spectrum is the suspension spectrum of a finite, based CW-complex, $\mathbb{X} = \Sigma^\infty(X)$. This example is quite general because of the result of the following exercise:

Exercise. Show that every finite spectrum $\mathbb{X}$ is weakly homotopy equivalent to an iterated suspension or desuspension of the suspension spectrum of a finite CW-complex.

Definition 10.28. *Two finite spectra $\mathbb{X}$ and $\mathbb{Y}$ are said to be "Spanier-Whitehead dual" to each other, (or simply $\mathbb{S}-$ dual) if there is a pairing of spectra*

$$\mathbb{X} \wedge \mathbb{Y} \to \mathbb{S}$$

whose adjoints define weak homotopy equivalences,

$$\mathbb{Y} \xrightarrow{\simeq} Map(\mathbb{X}, \mathbb{S}) \quad \textit{and}$$
$$\mathbb{X} \xrightarrow{\simeq} Map(\mathbb{Y}, \mathbb{S}).$$

An equivalent definition is that $\mathbb{X}$ and $\mathbb{Y}$ are said to be Spanier-Whitehead dual if there are maps of spectra

$$\mu : \mathbb{X} \wedge \mathbb{Y} \to \mathbb{S} \quad \textit{and} \quad \eta : \mathbb{S} \to \mathbb{Y} \wedge \mathbb{X}$$

so that compositions

$$\mathbb{X} = \mathbb{X} \wedge \mathbb{S} \xrightarrow{1\wedge\eta} \mathbb{X} \wedge \mathbb{Y} \wedge \mathbb{X} \xrightarrow{\mu\wedge 1} \mathbb{S} \wedge \mathbb{X} = \mathbb{X} \quad \textit{and}$$
$$\mathbb{Y} = \mathbb{S} \wedge \mathbb{Y} \xrightarrow{\eta\wedge 1} \mathbb{Y} \wedge \mathbb{X} \wedge \mathbb{Y} \xrightarrow{1\wedge\mu} \mathbb{Y} \wedge \mathbb{S} = \mathbb{Y}$$

are homotopic to the identity.,

Exercise. Show that these two definitions are equivalent.

If $\mathbb{X}$ is a finite spectrum, we denote it's Spanier-Whitehead dual by $D\mathbb{X}$.

Observations.

1. The sphere spectrum $\mathbb{S}$ is Spanier-Whitehead dual to itself, via the identity map

$$\mathbb{S} \wedge \mathbb{S} \xrightarrow{=} \mathbb{S}.$$

2. If $\mathbb{X}$ is Spanier-Whitehead dual to $\mathbb{Y}$, then the iterated suspensions $\Sigma^k\mathbb{X}$ and $\Sigma^{-k}\mathbb{Y}$ are also Spanier-Whitehead dual.

Exercises.

1. Let $\mathbb{X}$ be a finite spectrum, and let $\mathbb{E}$ be a connective spectrum. (Recall that a connective spectrum is one which has zero homotopy groups in negative dimensions.) Prove that there is a weak homotopy equivalence of spectra

$$Map(\mathbb{X}, \mathbb{S}) \wedge \mathbb{E} \xrightarrow{\simeq} Map(\mathbb{X}, \mathbb{E}).$$

2. Suppose that $\mathbb{X}$ and $\mathbb{Y}$ are finite spectra that are Spanier-Whitehead dual to each other. Suppose that $\mathbb{E}$ is a connective spectrum representing cohomology and homology theories E^* and E_*, then

$$E^q(\mathbb{X}) \cong E_{-q}(\mathbb{Y}) \quad \text{and}$$
$$E^q(\mathbb{Y}) \cong E_{-q}(\mathbb{X})$$

for all $q \in \mathbb{Z}$.

3. Show that if $\mathbb{X}$ and $\mathbb{Y}$ are finite spectra,

$$D(\mathbb{X} \wedge \mathbb{Y}) \simeq D\mathbb{X} \wedge D\mathbb{Y}.$$

4. Show that the dual of the dual is the original spectrum. That is, if $\mathbb{X}$ is a finite spectrum then $DD\mathbb{X} \simeq \mathbb{X}$.

Note. In exercises 3 and 4 the equivalences mean the same weak homotopy type).

5. If $f : \mathbb{X} \to \mathbb{Y}$ is a map of finite spectra, then there is a natural map $D(f) : D(\mathbb{Y}) \to D(\mathbb{X})$ with $DD(f) = f : \mathbb{X} \to \mathbb{Y}$.

6. If $\mathbb{X} \xrightarrow{f} \mathbb{Y} \xrightarrow{g} \mathbb{Z}$ is a cofibration sequence of finite spectra, then $D(\mathbb{X}) \xleftarrow{D(f)} D(\mathbb{Y}) \xleftarrow{D(g)} D(\mathbb{Z})$ is also a cofibration sequence of spectra.

Let X be a finite CW-complex, and assume that X is embedded, in a nonsurjective way, in the sphere S^n, such that the complement $S^n - X$ has the homotopy type of a finite CW-complex. We actually assume that this embedding has a regular neighborhood η, meaning an open subset of S^n which contains the image of X as a deformation retract. For example if X is a smooth manifold smoothly and regularly embedded, then η can be taken to be a tubular neighborhood. Every finite CW complex does have such a "regular embedding" in a sphere of sufficiently high dimension. See, for example, [41]. Then the following gives a more general form of Alexander duality:

Theorem 10.43. *The Spanier-Whitehead dual of the suspension spectrum of X, which we denote by DX, is given by the $(n-1)$-fold desuspension of the suspension spectrum of the complement:*

$$DX \simeq \Sigma^{-(n-1)}\Sigma^{\infty}(S^n - X).$$

Notice that in this setting the complement $S^n - X$ has the homotopy type of the complement of a regular neighborhood, $S^n - \eta$.

Proof. We think of the sphere S^n as the one-point compactification, $S^n = \mathbb{R}^n \cup \infty$. By rotating S^n if necessary, we may assume without loss of generality that $X \subset \mathbb{R}^n \subset S^n$. Consider the map

$$\alpha : (\mathbb{R}^n - X) \times X \to S^{n-1}$$
$$(v, x) \to \frac{v - x}{\|v - x\|}$$

Now suspend that map:

$$\Sigma\alpha : \Sigma((\mathbb{R}^n - X) \times X) \to \Sigma S^{n-1} = S^n.$$

We now need the following basic homotopy theoretic lemma.

Lemma 10.44. *Let A and B be have the homotopy type of CW-complexes. Then there is a natural "splitting" of the suspension of the product,*

$$\Sigma(A \times B) \simeq \Sigma A \vee \Sigma B \vee \Sigma(A \wedge B).$$

Proof. We leave the proof of this lemma as an exercise for the reader. Use the following hint:

Hint. Consider the natural projection maps $p_A : \Sigma(A \times B) \to \Sigma A$, $p_B : \Sigma(A \times B) \to \Sigma B$, and $p_{A\wedge B} : \Sigma(A \times B) \to \Sigma(A \wedge B)$. Use the (iterated) "pinch map" $S^1 \to S^1 \vee S^1 \vee S^1$ in the suspension coordinate to define a map

$$\Sigma(A\times B) \to \Sigma(A\times B)\vee\Sigma(A\times B)\vee\Sigma(A\times B) \xrightarrow{p_A\vee p_B\vee p_{A\wedge B}} \Sigma A\vee\Sigma B\vee\Sigma(A\wedge B)$$

Show that this map is a homotopy equivalence. It might be easiest to first show that it induces an isomorphism in homology. □

We now return to the proof of Theorem 10.43. Using Lemma 10.44 we have a natural map which gives an inclusion of a wedge summand,

$$\iota : \Sigma((\mathbb{R}^n - X) \wedge X) \to \Sigma((\mathbb{R}^n - X) \times X).$$

We therefore may consider the composition

$$\Sigma((\mathbb{R}^n - X) \wedge \Sigma X = \Sigma((\mathbb{R}^n - X) \wedge X) \xrightarrow{\iota} \Sigma((\mathbb{R}^n - X) \wedge X) \xrightarrow{\Sigma\alpha} S^n.$$

Taking the n-fold iterated desuspension of the corresponding map of suspension spectra we produce a map of spectra,

$$\bar{\mu} : \Sigma^{-(n-1)}\Sigma^\infty(\mathbb{R}^n - X) \wedge \Sigma^\infty X \to \mathbb{S} = \Sigma^\infty S^0.$$

Taking adjoints we get a map of spectra

$$\mu : \Sigma^{-(n-1)}\Sigma^\infty(\mathbb{R}^n - X) \to D(X). \tag{10.19}$$

Our goal is to show the map μ is a weak equivalence of spectra for every finite CW complex X. We will use an induction argument on the skeleta of X. We begin by showing that μ is an equivalence when X is a sphere S^k embedded in S^n. Now since we are assuming that $S^k \hookrightarrow S^n$ has a regular neighborhood η, then $S^n - S^k$ is homotopy equivalent to $S^k - \eta$. Since we may take an arbitrarily small perturbation of the embedding and make it smooth, we may assume that η is a tubular neighborhood of a smooth embedding of S^k in S^n. Indeed, since the embedding is not surjective, we may assume that its image does not include $\infty \in \mathbb{R}^n \cup \infty = S^n$. In this case we have that

$$\Sigma^\infty\left(\mathbb{R}^n/\mathbb{R}^n - \eta\right) = \Sigma^\infty T(\eta),$$

where by abuse of notation $T(\eta)$ refers to the Thom space of the normal bundle. (Our admittedly bad notation is identifying the tubular neighborhood with the normal bundle.) But the spectrum $\Sigma^\infty T(\eta)$ is the $(n-k)$-fold suspension of the Thom spectrum $\Sigma^{n-k}(S^k)^\eta$. But the stable normal bundle of S^k is trivial, so we have that

$$\Sigma^\infty\left(\mathbb{R}^n/\mathbb{R}^n - \eta\right) = \Sigma^{n-k}(S^k)^\eta \simeq \Sigma^\infty(S^n \vee S^{n-k}). \tag{10.20}$$

Now notice that

$$\mathbb{R}^n/\mathbb{R}^n - \eta = S^n/S^n - \eta, \tag{10.21}$$

so that

$$\Sigma^\infty \left(S^n/S^n - \eta\right) \simeq \Sigma^\infty(S^{n-k} \vee S^n).$$

To finish the argument for the case of $S^k \subset S^n$, we study the following diagram of homotopy cofibration sequences of spectra

$$\begin{array}{ccccccc}
\Sigma^\infty S^{n-1} & \longrightarrow & \Sigma^\infty \mathbb{R}^n & \longrightarrow & \Sigma^\infty S^n & \xrightarrow{=} & \Sigma^\infty S^n \\
{\scriptstyle =}\uparrow & & \uparrow & & {\scriptstyle .}\uparrow & & \uparrow{\scriptstyle =} \\
\Sigma^\infty S^{n-1} & \longrightarrow & \Sigma^\infty(\mathbb{R}^n - S^k) & \longrightarrow & \Sigma^\infty(S^n - S^k) & \longrightarrow & \Sigma^\infty S^n \\
 & & {\scriptstyle \simeq}\uparrow & & \uparrow & & \\
 & & \Sigma^{-1}\Sigma^\infty\left(\mathbb{R}^n/\mathbb{R}^n - S^k\right) & \xrightarrow[=]{} & \Sigma^{-1}\Sigma^\infty\left(S^n/S^n - S^k\right) & &
\end{array}$$

The vertical map $\Sigma^{-1}\Sigma^\infty\left(\mathbb{R}^n/\mathbb{R}^n - S^k\right) \to \Sigma^\infty(\mathbb{R}^n - S^k)$ is an equivalence, because its cofiber, $\Sigma^\infty\mathbb{R}^n$ is contractible. Combining this with (10.35) implies that

$$\Sigma^\infty(\mathbb{R}^n - S^k) \simeq \Sigma^{-1}(\Sigma^\infty(S^{n-k} \vee S^n)) = \Sigma^\infty(S^{n-k-1} \vee S^{n-1}). \tag{10.22}$$

Now the horizontal map $\Sigma^\infty(S^n - S^k) \to \Sigma^\infty(S^n)$ is null homotopic since the inclusion $S^n - S^k \hookrightarrow S^n$ is not surjective, and hence its image lies in $S^n - point$ which is contractible. This implies there is a splitting $\sigma : \Sigma^\infty(S^n - S^k) \to \Sigma^\infty(\mathbb{R}^n - S^k)$. (By a *"splitting"* we mean that the composition $\Sigma^\infty(S^n - S^k) \xrightarrow{\sigma} \Sigma^\infty(\mathbb{R}^n - S^k) \to \Sigma^\infty(S^n - S^k)$ is homotopic to the identity,). This means that there is an equivalence

$$\begin{aligned}
\Sigma^\infty(\mathbb{R}^n - S^k) &\simeq \Sigma^\infty S^{n-1} \vee \Sigma^\infty(S^n - S^k), \quad \text{and by (10.22)} \\
&\simeq \Sigma^\infty S^{n-1} \vee \Sigma^\infty S^{n-k-1}
\end{aligned}$$

From this it is easy to conclude that

$$\Sigma^\infty(S^n - S^k) \simeq \Sigma^\infty S^{n-k-1} \simeq \Sigma^{n-1}D(S^k) \tag{10.23}$$

and that this equivalence is induced by the duality map described above (10.19).

We now continue our proof of Theorem 10.43 using an induction argument on the skeleta of a finite CW-complex X. It is an easy exercise to see that the theorem holds if X is a zero-dimensional finite complex, meaning it is a finite collection of points. So assume the theorem is true for complexes of dimension less than q, and let X be a q-dimensional finite complex. Let $X^{(q-1)}$ be its $(q-1)$-dimensional skeleton, and assume we have an embedding $X \hookrightarrow S^n$. Then by our inductive assumption we have that

$$DX^{(q-1)} \simeq \Sigma^{-(n-1)}\Sigma^\infty(S^n - X^{(q-1)})$$

and that the equivalence is induced by the pairing 10.19.

Now write

$$X = X^{(q-1)} \cup_{\alpha_1} D^q \cup_{\alpha_2} \cdots \cup_{\alpha_r} D^q$$

where $\alpha_1, \cdots, \alpha_r : \partial D^q = S^{q-1} \to X^{(q-1)}$ are the attaching maps.

For ease of notation we will assume that $r = 1$, which is to say $X = X^{(q-1)} \cup_\alpha D^q$. The general case, i.e when X has an arbitrary finite number q-cells can be handled in the same way.

Let $\tilde{X}$ be the space obtained from $X^{(q-1)}$ by attaching a thin cylinder $S^{q-1} \times [1-\epsilon, 1]$ via the map $\alpha : S^{q-1} \times \{1\} \to X^{(q-1)}$. $\tilde{X}$ is homotopy equivalent to $X^{(q-1)}$, so we know that

$$D\tilde{X} \simeq \Sigma^{-(n-1)}\Sigma^\infty(S^n - \tilde{X}). \tag{10.24}$$

If we let $\tilde{\alpha} : S^{q-1} \to \tilde{X}$ be the inclusion

$$\tilde{\alpha} : S^{q-1} \times \{1-\epsilon\} \subset S^{q-1} \times [1-\epsilon, 1] \subset \tilde{X}$$

we then have a description

$$X = \tilde{X} \cup_{\tilde{\alpha}} D^q.$$

In particular the composition $S^{q-1} \xrightarrow{\tilde{\alpha}} \tilde{X} \subset X \subset S^n$ is an embedding.

Now consider the cofibration sequence

$$S^{q-1} \xrightarrow{\tilde{\alpha}} \tilde{X} \to X$$

It's Spanier - Whitehead dual gives a cofibration sequence of spectra

$$D(S^{q-1}) \xleftarrow{D(\tilde{\alpha})} D(\tilde{X}) \leftarrow D(X)$$

We also have the commutative diagram of spectra

$$\begin{array}{ccccc}
\Sigma^{n-1}\Sigma^\infty(S^n - \tilde{\alpha}(S^{q-1})) & \longleftarrow & \Sigma^{n-1}\Sigma^\infty(S^n - \tilde{X}) & \longleftarrow & \Sigma^{n-1}\Sigma^\infty(S^n - X) \\
\simeq \downarrow \mu & & \simeq \downarrow \mu & & \downarrow \mu \\
D(S^{q-1}) & \xleftarrow{D(\tilde{\alpha})} & D(\tilde{X}) & \longleftarrow & D(X)
\end{array}$$

In order to prove that the right vertical map $\mu : \Sigma^{n-1}\Sigma^\infty(S^n - X) \to D(X)$ is a weak homotopy equivalence, it suffices to show that the top row

$$\Sigma^{n-1}\Sigma^\infty(S^n - \tilde{\alpha}(S^{q-1})) \leftarrow \Sigma^{n-1}\Sigma^\infty(S^n - \tilde{X}) \leftarrow \Sigma^{n-1}\Sigma^\infty(S^n - X)$$

is a cofibration sequence of spectra. This is because if that were the case, then the above diagram would be a map between cofibration sequences, where two of the terms are equivalences. This would imply that the third term is an equivalence.

We can see that this sequence is a cofibration sequence at the level of spaces. Namely, since $\tilde{\alpha} : S^{q-1} \to X \to X \to S^n$ is an embedding, it's complement can be described by

$$S^n - \tilde{\alpha}(S^{q-1}) = (S^n - \tilde{X}) \sqcup_{S^n - X} (S^n - D^q).$$

Thus the cofiber of $(S^n - \tilde{X}) \to (S^n - \tilde{\alpha}(S^{q-1}))$ is the quotient $(S^n - D^q)/(S^n - X)$. But since $S^n - D^q$ is contractible, we have $\Sigma^\infty(S^n - D^q)/(S^n - X) \simeq \Sigma^\infty\Sigma(S^n - X)$. In other words, $\Sigma^{n-1}\Sigma^\infty(S^n - \tilde{\alpha}(S^{q-1})) \leftarrow \Sigma^{n-1}\Sigma^\infty(S^n - \tilde{X}) \leftarrow \Sigma^{n-1}\Sigma^\infty(S^n - X)$ is a cofibration sequence of spectra. As mentioned before this is what was needed to complete the proof. □

We often have a situation where the embedding of a finite complex X is given inside a Euclidean space, $X \subset \mathbb{R}^n \subset \mathbb{R}^n \cup \infty = S^n$. So it is natural to ask how the the homotopy type of the complement $\mathbb{R}^n - X$ is related to the Spanier-Whitehead dual. For this notice that

$$\mathbb{R}^n - X = S^n - (X_+))$$

where the disjoint basepoint in X_+ is embedded in S^n as the point at ∞. So Theorem 10.43 has the following corollary.

Corollary 10.45. *Let X be a finite CW-complex regularly embedding in $\mathbb{R}^n$. then there are weak homotopy equivalences of spectra*

$$\begin{aligned}\Sigma^{-(n-1)}\Sigma^\infty(\mathbb{R}^n - X) = \Sigma^{-(n-1)}\Sigma^\infty(S^n - X_+) &\simeq D(X_+)\\ &\simeq D(X_+)\\ &= D(X) \vee \mathbb{S}\end{aligned}$$

An important result regarding the topology of manifolds, proved by Atiyah in [6], relates the Thom spectrum of the normal bundle of an embedding into Euclidean space, $e : M^n \hookrightarrow \mathbb{R}^N$, to the Spanier-Whitehead dual of M^n. This duality property is sometimes known as "Atiyah duality", and it now follows quickly from the generalized version of Alexander duality that we've proved (Theorem 10.43) and its Corollary 10.45.

Theorem 10.46. *(Atiyah [6]) Let M^n be a closed n-dimensional manifold and $e : M^n \hookrightarrow \mathbb{R}^{n+k}$ an embedding with normal bundle $\nu_e^k \to M^n$ and tubular neighborhood η_e. Then there is a weak homotopy equivalence of spectra*

$$\Sigma^\infty T(\nu_e^k) \simeq \Sigma^{n+k} D(M^n_+)$$

Proof. Recall that we have a homeomorphism of the Thom space of the normal bundle,

$$(M^n)^{\nu_e^k} = \nu_e^k \cup \infty \cong \eta_e \cup \infty \cong \mathbb{R}^{n+k}/(\mathbb{R}^{n+k} - \eta_e).$$

But since $\mathbb{R}^{n+k}$ is contractible there is a homotopy equivalence

$$\mathbb{R}^{n+k}/(\mathbb{R}^{n+k} - \eta_e) \simeq \Sigma(\mathbb{R}^{n+k} - \eta_e).$$

So we have

$$(M^n)^{\nu_e^k} \simeq \Sigma(\mathbb{R}^{n+k} - \eta_e) \simeq \Sigma(\mathbb{R}^{n+k} - M^n).$$

The theorem now follows from Corollary 10.45. □

As in the constructions of the maps yielding Alexander duality (Theorem 10.43) one can give a conceptual, explicit map yielding Atiyah duality, Suppose the tubular neighborhood η_e of M^n in $\mathbb{R}^{n+k}$ is small enough so that every point $y \in \eta_e$ has Euclidean distance less that some number $\epsilon > 0$. Now consider the subtraction map

$$\begin{aligned} M^n \times \mathbb{R}^{n+k} &\xrightarrow{\alpha} \mathbb{R}^{n+k} \\ (x, v) &\to e(x) - v \end{aligned} \tag{10.25}$$

This map restricts to give a map

$$M^n \times \left(\mathbb{R}^{n+k} - \eta_e\right) \xrightarrow{\alpha} \mathbb{R}^{n+k} - B_\epsilon^{n+k}$$

where B_ϵ^{n+k} is the open ball around the origin $\mathbb{R}^{n+k}$ of radius ϵ. We therefore have a map of the quotient space

$$M^n \times \mathbb{R}^{n+k}/M^n \times \left(\mathbb{R}^{n+k} - \eta_e\right) \xrightarrow{\alpha} \mathbb{R}^{n+k}/\left(\mathbb{R}^{n+k} - B_\epsilon^{n+k}\right) \cong S^{n+k}. \tag{10.26}$$

The left hand quotient space is equal to the smash product with a disjoint basepoint, $M_+^n \wedge \mathbb{R}^{n+k}/(\mathbb{R}^{n+k} - \eta_e)$ which in turn, via the tubular neighborhood theorem, is homeomorphic to $M_+^n \wedge M^{\nu^k}$. Thus this subtraction map defines a map

$$\alpha : M_+^n \wedge M^{\nu^k} \to S^{n+k}$$

and therefore map of spectra, which by abuse of notation we still call α,

$$\alpha : M_+^n \wedge \Sigma^{-(n+k)}\Sigma^\infty(M^{\nu^k}) \to \Sigma^\infty(S^0) = \mathbb{S}. \tag{10.27}$$

We leave it to the reader to verify, by running through the above proof, that this subtraction map α yields the Spanier-Whitehead duality between M_+^n and the Thom space of the normal bundle $\Sigma^{-(n+k)}\Sigma^\infty(M^{\nu^k})$.

Inspired by this we make the following definition.

Definition 10.29. *Let M^n be a closed n-dimensional manifold embedded in Euclidean space $M^n \subset \mathbb{R}^{n+k}$ with normal bundle $\nu^k \to M^n$. Define the spectrum M^{-TM} to be the desuspension of the Thom space*

$$M^{-TM} = \Sigma^{-(n+k)}\Sigma^\infty(M^{\nu^k}).$$

Exercise. Show that the homotopy type of the spectrum M^{-TM} does not depend on the choice of embedding.

Atiyah duality can then be restated as follows.

Corollary 10.47. *There is an equivalence of spectra*

$$\begin{aligned} M^{-TM} &\simeq D(M^n_+) \\ &= Map(M^n_+, \mathbb{S}) \end{aligned}$$

We now observe that the Spanier-Whitehead dual of any space X with a disjoint basepoint, $D(X_+) = Map(X_+, \mathbb{S})$ has a natural ring spectrum structure. This is because there is a natural diagonal map

$$\Delta : X_+ \to (X \times X)_+ = X_+ \wedge X_+ \tag{10.28}$$

and of course a (commutative) ring structure on the sphere spectrum

$$\mathbb{S} \wedge \mathbb{S} \xrightarrow{=} \mathbb{S}.$$

This allows us to make the following ring structure on $D(X_+) = Map(X_+, \mathbb{S})$:

$$\begin{aligned} \mu : Map(X_+, \mathbb{S}) \wedge Map(X_+, \mathbb{S}) &\xrightarrow{\gamma} Map(X_+ \wedge X_+, \mathbb{S} \wedge \mathbb{S}) \\ &\xrightarrow{\Delta^*} Map(X_+, \mathbb{S} \wedge \mathbb{S}) \xrightarrow{=} Map(X_+, \mathbb{S}) \end{aligned} \tag{10.29}$$

where $\gamma(\phi_1 \wedge \phi_2)(x_1 \wedge x_2) = \phi_1(x_1) \wedge \phi_2(x_2)$ and $\Delta^*(\psi)(x) = \psi(\Delta(x)) = \psi(x \wedge x)$.

Actually this ring structure is commutative in the sense of [48] essentially because the diagonal map is cocommutative and because the ring structure on $\mathbb{S}$ is commutative. We refer the reader to [48] for a discussion of commutative ring (symmetric) spectra.

Notice that by Corollary 10.47 this ring structure translates to give the Thom spectrum M^{-TM} the structure of a commutative ring structure. In [22] he author described an explicit ring structure on M^{-TM} defined in terms of an embedding $M^n \hookrightarrow \mathbb{R}^{n+k}$.

10.9 Eilenberg-MacLane spectra and the Steenrod algebra

When we first introduced the notion of a spectrum toward the beginning of this chapter, one of the first examples described was that of the Eilenberg-MacLane spectrum $\mathbb{H}G$, where G is an abelian group. The n^{th} space in this spectrum $(\mathbb{H}G)_n$ is an Eilenberg-MacLane space

$$(\mathbb{H}G)_n = K(G, n).$$

The main reason that these spaces, and the resulting spectrum are so important is that they classify ordinary (co)homology, In particular, if X is any based space of the homotopy type of a CW-complex

$$H^n(X;G) \cong [X, K(G,n)]. \tag{10.30}$$

Because the Eilenberg-Steenrod axioms are satisfied, this means that, via Brown's Representability Theorem 10.16 and Whitehead's Theorem 10.18, that the Eilenberg-MacLane spectrum $\mathbb{H}G$ represents ordinary (co)homology with G-coefficients. This means that given any pair of spaces $A \subset X$ of the homotopy type of CW complexes

$$H^q(X,A;G) \cong [X/A, \Sigma^q \mathbb{H}G] \quad \text{and} \quad H_q(X,A;G) \cong \pi_q(X/A \wedge \mathbb{H}G).$$

In fact if $\mathbb{E}$ is a spectrum its homology and cohomology also satisfy

$$H^q(\mathbb{E};G) \cong [\mathbb{E}, \Sigma^q \mathbb{H}G] \quad \text{and} \quad H_q(\mathbb{E};G) \cong \pi_q(\mathbb{E} \wedge \mathbb{H}G).$$

In particular notice that

$$H^*(\mathbb{H}G;G) \cong [\mathbb{H}G, \mathbb{H}G]^*$$

where the superscript $*$ represents the degree of the maps to be taken. That is, $[\mathbb{H}G, \mathbb{H}G]^q = [\mathbb{H}G, \Sigma^q \mathbb{H}G]$.

This observation describes a special case of the generalized cohomology of a representing spectrum. Namely, suppose $\mathbb{E}$ is a spectrum representing a generalized cohomology theory E^*. Then

$$E^*(\mathbb{E}) = [\mathbb{E}, \mathbb{E}]^*.$$

When $\mathbb{E}$ is a ring spectrum these cohomology groups form a ring via composition. Indeed they form an algebra over the ground ring $E_* = E_*(point) = \pi_0(\mathbb{E})$. As we will see, the importance of this algebra is due to the fact that it forms the algebra of E^*-cohomology operations, in a sense that we will now make precise.

10.9.1 Cohomology operations

Recall that according to Definition 10.8, a generalized cohomology theory E^* consists of a collection of functors from the category of CW-pairs CW_2 to the category of abelian groups $\mathcal{G}$, as well as a collection of natural "coboundary" homomorphisms $\delta^q : E^q(A) \to E^{q+1}(X,A)$ for any CW-pair (X,A). satisying the Homotopy, Excision, and Exactness Eilenberg-Steenrod axioms.

Definition 10.30. *Let E^* be a generalized cohomology theory. An E^*-cohomology operation of degree k is a collection of natural transformations*

$\alpha^q : E^q \to E^{q+k}$ that respect the coboundary homomorphisms. That is, for any CW pair (X, A),

$$\delta^{q+k}\alpha^q(x) = \alpha^{q+1}\delta^q(x)$$

for any $x \in E^q(A)$.

Given a cohomology operation $\alpha = \{\alpha^q\}$ of degree k, we simply write

$$\alpha : E^* \to E^{*+k}.$$

Remark. The type of cohomology operations we are considering are sometimes referred to as "stable cohomology operations" since our definition implies that such a cohomology operation commutes with the suspension isomorphism $E^*(X) \cong E^*(\Sigma X, point)$.

Exercise. Verify this statement. That is, verify that according to our definition of a cohomology operation, such an operation commutes with the suspension isomorphism.

In this book we are mostly concerned with cohomology operations for ordinary cohomology with coefficients in $\mathbb{Z}/p$ where p is a prime number. Notice that the set of these operations form an algebra over the field $\mathbb{Z}/p$. The multiplication is given by composition of cohomology operations. This algebra is called the "**mod p Steenrod algebra"** which we denote by $\mathcal{A}_p$.

Notice that if (X, A) is any pair in CW_2, its cohomology $H^*(X, A; \mathbb{Z}/p)$ forms a module over the Steenrod algebra $\mathcal{A}_p$, under application of cohomology operations. This structure is extremely important in homotopy theory, and so we explore it further here.

The following is the basic connection between the Steenrod algebra $\mathcal{A}_p$ of mod p cohomology operations, and the mod p Eilenberg-MacLane spectrum.

Theorem 10.48. *There is an isomorphism of algebras over $\mathbb{Z}/p$*

$$\phi : \mathcal{A}_p \xrightarrow{\cong} H^*(\mathbb{HZ}/p; \mathbb{Z}/p) \cong [\mathbb{HZ}/p, \mathbb{HZ}/p]^*.$$

Proof. (Sketch). In some ways the proof of this theorem is formal. We suggest the book by Mosher and Tangora [78] for details.

Let $a \in \mathcal{A}_p$ be an element of degree k. Since a is a cohomology operation, it acts on the mod p cohomology of every space, and in particular of Eilenberg-MacLane spaces. So $a \in \mathcal{A}_p$ defines homomorphisms

$$a_n : H^n(K(\mathbb{Z}/p, n); \mathbb{Z}/p) \to H^{n+k}(K(\mathbb{Z}/p; n); \mathbb{Z}/p)$$

for every $n \geq 0$.

Now $H^n(K(\mathbb{Z}/p, n); \mathbb{Z}/p) = \mathbb{Z}/p$ so consider the image of the generator $a_n(\iota) \in H^{n+k}(K(\mathbb{Z}/p; \mathbb{Z}/p)$. Since cohomology is classified by Eilenberg-MacLane spaces, we can represent these cohomology classes by maps which are well-defined up to homotopy, which by abuse of notation we call

$$a_n : K(\mathbb{Z}/p, n) \to K(\mathbb{Z}/p, n+k)$$

for each n.

Furthermore, and the reader should check this, because the cohomology operation $a \in \mathcal{A}_p$ respects the suspension homomorphism, the following diagrams homotopy commutes:

$$\begin{array}{ccc} K(\mathbb{Z}/p, n) & \xrightarrow{a_n} & K(\mathbb{Z}/p, n+k) \\ \simeq \downarrow & & \downarrow \simeq \\ \Omega K(\mathbb{Z}/p, n+1) & \xrightarrow[\Omega a_{n+1}]{} & \Omega K(\mathbb{Z}/p, n+k+1). \end{array}$$

The a_n's then fit together to give a map of ω-spectra, which by abuse of notation we again call

$$a : \mathbb{HZ}/p \to \Sigma^k \mathbb{HZ}/p,$$

We leave it to the reader to check that this map of spectra is well-defined up to homotopy, and this correspondence defines a map of graded algebras,

$$\phi : \mathcal{A}_p \to [\mathbb{HZ}/p; \mathbb{HZ}/p]^*.$$

To see that this map is an isomorphism, we note that the above procedure is completely reversible. Namely, given $\alpha \in [\mathbb{HZ}/p; \mathbb{HZ}/p]^k$, we represent α by a map of ω-spectra, which defines maps

$$\alpha_n : K(\mathbb{Z}/p, n) \to K(\mathbb{Z}/p, n+k)$$

such that the following diagrams commute:

$$\begin{array}{ccc} K(\mathbb{Z}/p, n) & \xrightarrow{\alpha_n} & K(\mathbb{Z}/p, n+k) \\ \simeq \downarrow & & \downarrow \simeq \\ \Omega K(\mathbb{Z}/p, n+1) & \xrightarrow[\Omega \alpha_{n+1}]{} & \Omega K(\mathbb{Z}/p, n+k+1). \end{array} \tag{10.31}$$

If $(X, A)$is any CW-pair, then by composition the maps α_n then define maps

$$[X/A, K(\mathbb{Z}/p, n)] \xrightarrow{\alpha_n} [X/A, K(\mathbb{Z}/p, n+k)]$$
$$H^n(X/A; \mathbb{Z}/p) \xrightarrow{\alpha_n} H^{n+k}(X/A; \mathbb{Z}/p)$$

We leave it to the reader to check that the commutativity of the squares (10.31) says that these operations are homomorphisms commute with the suspension homomorphism, and therefore define a cohomology operation $\psi(\alpha) \in \mathcal{A}_p$, We leave it to the reader to fill in the details of this argument and to check that $\phi : \mathcal{A}_p \to [\mathbb{HZ}/p, \mathbb{HZ}/p]^*$ and $\psi : [\mathbb{HZ}/p, \mathbb{HZ}/p]^* \to \mathcal{A}_p$ are both algebra homomorphisms that are inverse to each other.

□

Historically, the Steenrod algebra $\mathcal{A}_p$ was discovered in two main steps. First, in approximately 1950, N. Steenrod described cohomology operations Sq^i with coeffiencts in $\mathbb{Z}/2$ that became known as "Steenrod squares" and he studied many of their properties. His student J. Adem found the multiplicative relations the Steenrod operations satisfied. Steenrod produced similarly defined cohomology operations $\mathcal{P}^i$ with coefficients in $\mathbb{Z}/p$ for p an odd prime, which he called "reduced powers". The reduced powers were shown to satisfy similar "Adem relations".

The second step, which we see is necessary by Theorem 10.48, is a calculation of $[\mathbb{HZ}/p, \mathbb{HZ}/p]^*$. To do this one needs to compute the cohomology of the Eilenberg-MacLane spaces, $H^*(K(\mathbb{Z}/p, n); \mathbb{Z}/p)$. This was carried out by Cartan and Serre. A very nice account of that calculation is given in [78]. It is a beautiful example of a calculation using Serre's spectral sequence. In any case, the result of these calculations was that the Steenrod algebra $\mathcal{A}_p (= [\mathbb{HZ}/p, \mathbb{HZ}/p]^*)$ is precisely the algebra generated by the Steenrod squares at $p = 2$, and by the reduced powers together with the "Bockstein operator" $\beta : H^q(X, A; \mathbb{Z}/p) \to H^{q+1}(X, A; \mathbb{Z}/p)$ when p is odd.

To understand the Bockstein operator, recall that given any short exact sequence of abelian groups

$$0 \to H \xrightarrow{\iota} G \xrightarrow{p} K \to 0 \tag{10.32}$$

there is an associate long exact sequence of cohomology groups,

$$\xrightarrow{\delta} H^q(X, A; H) \xrightarrow{\iota_*} H^q(X, A; G) \xrightarrow{p_*} H^q(X, A; K) \xrightarrow{\delta} H^{q+1}(X, A; H) \xrightarrow{\iota_*} \dots$$

The connecting homomorphism $\delta : H^q(X, A; K) \to H^{q+1}(X, A; H)$ is known as the *"Bockstein operator"* associated to the short exact sequence (10.32). Of particular importance are the Bockstein operators associated to the short exact sequence

$$0 \to \mathbb{Z}/p \to \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0$$

for p a prime.

Exercise. Show that the Bockstein operator $\beta : H^q(X, A; \mathbb{Z}/p) \to H^{q+1}(X, A; \mathbb{Z}/p)$ associated to this short exact sequence is a cohomology operation in the sense of Definition 10.30.

10.9.2 The axioms and some consequences

In the famous book by Steenrod and Epstein on cohomology operations [91], they showed that the Steenrod squaring operations satisfy the following axioms, and moreover, they are completely characterized by these axioms.

Axioms. (10.33)

1. There are cohomology operations in the sense of Definition 10.30 known as "Steenrod squares"
$$Sq^i : H^n(-;\mathbb{Z}/2) \to H^{n+i}(-;\mathbb{Z}/2)$$
for all integers $i \geq 0$.
2. $Sq^0 = 1$ the identity transformation
3. $Sq^i(x) = 0$ if the dimension of x is less than i
4. $Sq^i(x) = x^2$ if the dimension of x equals i
5. The Steenrod squares satisfy the product formula known as the "Cartan formula":
$$Sq^i(xy) = \sum_j (Sq^j x)(Sq^{i-j} y).$$
6. Sq^1 is the Bockstein homomorphism associated to the coefficient sequence
$$0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0.$$
7. The Steenrod squares satisfy the "Adem relations":
For $a < 2b$,
$$Sq^a Sq^b = \sum_j \binom{b-j-1}{a-2j} Sq^{a+b-j} Sq^j$$
where the binomial coefficients are taken mod 2.

Axioms (6) and (7) can be shown to be consequences of axioms (1)-(5). Since they commute with the suspension isomorphism, the Steenrod operations act on the cohomology of spectra as well as spaces.

A consequence of Cartan and Serre's calculation of the cohomology of the Eilenberg-MacLane spaces $H^*(K(\mathbb{Z}/2,n);\mathbb{Z}/2)$ and the resulting calculation of the cohomology of the Eilenberg-MacLane spectra, $H^*(\mathbb{HZ}/2;\mathbb{Z}/2) = [\mathbb{HZ}/2, \mathbb{HZ}/2]^*$ one has the following theorem.

Theorem 10.49. *The algebra of $\mathbb{Z}/2$-cohomology operations $\mathcal{A}_2 = [\mathbb{HZ}/2, \mathbb{HZ}/2]^*$ is the algebra over $\mathbb{Z}/2$ generated by the Steenrod squaring operations Sq^i subject to the Adem relations.*

In this book we will mostly be concerned with mod 2 cohomology operations, but in Steenrod and Epstein's book [91] they also describe the following mod p cohomology operations for p an odd prime.

Let p be an odd prime and let

$$\beta : H^q(X, A; \mathbb{Z}/p) \to H^{q+1}(X, A; \mathbb{Z}/p)$$

be the Bockstein operator associated to the short exact sequence

$$0 \to \mathbb{Z}/p \to \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0.$$

We have the following axioms:

Axioms. (10.34)

1. There are cohomology operation in the sense of Definition 10.30

$$P^i : H^q(X, A; \mathbb{Z}/p) \to H^{q+2i(p-1)}(X, A; \mathbb{Z}/p)$$

   known as "Steenrod reduced power operations" for all integers $i \geq 0$.

2. $P^0 = 1$ the identity transformation
3. If $dim(x) = 2k$, then $P^k(x) = x^p$.
4. If $2k > dim(x)$, then $P^k(x) = 0$.
5. The reduced power operations satisfy a product formula known as the "Cartan formula":

$$P^k(xy) = \sum_i P^i(x)P^{k-i}(y).$$

6. The reduced powers satisfy the "Adem relations"": If $a < pb$ then

$$P^aP^b = \sum_{t=0}^{[a/p]} (-1)^{a+t} \binom{(p-1)(b-t)-1}{a-pt} P^{a+b-t}P^t.$$

   If $a \leq b$ then

$$P^a\beta P^b = \sum_{t=0}^{[a/p]} (-1)^{a+t} \binom{(p-1)(b-t)}{a-pt} \beta P^{a+b-t}P^t + \sum_{t=0}^{[(a-1)/p]} (-1)^{a+t-1} \binom{(p-1)(b-t)-1}{a-pt-1} P^{a+b-t}\beta P^t.$$

Again, by the calcuation of Cartan and Serre of the cohomology of the Eilenberg-MacLane spaces $H^*(K(\mathbb{Z}/p, n); \mathbb{Z}/p)$ and the resulting calculation of the cohomology of the Eilenberg-MacLane spectrum, $H^*(\mathbb{HZ}/p; \mathbb{Z}/p) \cong [\mathbb{HZ}/p, \mathbb{HZ}/p]^*$ one has the following theorem:

Theorem 10.50. *For p an odd prime, the algebra of $\mathbb{Z}/p$-cohomology operations $\mathcal{A}_p = [\mathbb{HZ}/p, \mathbb{HZ}/p]^*$ is the algebra over $\mathbb{Z}/p$ generated by the Steenrod reduced power operations P^i and the Bockstein operator β, subject to the Adem relations.*

As it turns out, the axioms for the Steenrod squares and Steenrod's reduced power operations completely characterize these cohomology operations (see [91] for a verification). We now observe that calculations can be directly made using these axioms.

Proposition 10.51. *. Let X be a space and $u \in H^1(X; \mathbb{Z}/2)$. Then*

$$Sq^i(u^k) = \binom{k}{i} u^{k+i}.$$

Proof. If $k = 0$, then the proposition follows immediately from Axioms 2 and 3 given in (10.33). Now we use induction on k, and observe that

$$\begin{aligned} Sq^i(u^k) &= Sq^i(u \cdot u^{k-1}) = Sq^0(u) \cdot Sq^i(u^{k-1}) + Sq^1 u \cdot Sq^{i-1}(u^{k-1}) \\ &= \left[\binom{k-1}{i} + \binom{k-1}{i-1}\right] u^{k+i} = \binom{k}{i} u^{k+i}. \end{aligned}$$

□

For ease of notation let $\mathbb{P}$ denote the infinite dimensional real projective space $\mathbb{P} = \mathbb{RP}^\infty$. Recall that its cohomology is the polynomial algebra,

$$H^*(\mathbb{P}; \mathbb{Z}/2) \cong \mathbb{Z}/2[a]$$

where $a \in H^1(\mathbb{P}; \mathbb{Z}/2)$. Proposition 10.51 then gives a complete calculation of $H^*(\mathbb{P}; \mathbb{Z}/2)$ as a module over $\mathcal{A}_2$.

10.9.3 Basic algebraic properties

We now discuss some basic properties of the Steenrod algebra $\mathcal{A}_p$. For a more detailed discussion we refer the reader to the book by Steenrod and Epstein [91].

We begin with a purely combinatorial identity which is extremely useful in making calculations with the Steenrod algebra.

Proposition 10.52. *Let p be a prime and let $a = \sum_{i=0}^k a_i p^i$ and $b = \sum_{i=0}^k b_i, p^i \quad (0 \leq a_i, b_i < p)$. Then*

$$\binom{b}{a} = \prod_{i=0}^{m} \binom{b_i}{a_i} \quad \textit{mod } p.$$

We leave the proof of this proposition as an exercise for the reader, or the reader can refer to [91] for a proof. The main observation needed for the proof is that because for $0 < i < p$, the binomial coefficient $\binom{p}{i}$ is congruent to zero mod p, and so

$$(1+x)^p = 1 + x^p \quad \text{mod } p,$$

and by induction,

$$(1+x)^{p^i} = 1 + x^{p^i} \quad \text{mod } p$$

for all i.

We now focus our attention on the mod 2 Steenrod algebra, $\mathcal{A}_2$.

Given a finite sequence of nonnegative integers, $I = (i_1, \cdots, i_k)$, k is called the *length* of I, $k = \ell(I)$. We write

$$Sq^I = Sq^{i_1} \cdots Sq^{i_k}.$$

We say that a sequence I is *admissible* if $i_q \geq 2i_{q+1}$ for $q = 1, \cdots, k-1$ and if $i_k \geq 1$.

Theorem 10.53. *The collection $\{Sq^I : I \text{ is admissible}\}$ forms a $\mathbb{Z}/2$-vector space basis for $\mathcal{A}_2$.*

Proof. Given a sequence $I = (i_1, \cdots, i_k)$ we define its *moment* to be

$$m(I) = \sum_{q=1}^{k} k i_k.$$

We first show that any Sq^I, for any inadmissible sequence I is a sum of Sq^J's where the sequences J have smaller moment than I. This will show that the admissible monomials span the Steenrod algebra.

Let $I = (i_1, \cdots, i_k)$ be an inadmissible sequence with no zeros. Then for some q, $i_q < 2i_{q+1}$. Now by the Adem relations,

$$Sq^I = Sq^L Sq^{i_q} Sq^{i_{q+1}} Sq^M = \sum_j a_j Sq^L Sq^{i_q + i_{q+1} - j} Sq^j Sq^M$$

where $a_j \in \mathbb{Z}/2$. It is easy to check that each of the monomials in the sum have smaller moment than $m(I)$. Thus the admissible monomials span $\mathcal{A}_2$

In order to prove that the admissible monomials in $\mathcal{A}_2$ are linearly independent we need the following lemma, which involves an independently interesting calculation.

Let $\mathbb{P}^n$ denote the n-fold cartesian product of the infinite dimensional projective space $\mathbb{P}$ with itself. Let $w = a \times \cdots \times a \in H^n(\mathbb{P}^n; \mathbb{Z}/2)$. Notice that the following lemma will prove that the admissible monomials are linearly independent, and will complete the proof of this theorem.

Lemma 10.54. *The map $\mathcal{A}_2 \to H^*(\mathbb{P}^n; \mathbb{Z}/2)$ defined by*

$$Sq^I \to Sq^I(w)$$

sends admissible monomials of dimension $\leq n$ to linearly independent elements.

Proof. We prove this lemma by induction on n. For $n = 1$ it follows from the fact that $Sq^1(a) = a^2 \neq 0$. So we now assume the lemma is true for $n-1$. Our goal is to prove it for n. So suppose that

$$\sum_I a_I Sq^I(w) = 0$$

where the sum is taken over monomials of a fixed dimension q, where $q \leq n$. Our job is to prove that this implies that the coefficients a_I are all zero. We do this by decreasing induction on the length $\ell(I)$. Suppose that $a_I = 0$ for $\ell(I) > k$. We can rewrite the above equality as

$$\sum_{\ell(I)=k} a_I Sq^I(w) + \sum_{\ell(I)<k} a_I Sq^I(w) = 0. \tag{10.35}$$

Now the Kunneth formula says that

$$H^{q+n}(\mathbb{P}^n; \mathbb{Z}/2) \cong \sum_s H^s(\mathbb{P}; \mathbb{Z}/2) \otimes H^{q+n-s}(\mathbb{P}^{n-1}; \mathbb{Z}/2).$$

Let π be the projection onto the summand with $s = 2^k$. Let $w = u \times w'$, where $w' \in H^{n-1}(\mathbb{P}^{n-1}; \mathbb{Z}/2)$ is the generator. Then by the Cartan formula

$$Sq^I(w) = Sq^I(a \times w') = \sum_{I \leq J} Sq^J(u) \times Sq^{I-J}(w') \tag{10.36}$$

where $J \leq I$ means that $0 \leq j_r \leq i_r$ for all r. Let J_k be the sequence $(2^{k-1}, \cdots, 2^1, 2^0)$.

We claim that

$$\pi Sq^I(w) = \begin{cases} 0 \quad \text{if} \quad \ell(I) < k \\ a^{2^k} \times Sq^{I-J_k}(w') \quad \text{if} \quad \ell(I) = k. \end{cases} \tag{10.37}$$

To see this, notice that since $dim\, a = 1$, $Sq^J(a) = 0$ unless J has the form $(2^{q-1}, 2^{q-2}, \cdots, 2, 1)$ for some q. We call this sequence J_q. Notice furthermore $Sq^{J_q}(a) = a^{2^q}$.

To prove (10.37), notice that if $\ell(I) < k$ then $J \leq I$ implies that $\ell(J) < k$ and so $\pi Sq^i(w) = 0$. If $\ell(I) = k$, then $\pi(Sq^J(a) \times Sq^{I-J}(w')) = 0$ unless $J = J_k \leq I$. This verifies (10.37).

If we apply π to equation (10.35) and use (10.37), we find that

$$a^{2^k} \times \sum_{\ell(I)=k} a_I Sq^{I-J_k}(w') = 0. \tag{10.38}$$

Now one can easily check that as I ranges over all admissible sequences of length k and dimension q, $I - J_k$ will range over all admissible sequences of length $\leq k$, and dimension $q - 2^k + 1$, and the correspondence is one-to-one. Since $k \geq 1$, we have that $q - 2^k + 1 \leq n - 1$. So the inductive assumption on n implies that each coefficient in equation (10.38) is zero. Thus $a_I = 0$ for $\ell(I) = k$. This completes the proof of the lemma and therefore of Theorem 10.53. □

□

We now have an additive basis for the Steenrod algebra $\mathcal{A}_2$. Our next goal is to find a convenient set of multiplicative generators.

Recall that if A is an associative graded algebra, the set of decomposable elements of A is the image of the multiplication map,

$$\mu : A \otimes A \to A.$$

This image is a two-sided ideal, and the quotient,

$$Q(A) = A/\mu(A \otimes A)$$

is called the set of *indecomposable elements* of A. Our next goal is to compute the set of indecomposable elements in the Steenrod algebra, $\mathcal{A}_2$.

Lemma 10.55. *The Steenrod square Sq^i is indecomposable if and only if i is not a power of 2.*

Proof. We write the Adem relations in the form

$$\binom{b-1}{a} Sq^{a+b} = Sq^a Sq^b + \sum_{j>0} \binom{b-1-j}{a-2j} Sq^{a+b-j} Sq^j$$

where $0 < a < 2b$. One then immediately sees that if $\binom{b-1}{a} = 1 \in \mathbb{Z}/2$, then Sq^{a+b} is decomposable. Now suppose that i is not a power of 2. Then there is a unique k such that $i = a + 2^k$, $0 < a < 2^k$. Let $b = 2^k$. Then $b - 1 = 1 + 2 + \cdots + 2^{k-1}$. But as is immediate from Proposition 10.52, $\binom{b-1}{a} = 1 \in \mathbb{Z}/2$. Thus Sq^i is decomposable.

To prove the converse, let $i = 2^k$. Suppose by way of contradiction that Sq^{2^k} is decomposable, so we can write

$$Sq^{2^k} = \sum_{j=1}^{2^k-1} m_j Sq^j.$$

Then if $u \in H^1(\mathbb{P}; \mathbb{Z}/2) = \mathbb{Z}/2$ is the generator, we would have that

$$u^{2^{k+1}} = Sq^{2^k} u^{2^k} = \sum_{j=1}^{2^k-1} m_j Sq^j(u^{2^k}) = 0$$

since for $1 \leq j \leq 2^k - 1$, $dim\, Sq^j < dim\, u^{2^k}$. This contradiction completes the proof of the lemma. □

From this we now have a set of multiplicative generators of $\mathcal{A}_2$:

Theorem 10.56. *The elements Sq^{2^k} multiplicatively generate $\mathcal{A}_2$.*

10.9.4 The Hopf Invariant

We now describe a classical application of the the Steenrod algebra to the homotopy groups of spheres. In particular we study the question of the existence of elements of the homotopy groups of spheres having *Hopf invariant* one.

Given a map $\phi : S^{2n-1} \to S^n$, it's *Hopf invariant*, $h(\phi)$ is defined as follows. Consider the mapping cone,

$$C(\phi) = S^n \cup_\phi D^{2n}$$

where here D^{2n} represents the closed disk of dimension $2n$ which is attached to S^n along its boundary $\partial D^{2n} = S^{2n-1}$ via the map ϕ. That is, $C(\phi)$ is the CW complex built out of the union of S^n with D^{2n}, subject to the identification of $x \in S^{2n-1} = \partial D^{2n}$ with $\phi(x) \in S^n$.

Now compute in mod 2 cohomology

$$\tilde{H}^q(C(\phi); \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & \text{if } q = n \text{ or } 2n \\ 0 & \text{otherwise} \end{cases}$$

Let $\sigma_n \in H^n(C(\phi); \mathbb{Z}/2)$ and $\sigma_{2n} \in H^{2n}(C(\phi); \mathbb{Z}/2)$ be the generators. Now take the cup square,

$$\sigma_n^2 = \epsilon \cdot \sigma_{2n} \in H^{2n}(C(\phi)\mathbb{Z}/2)$$

where $\epsilon \in \mathbb{Z}/2$. Then the Hopf invariant of ϕ is defined to be the coefficient

$$h(\phi) = \epsilon \in \mathbb{Z}/2.$$

Notice that we could have equivalently defined the Hopf invariant $h(\phi)$ by

$$Sq^n(\sigma_n) = h(\phi)\sigma_{2n}.$$

Exercises.

1. Show that the Hopf invariant is a homotopy invariant. That is, if ϕ_1 and $\phi_2 : S^{2n-1} \to S^n$ are homotopic, then $h(\phi_1) = h(\phi_2)$, and moreover,

$$h : \pi_{2n-1}(S^n) \to \mathbb{Z}/2$$

is a homomorphism.

2. Extend the definition of the Hopf invariant(s) to the stable homotopy groups of spheres

$$h_k : \pi_{k-1}(\mathbb{S}) \to \mathbb{Z}/2,$$

where if $\psi : \Sigma^{k-1}\mathbb{S} \to \mathbb{S}$ represents a class in $\pi_{k-1}(\mathbb{S})$, and it has mapping cone $C(\psi)$, then define $h_k(\psi)$ by the equation in cohomology

$$Sq^k(\sigma_0) = h_k(\psi) \cdot \sigma_k$$

where σ_0 and σ_k are the generators of $H^q(C(\psi)\mathbb{Z}/2)$ in dimensions *zero* and k respectively.

Show that $h_k : \pi_{k-1}(\mathbb{S}) \to \mathbb{Z}/2$ is well-defined.

3. Consider the self map of the sphere spectrum $t : \mathbb{S} \to \mathbb{S}$ of degree 2. That is, $t \in \pi_0(\mathbb{S}) = \mathbb{Z}$ represents $2 \in \mathbb{Z}$. Show that t has Hopf invariant one,

$$h_1(t) = 1 \in \mathbb{Z}/2.$$

We now describe an immediate application of the Steenrod algebra to the problem of the existence of elements of the stable homotopy groups of spheres having Hopf invariant one.

Theorem 10.57. *If there exists an element $\phi \in \pi_{k-1}(\mathbb{S})$ with Hopf invariant $h_k(\phi) = 1 \in \mathbb{Z}/2$, then k is a power of 2.*

Proof. Suppose $\phi \in \pi_{k-1}(\mathbb{S})$ has Hopf invariant one. Then in the mapping cone $C(\phi)$, $Sq^k(\sigma_0) = \sigma_k \in H^i(C(\phi); \mathbb{Z}/2)$. Suppose k is not a power of 2. The by Lemma 10.55, Sq^k is indecomposable. So we may write

$$Sq^k = \sum_{j=1}^{k-1} a_j b_j$$

where for each j, the dimension of b_j is equal to j, where $1 \le j \le k-1$. Now $Sq^k(\sigma_0) \neq 0$ implies that $b_j(\sigma_0) \neq 0$ for some j. But $b_j(\sigma_0) \in H^j(C(\phi); \mathbb{Z}/2) = 0$ since $1 \le j \le k-1$. This contradiction implies the theorem. $\square$

This result has an important application to the question of the existence of certain multiplicative structures on spheres and on Euclidean spaces.

Let S_i, $i = 1, 2, 3$ be spheres of dimension $n-1$, and suppose one has a pairing

$$\mu : S_1 \times S_2 \to S_3.$$

We say that μ has bidegree (α, β) if the restriction of μ to $S_1 \times x_2$ has degree α and the restriction of μ to $x_1 \times S_2$ has degree β, where $x_i \in S_i$ are base-points. Notice that the degree is independent of the choices of $x_i \in S_i$. We observe that if we think of $S^1 \in \mathbb{C}$ as the unit complex numbers, then complex multiplication defines a map

$$\mu_1 : S^1 \times S^1 \to S^1$$

of bidegree $(1, 1)$. Similiarly multiplication of quaternions defines a map $\mu_3 : S^3 \times S^3 \to S^3$ and multiplication of the octonians defines a map $\mu_7 : S^7 \times S^7 \to S^7$, both having bidegree $(1, 1)$.

Now go back to the general case of a map $\mu : S_1 \times S_2 \to S_3$ of bidegree (α, β). Let D_i, $i = 1, 2, 3$ be closed n-dimensional disks so that

$$\partial D_i = S_i.$$

Notice that $\partial(D_1 \times D_2) = (S_1 \times D_2) \cup (D_1 \times S_2)$ which is a $(2n-1)$ dimensional sphere, and that

$$(D_1 \times S_2) \cap (S_1 \times D_2) = S_1 \times S_2.$$

Consider the suspension ΣS_3 which is an n-dimensional sphere. This suspension consists of an upper and lower cone which we denote by C_+ and C_-. These are n-dimensional cells with $C_+ \cap C_- = S_3$. We extend the map $\mu : S_1 \times S_2 \to S_3$ to a map

$$C(\mu) : (D_1 \times S_2) \cup (S_1 \times D_2) \to C_+ \cup C_- = \Sigma S_3 \cong S^n$$

in such a way that $C(\mu)(D_1 \times S_2) \subset C_+$ and $C(\mu)(S_1 \times D_2) \subset C_-$. (We leave it to the reader to verify that such and extension can be produced.) Then $C(\mu)$ is a map

$$C(\mu) : S^{2n-1} \to S^n.$$

We now prove the following theorem about this construction.

Theorem 10.58. *The Hopf invariant of the map $C(\mu) : S^{2n-1} \to S^n$ is the product of the components of the bidegree mod 2:*

$$h(C(f)) = \alpha\beta \in \mathbb{Z}/2.$$

Proof. (See [91]) The product of the disks $D_1 \times D_2$ has boundary equal to $(D_1 \times S_2) \cup (S_1 \times D_2)$. So may consider the space

$$X = (D_1 \times D_2) \cup_{C(\mu)} S^n$$

where the attaching is along the boundary $\partial(D_1 \times D_2)$ via the map $C(\mu)$. Notice that the attaching map gives rise to a map of triples

$$g : (D_1 \times D_2, D_1 \times S_2, S_1 \times D_2) \to (X, C_+, C_-).$$

Let $u \in H^n(X; \mathbb{Z}/2) = \mathbb{Z}/2$ be the generator. Define u_+ and u_- to be the inverse images of u under the isomorphisms $H^n(X, C_+; \mathbb{Z}/2) \xrightarrow{\cong} H^n(X; \mathbb{Z}/2)$ and $H^n(X, C_-; \mathbb{Z}/2) \xrightarrow{\cong} H^n(X; \mathbb{Z}/2)$ respectively. Consider the commutative diagram (all coefficients are taken to be $\mathbb{Z}/2$)

$$\begin{array}{ccc} H^n(X) \otimes H^n(X) & \xrightarrow{\times} & H^{2n}(X) \\ \cong\uparrow & & \uparrow\cong \\ H^n(X, C_+) \otimes H^n(X, C_-) & \xrightarrow{\times} & H^{2n}(X, \Sigma S_3) \end{array}$$

Thus the cup product $u_+ \cup u_-$ has image u^2 under the map $H^{2n}(X; \Sigma S_3) \to H^{2n}(X)$.

Now easy diagram chases, that we leave to the reader (or refer to [91]) show that the map of triples g restricts to maps in cohomology

$$g^* : H^n(X, C_-) \to H^n(D_1 \times D_2, S_1 \times D_2) \quad \text{and}$$
$$g^* : H^n(X, C_+) \to H^n(D_1 \times D_2, D_1 \times S_2)$$

such that $g^*(u_+) = \alpha v_+$ and $g^*(u_-) = \beta v_-$ where $v_+ \in H^n(D_1 \times D_2, S_1 \times D_2)$ and $v_- \in H^n(D_1 \times D_2, D_1 \times S_2)$ are the generators. Notice that v_+ determines a class $v_1 \in H^n(D_1, S_1)$ and v_- determines a class $v_2 \in H^n(D_2, S_2)$ under the obvious projection maps. Now

$$v_+ \cup v_- = (v_1 \times 1) \cup (1 \times v_2) = v_1 \times v_2.$$

Therefore

$$g^*(u_+) \cup g^*(u_-) = \alpha\beta(v_1 \times v_2)$$

and $(v_1 \times v_2)$ generates $H^{2n}(D_1 \times D_2; D_1 \times S_2 \cup S_1 \times D_2)$.

Now $g : (D_1 \times D_2, D_1 \times S_2 \cup S_1 \times D_2) \to (X, \Sigma S_3)$ is a relative homeomorphism and so induces an isomorphism in cohomology. So we have isomorphisms

$$H^{2n}(X) \xleftarrow{\cong} H^{2n}(X, \Sigma S_3) \xrightarrow{\cong} H^{2n}(D_1 \times D_2, D_1 \times S_2 \cup S_1 \times D_2)$$

Under these isomorphisms, $u^2 \in H^{2n}(X)$ corresponds to $u_- \cup u_+ \in H^{2n}(X; \Sigma S_3)$ and to $\alpha\beta(u_1 \times u_2) \in H^{2n}(D_1 \times D_2, D_1 \times S_2 \cup S_1 \times D_2)$. Since $(u_1 \times u_2) \in H^{2n}(D_1 \times D_2, D_1 \times S_2 \cup S_1 \times D_2)$ corresponds to the generator of $H^{2n}(X)$ under these isomorphisms, this completes the proof of the theorem. □

Notice that Theorem 10.58 says that if there is a pairing

$$\mu : S^{n-1} \times S^{n-1} \to S^{n-1}$$

that has bidegree $(1,1)$ then the resulting construction $C(\mu) : S^{2n-1} \to S^n$ has Hopf invariant one. But by Theorem 10.57 we know that this cannot happen unless n is a power of 2. That is to say we have the following application of these results.

Corollary 10.59. *If there is a pairing $\mu : S^{n-1} \times S^{n-1} \to S^{n-1}$ of bidegree $(1,1)$, then $n = 2^k$ for some $k \geq 0$.*

Finally we remark that if $\mathbb{R}^n$ has the structure of a division algebra (even a non-associative one) then its unit sphere S^{n-1} would admit a pairing of bidigree $(1,1)$. This is given by the restriction of the multiplication map

$$\mu : S^{n-1} \times S^{n-1} \subset (\mathbb{R}^n - \{0\}) \times (\mathbb{R}^n - \{0\}) \xrightarrow{multiply} \mathbb{R}^n - \{0\} \xrightarrow{\simeq} S^{n-1}$$

where the last map is the homotopy equivalence given by radial retraction of $\mathbb{R}^n - \{0\}$ onto the unit sphere. Notice that the image of the multiplication of two nonzero elements of $\mathbb{R}^n$ is nonzero is because a division algebra contains no zero divisors.

Exercise. Show that the pairing of the unit sphere S^{n-1} described above when $\mathbb{R}^n$ is a not-necessarily commutative or associative division algebra, has bidegree $(1,1)$.

From these arguments we know that the only dimensions in $\pi_{n-1}(\mathbb{S})$ that can possibly contain elements of Hopf invariant one are when $n = 2^k$ for some $k \geq 0$. In one of the most striking algebraic topology results of the 20th century, J. F. Adams showed that there are no elements of $\pi_{n-1}(\mathbb{S})$ of Hopf invariant one unless $n = 1, 2, 4, 8$ [2]. In particular this means that the only dimensions n for which $\mathbb{R}^n$ can have the structure of a division algebra are $n = 1, 2, 4$, or 8. Of course such structures in these dimensions are well known: the real numbers when $n = 1$, the complex numbers when $n = 2$, the Hamiltonians when $n = 4$, and the octonions when $n = 8$. What was startling was that sophisticated techniques from algebraic topology could be used to show that no such structures exist in other dimensions.

Adams's technique for the solution of this problem is what became known as the Adams spectral sequence. We will say more about this spectral sequence later in this book.

10.9.5 Definitions

So far our discussion of the Steenrod algebra and its applications were based on the assumption that the Steenrod squares and reduced powers exist, and satisfy the axioms 10.33 and 10.34. We now give a quick definition of the Steenrod squaring operations.

Given a space X, consider the diagonal mapping $\Delta : X \to X \times X$. This map is clearly equivariant with respect to the trivial $\mathbb{Z}/2$- action on the source X and the action on the target $X \times X$ given by permuting the coordinates. Indeed Δ embeds X as the subspace of fixed points of this action. By this equivariance we can extend this map (which by abuse of notation we also call Δ),

$$\Delta : E\mathbb{Z}/2 \times_{\mathbb{Z}/2} X \to E\mathbb{Z}/2 \times_{\mathbb{Z}/2} (X \times X) \tag{10.39}$$
$$B\mathbb{Z}/2 \times X \xrightarrow{\Delta} E\mathbb{Z}/2 \times_{\mathbb{Z}/2} (X \times X).$$

In this notation $E\mathbb{Z}/2$ refers to the total space of the universal principal bundle

$$\mathbb{Z}/2 \to E\mathbb{Z}/2 \to B\mathbb{Z}/2$$

a model of which can be taken to be the $\mathbb{Z}/2$-covering space

$$\mathbb{Z}/2 \to S^\infty \to \mathbb{RP}^\infty.$$

The subscript $\mathbb{Z}/2$ under the product sign means taking the orbit space of the induced diagonal action on the product space. Notice that since in the source space the action on X is trivial,

$$E\mathbb{Z}/2 \times_{\mathbb{Z}/2} X = B\mathbb{Z}/2 \times X \simeq \mathbb{RP}^\infty \times X.$$

One way to define the mod 2 Steenrod squares is by computing this map in cohomology. To do this, we recall that S^∞ has a $\mathbb{Z}/2$-equivariant cell decomposition with two cells e_i and e_i' in each dimension i. The $\mathbb{Z}/2$-action interchanges these two cells. Let $C_*(S^\infty)$ be the resulting cellular chain complex with coefficients in $\mathbb{Z}/2$. Since S^∞ is contractible and its $\mathbb{Z}/2$-action is free, $C_*(S^\infty)$ is a free acyclic resolution of the ground field $\mathbb{Z}/2$ as a module over the group ring $\mathcal{R}_2 = \mathbb{Z}/2[\mathbb{Z}/2]$. Explicitly it is the complex

$$\to \cdots \to C_i \xrightarrow{\partial_i} C_{i-1} \xrightarrow{\partial_{i-1}} \cdots \xrightarrow{\partial_0} C_0 \xrightarrow{\epsilon} \mathbb{Z}/2$$

where C_i is the 2-dimensional vector space generated by e_i and e_i' where the $\mathbb{Z}/2$ action interchanges these generators. That is, if $t \in \mathbb{Z}/2$ is the nonzero element, the the module structure of C_i is given by $t \cdot e_i = e_i'$ and $t \cdot e_i' = e_i$. The boundary homomorphism is given by

$$\partial_i(x) = (1+t)x$$

for every i.

Exercise. Show that this complex is a free acyclic resolution of the ground field $\mathbb{Z}/2$ over $\mathcal{R}_2$.

If we let $\mathcal{S}_*(X)$ and $\mathcal{S}^*(X)$ respectively denote the singular chains and cochains with coefficients in $\mathbb{Z}/2$ of a space X, then using the Alexander-Whitney correspondence, which gives a chain equivalence $\mathcal{S}_*(X \times Y) \xrightarrow{\simeq}$

$\mathcal{S}_*(X) \otimes \mathcal{S}_*(Y)$, one sees that the cohomology $H^*(E\mathbb{Z}/2 \times_{\mathbb{Z}/2} (X \times X); \mathbb{Z}/2)$ can be computed using the cochain complex $C^*(S^\infty) \otimes_{\mathcal{R}_2} \mathcal{S}^*(X) \otimes \mathcal{S}^*(X)$. If $\alpha \otimes \beta \in \mathcal{S}^*(X) \otimes \mathcal{S}^*(X)$ then the $\mathcal{R}_2$ action is given by $t(\alpha \otimes \beta) = \beta \otimes \alpha$.

Exercises.

1. Verify this claim. That is, show that the cochain complex $C^*(S^\infty) \otimes_{\mathcal{R}_2} \mathcal{S}^*(X) \otimes \mathcal{S}^*(X)$ computes $H^*(E\mathbb{Z}/2 \times_{\mathbb{Z}/2} (X \times X); \mathbb{Z}/2)$.

2. Show that for any cohomology class $\alpha \in H^q(X; \mathbb{Z}/2)$ represented by a cocycle $\tilde{\alpha} \in \mathcal{S}^q(X)$, the class $1 \otimes \tilde{\alpha} \otimes \tilde{\alpha} \in C^*(S^\infty) \otimes_{\mathcal{R}_2} \mathcal{S}^*(X) \otimes \mathcal{S}^*(X)$ is a cocycle and so represents an element

$$1 \otimes \alpha \otimes \alpha \in H^{2q}(E\mathbb{Z}/2 \times_{\mathbb{Z}/2} (X \times X); \mathbb{Z}/2).$$

Verify that this correspondence gives a well-defined homomorphism

$$\omega : H^q(X; \mathbb{Z}/2) \to H^{2q}(E\mathbb{Z}/2 \times_{\mathbb{Z}/2} (X \times X); \mathbb{Z}/2).$$

Now consider the equivariant diagonal map in cohomology:

$$\Delta^* : H^*(E\mathbb{Z}/2 \times_{\mathbb{Z}/2} (X \times X); \mathbb{Z}/2) \to H^*(B\mathbb{Z}/2 \times X; \mathbb{Z}/2).$$

For $\alpha \in H^q(X; \mathbb{Z}/2)$, the Kunneth theorem says that we can write

$$\Delta^*(\omega(\alpha)) = \sum_{i=0}^{2q} a^i \otimes \beta_i$$

where $a \in H^1(B\mathbb{Z}/2; \mathbb{Z}/2) = H^1(\mathbb{RP}^\infty; \mathbb{Z}/2) = \mathbb{Z}/2$ is the generator, $a^i \in H^i(\mathbb{RP}^\infty; \mathbb{Z}/2)$ is the i-fold cup product, and $\beta_i \in H^{2q-i}(X; \mathbb{Z}/2)$ is some cohomology class. We define the Steenrod square Sq^{q-i} by letting

$$Sq^{q-i}(\alpha) = \beta_i.$$

A shorthand description of the above definition is

$$\Delta^*(1 \otimes \alpha \otimes \alpha) = \sum_{i=0}^{2q} a^i \otimes Sq^{q-i}(\alpha). \tag{10.40}$$

Exercise Show that this definition satisfies the following axiom from (10.33):
Axiom: If $\alpha \in H^q(X; \mathbb{Z}/2)$, $Sq^q(\alpha) = \alpha^2 \in H^{2q}(Xl\mathbb{Z}/2)$.

We now have a map $Sq^k : H^q(X; \mathbb{Z}/2) \to H^{q+k}(X; \mathbb{Z}/2)$ for each space X and for each q and k. Of course now one needs to check that this defines a cohomology operation satisfying all the Axioms 10.33. This is essentially done in [91]. By "essentially" we mean that the above description is a topological version of an algebraic definition of the Steenrod squares given by Steenrod and Epstein in [91]. A closely related approach using the notion of "cup - i" products is given in the book by Mosher and Tangora [78]. An elegant, more general approach to Steenrod operations is given by J. P. May in [62]. We encourage the reader to consult these sources for more thorough developments of the Steenrod algebras.

10.9.6 Free modules over $\mathcal{A}_p$

We end this section on the Steenrod algebra with an observation about what it means for the cohomology of a spectrum to be a *free module* over $\mathcal{A}_p$. From Theorem 10.48 we know that $H^*(\mathbb{HZ}/p; \mathbb{Z}/p)$ is isomorphic to the Steenrod algebra $\mathcal{A}_p$. More generally we can conclude the following:

Corollary 10.60. *Let $\mathbb{E}$ be a spectrum that is weakly homotopy equivalent to a wedge of suspensions of the Eilenberg-MacLane spectrum $\mathbb{HZ}/p$. More precisely, suppose there is a graded $\mathbb{Z}/p$ - vector space with basis $\mathcal{B} = \{\eta\}$ such that*

$$\mathbb{E} \simeq \bigvee_{\eta \in \mathcal{B}} \Sigma^{|\eta|} \mathbb{HZ}/p$$

where $|\eta|$ denotes the grading (dimension) of a basis element $\eta \in \mathcal{B}$. Then $H^*(\mathbb{E}; \mathbb{Z}/p)$ is a free module over the Steenrod algebra $\mathcal{A}_p$ with basis $\mathcal{B}$. That is,

$$H^*(\mathbb{E}; \mathbb{Z}/p) \cong \bigoplus_{\eta \in \mathcal{B}} \Sigma^{|\eta|} \mathcal{A}_p.$$

We now observe that the converse to this corollary is also true.

Theorem 10.61. *Let p be a prime and suppose $\mathbb{E}$ is a spectrum such that $H^*(\mathbb{E}; \mathbb{Z}/q) = 0$ for all primes $q \neq p$ and that $H^*(\mathbb{E}; \mathbb{Q}) = 0$. (Such a spectrum is called "p-local".) Suppose furthermore that $H^*(\mathbb{E}; \mathbb{Z}/p)$ is a free module over the Steenrod algebra $\mathcal{A}_p$, with a countable basis. Then $\mathbb{E}$ is weakly homotopy equivalent to a wedge of suspensions of the Eilenberg-MacLane spectrum $\mathbb{HZ}/p$.*

Proof. Let $\mathcal{B} = \{\eta\}$ be a basis for $H^*(\mathbb{E}; \mathbb{Z}/p)$ as an $\mathcal{A}_p$-module. By assumption it is countable. As cohomology classes these classes can be represented by maps of spectra, which we call

$$b_\eta : \mathbb{E} \to \Sigma^{|\eta|} \mathbb{HZ}/p.$$

Now recall that any spectrum $\mathbb{X}$ has a "pinch map" $\mathbb{X} \to \mathbb{X} \vee \mathbb{X}$. Since one can suspend and desuspend a spectrum, this pinch map can be viewed as being induced by applying the pinch map $S^1 \to S^1 \vee S^1$ in the suspension coordinate. (Check that the homotopy type of the pinch map on a spectrum $\mathbb{X}$ is well-defined.) By iterating the pinch map a countable number of times one then has a map

$$\bigvee_{\eta \in \mathcal{B}} b_\eta : \mathbb{E} \to \bigvee_{\eta \in \mathcal{B}} \Sigma^{|\eta|} \mathbb{HZ}/p.$$

It is immediate that this map induces an isomorphism in (co)homology with $\mathbb{Z}/p$-coefficients. Since both the source and target of this map have zero (co)homology with $\mathbb{Z}/q$-coefficients for q any prime other than p, and also have zero rational (co)homology, this means that this map induces an isomorphism in integral homology, and therefore by the Hurewicz theorem, in homotopy groups. □

11

Cobordism theory

Cobordism is a theory coming out of work of Pontrjagin and Thom which gives one of the most important connections between differential topology and stable homotopy theory. The Pontrjagin-Thom theorem basically says that to classify smooth manifolds up to "cobordism", perhaps with structure (eg an orientation, almost complex structure, framing), one needs to study the homotopy type of a corresponding spectrum. This theorem supplied a tremendously important computational tool in differential topology, while at the same time served as an important stimulus for the development of stable homotopy theory. In this chapter we prove the Pontrjagin-Thom theorem, and use it as Thom did [94], to compute the unoriented cobordism ring. Along the way we show that the unoriented cobordism Thom spectrum, traditionally denoted $\mathbb{MO}$, is built out of mod 2 Eilenberg-MacLane spectra (the spectrum corresponding to ordinary mod 2 cohomology). These spectra and the algebra of natural transformations between them, namely the Steenrod algebra $\mathcal{A}_2$, were introduced and studied in the last chapter. We use this study to show that the unoriented cobordism ring turns out to be a polynomial algebra, and we give explicit examples of manifolds representing generators of this algebra. We will also discuss other cobordism rings (oriented cobordism, almost complex cobordism, framed cobordism). We describe Milnor's famous and beautiful calculation of the almost complex cobordism ring. This uses the "*Adams spectral sequence*" which we will also discuss in this chapter. A theory such as framed cobordism is much more difficult to compute, but homotopy theoretic techniques lead to fascinating geometric consequences.

Pontrjagin and Thom's theory studies cobordism classes of closed manifolds. This is an equivalence classes based on the following equivalence relation: two closed n-manifolds M^n and N^n are *cobordant* if there is an $(n+1)$-dimensional manifold with boundary W^{n+1} whose boundary is the disjoint union,

$$\partial W^{n+1} = M^n \sqcup N^n.$$

In recent years, an exciting area of research has developed around the study of "*cobordism categories*". In such a category the objects are closed n-manifolds, and the morphisms between, M^n and W^n are all possible cobordisms between them. Work of Madsen and Weiss [58], and Galatius, Madsen, Tillmann, and Weiss [34] has lead to work of Galatius and Randal-Williams [35] which uses cobordism categories to study the topology of diffeomorphisms of manifolds in a stable sense, that we will make precise. We give an overview of this exciting area of current research toward the end of this chapter.

11.1 Studying cobordism via stable homotopy: the Pontrjagin-Thom Theorem

In the last chapter we presented Theorem 10.32 which describes the"Thom functor", which is a monoidal functor from the category of spaces over BO to the category of symmetric spectra,

$$Th : \mathcal{C}_{BO} \to Sp^{\Sigma}$$

that takes a map $X \to BO$ to its Thom spectrum X^f. We mentioned the example of the Thom spectrum of the identity map

$$BO \to BO$$

which is denoted $\mathbb{MO}$. Since the Thom functor is monoidal, and since the identity map of BO obviously preserves its multiplicative structure, $\mathbb{MO}$ is a ring spectrum. It can be viewed as being built out of the spaces $\{MO(n), n \geq 0\}$, which are the Thom spaces of the universal vector bundles over the classifying spaces $\{BO(n), n \geq 0\}$. The structure maps of this spectrum are maps

$$\epsilon_n : \Sigma MO(n) \to MO(n+1)$$

which are maps of Thom spectra induced by the usual inclusion maps $BO(n) \to BO(n+1)$.

The critical feature of the Thom spectrum $\mathbb{MO}$ is that by the following remarkable theorem of Thom, it's homotopy type describes cobordism classes of manifolds.

Theorem 11.1. *(Thom, [94] (1954)) There is an isomorphism between the homotopy groups of the Thom spectrum,*

$$\pi_n(\mathbb{MO}) = \lim_{k \to \infty} \pi_{n+k}(MO(k))$$

and the set of cobordism classes of closed n-manifolds, η_n. This is defined to be the set of equivalence classes of n-dimensional closed manifolds, defined by saying M_1^n is cobordant to M_2^n if there is an $(n+1)$ dimensional manifold with boundary, W^{n+1}, with

$$\partial W^{n+1} = M_1^n \sqcup M_2^n.$$

The abelian group structure on η_n corresponding to the group structure on stable homotopy groups is simply induced by disjoint union of manifolds. The identity element in this group is the empty set $\emptyset$ (by convention $\emptyset$ can be viewed as a manifold of any dimension). Notice that this group consists entirely of elements of order 2. One sees this fact by observing that for any closed n-manifold M^n, the disjoint union $M^n \sqcup M^n$ is cobordant to the empty set $\emptyset$ since it is the boundary of $W^{n+1} = M^n \times [0,1]$. Furthermore, the graded abelian groups $\eta_* \cong \pi_*^s(\mathbb{MO})$ is a graded ring (and hence an algebra over $\mathbb{Z}/2$), since $\mathbb{MO}$ is a ring spectrum, with the induced product on $\eta_* = \oplus_n \eta_n$ given by cartesian product of manifolds.

The main goal of this section is to give a proof of this fundamental theorem and its natural generalizations. In particular we will describe the analogue of this theorem in the setting of almost complex and framed cobordism. The example of framed cobordism was actually proved considerably earlier by Pontrjagin, and so the generalization of this theorem that we will prove is often referred to as the *Pontrjagin - Thom Theorem.*

We begin with Pontrjagin's construction.

Definition 11.1. *Let M^n be a closed, smooth manifold, and N^{n+k} be a smooth $(n+k)$-dimensional manifold (not necessarily closed). Suppose $e : M^n \hookrightarrow N^{n+k}$ is an embedding. A* **framing** *of this embedding is a an extension of e to an embedding $\tilde{e} : M^n \times D^k \hookrightarrow N^{n+k}$ that is a diffeomorphism onto its image. Here D^k denotes the unit open disk in $\mathbb{R}^k$.*

Exercises

1. Show that an embedding $e : M^n \hookrightarrow N^{n+k}$ has a framing if and only if the normal bundle $\nu_e \to M^n$ is a trivial k-dimensional vector bundle.

2. Show that a framing $\tilde{e} : M^n \times D^k \hookrightarrow N^{n+k}$ determines, and is determined by a vector bundle isomorphism

$$\Phi : \nu_e \xrightarrow{\cong} M^n \times \mathbb{R}^k.$$

3. Show that the standard embedding $e : S^n \hookrightarrow \mathbb{R}^{n+1}$ as the unit sphere, has a framing.

4. Show that the inclusion embedding $e : \mathbb{RP}^n \hookrightarrow \mathbb{RP}^{n+1}$ does *not* have a framing.

Given a framed embedding $\tilde{e} : M^n \hookrightarrow \mathbb{R}^{n+k}$ one can perform the "Pontrjagin- Thom construction" to define a map $\alpha_{\tilde{e}} : S^{n+k} \to S^k$.

$$\begin{aligned} \alpha_{\tilde{e}} : S^{n+k} = \mathbb{R}^{n+k} \cup \infty &\to \mathbb{R}^{n+k}/(\mathbb{R}^{n+k} - \tilde{e}(M \times D^k)) \\ &\cong (M^n \times D^k) \cup \infty \xrightarrow{project} D^k \cup \infty = S^k \end{aligned}$$

Here, when we write $\cup\infty$ we mean the one-point compactification. The map $\alpha_{\tilde{e}} : S^{n+k} \to S^n$ determines a class in the stable homotopy groups of spheres,

$$[\alpha_{\tilde{e}}] \in \pi_n(\mathbb{S}).$$

Conversely, suppose $[\alpha] \in \pi_n \mathbb{S}$ is represented by a smooth, basepoint preserving map $\alpha : S^{n+k} \to S^k$ for some k sufficiently large. Think of S^k as the one-point compactification $S^k = \mathbb{R}^k \cup \infty$ and assume $0 \in \mathbb{R}^k \subset S^k$ is a regular point of α. One loses no generality in this assumption since if this were not the case, then choose a regular value $x_0 \in S^k$ of α near 0, and then by composing α with a degree one map of the sphere that send x_0 to the origin and keeps $\infty \in S^k$ fixed, one produces a map that is homotopic to α for which the origin is a regular value. So we continue with this assumption. Recall that the preimage $\alpha^{-1}(0) \subset S^{n+k}$ is a closed manifold of dimension n. Indeed by compactness we know that it lies in $(S^{n+k} - \infty) = \mathbb{R}^{n+k}$. So one has a manifold

$$M^n = \alpha^{-1}(0) \subset \mathbb{R}^{n+k}.$$

Notice that this embedding is in fact framed. To see this notice that if $\epsilon > 0$ is sufficiently small, and $B_\epsilon(0) \subset \mathbb{R}^k$ is the ball of radius ϵ, then

$$M^n \times \mathbb{R}^k = \alpha^{-1}(0) \times \mathbb{R}^k \cong \alpha^{-1}(B_\epsilon(0)) \subset \mathbb{R}^{n+k}$$

is a framed embedding.

Exercise. Prove this assertion. Namely, show that $M^n \times \mathbb{R}^k = \alpha^{-1}(0) \times \mathbb{R}^k \cong \alpha^{-1}(B_\epsilon(0))$.

These constructions lead to the famous result giving a correspondence between the stable homotopy group of spheres, $\pi_n(\mathbb{S})$ and the group of "framed cobordism classes of n-dimensional closed, (stably) framed manifolds", η_n^{fr}. Rather than immediately make this precise and provide a proof, we will first prove Thom's Theorem 11.1, and then show how it generalizes to describe the cobordism groups of manifolds with any type of stable normal structure (which we define), including a framing, in terms of the homotopy groups of a corresponding Thom spectrum.

We now proceed with a proof of the Pontrjagin-Thom Theorem 11.1.

Proof. By differentiating, we get for every $x \in M^n$, two subspaces, $De_x(T_x M^n) \subset \mathbb{R}^{n+k}$ and $De_x(T_x M^n)^\perp \subset \mathbb{R}^{n+k}$. Of course the first of these is the tangent space $T_x M^n$ linearly embedded in $\mathbb{R}^{n+k}$, and the second is the normal space $\nu_x^k \subset \mathbb{R}^{n+k}$. Letting x vary over M^n defines continuous maps to Grassmannians,

$$\tau_e : M^n \to Gr_n(\mathbb{R}^{n+k}) \quad \text{and} \quad \nu_e : M^n \to Gr_k(\mathbb{R}^{n+k}). \tag{11.1}$$

Allowing the ambient vector spaces to get large we get maps

$$\tau_{M^n} : M^n \to Gr_n(\mathbb{R}^\infty) \simeq BO(n) \quad \text{and} \quad \nu_{M^n,e} : M^n \to Gr_k(\mathbb{R}^\infty) \simeq BO(k)$$

that classify the tangent bundle and the normal bundle respectively. Notice that the homotopy class of the map $\nu_{M^n,e}$ depends on the embedding e, but, as we have seen earlier, by taking the colimit we obtain the stable normal bundle map, $\nu_{M^n} : M^n \to BO$ which is an invariant of the smooth manifold M^n.

Going back to the original embedding, $e : M^n \hookrightarrow \mathbb{R}^{n+k}$, we may let η_e be a tubular neighborhood. The we can perform the "Thom collapse map"

$$\pi_e : S^{n+k} = \mathbb{R}^{n+k} \cup \infty \to S^{n+k}/(S^{n+k} - \eta_e) = \eta_e \cup \infty \cong T(\nu_e) \tag{11.2}$$

where, as earlier $T(\nu_e)$ denotes Thom space of the normal bundle ν_e. Recall that we have seen this construction earlier, as well as similar constructions when we discussed Alexander duality (Theorem 10.41) and Atiyah duality (Theorem 10.46).

Composing with the map of Thom spaces induced by the classifying map of the normal bundle,

$$T(\nu_e) \to MO(k)$$

we get the map

$$\alpha_e : S^{n+k} \xrightarrow{\pi_e} T(\nu_e) \to MO(k). \tag{11.3}$$

This map is known as the *"Pontrjagin - Thom construction"* on the embedding $e : M^n \hookrightarrow \mathbb{R}^{n+k}$. The resulting homotopy class $[\alpha_e] \in \pi_{n+k}(MO(k))$ depends on the embedding e. However if we let the codimension of the embedding get large, we get an element in the homotopy group of the spectrum $\mathbb{MO}$

$$\alpha_{M^n} = [\alpha_e] \in \lim_{k\to\infty} \pi_{n+k} MO(k) = \pi_n(\mathbb{MO}) \tag{11.4}$$

which does not depend on the embedding, basically because all embeddings are isotopic in sufficiently large codimensions. The next result shows that Pontrjagin - Thom class gives a well-defined homomorphism from the cobordism group to the homotopy groups of $\mathbb{MO}$.

Proposition 11.2. *1. A cobordism W^{n+1} between two closed manifolds M^n and N^n defnes a homotopy between their Pontrjagin-Thom constructions*

$$\alpha_{M^n} : S^{n+k} \to MO(k) \quad \textit{and} \quad \alpha_{N^n} : S^{n+k} \to MO(k)$$

for k sufficiently large.

2. The Pontrjagin-Thom construction for a disjoint union of closed n-manifolds, $M_1^n \sqcup M_2^n$ is homotopic to the sum of the Pontrjagin-Thom constructions of each component

$$\alpha_{M_1^n \sqcup M_2^n} = \alpha_{M_1^n} + \alpha_{M_2^n} \in \pi_n(\mathbb{MO}).$$

Proof. For part (1), assume W^{n+1} is a smooth, compact manifold with boundary and $\partial W^{n+1} = M^n \sqcup N^n$. By a relative version of Whitney's embedding theorem, we can find an embedding of manifolds with boundary,

$$e : W^{n+1} \hookrightarrow \mathbb{R}^{n+k} \times [0,1]$$

where

$$e(W^{n+1}) \cap (\mathbb{R}^{n+k} \times \{0\}) = e(M^n) \quad \text{and} \quad e(W^{n+1}) \cap (\mathbb{R}^{n+k} \times \{1\}) = e(N^n).$$

We now do the Pontrjagin-Thom construction for the embedding e:

$$\rho_e : \big((\mathbb{R}^{n+k} \times [0,1]) \cup \infty\big) \to (\mathbb{R}^{n+k} \times [0,1]) / \big((\mathbb{R}^{n+k} \times [0,1]) - \eta_e\big) \to MO(k) \tag{11.5}$$

where η_e is a tubular neighborhood of the embedding e. The relative Tubular Neighborhood Theorem states that this neighborhood is homeomorphic to a k-dimensional normal bundle $\nu_e^k \to W^{n+1}$, and the last map in this composition is the induced map on Thom spaces of the classifying map of ν_e^k. Also observe that the tubular neighborhood η_e has the property that

$$\eta_e^0 = \eta_e \cap (\mathbb{R}^{n+k} \times \{0\}) \quad \text{and} \quad \eta_e^1 = \eta_e \cap (\mathbb{R}^{n+k} \times \{1\})$$

are tubular neighborhoods of the restrictions of the embedding e to M^n and N^n respectively.

Notice that the one point compactification of $\mathbb{R}^{n+k} \times [0,1]$ is given by

$$(\mathbb{R}^{n+k} \times [0,1]) \cup \infty = \big(S^{n+k} \times [0,1]\big) / (\infty \times [0,1]),$$

where, as usual, we are thinking of S^{n+k} as $\mathbb{R}^{n+k} \cup \infty$. We can therefore think of the Pontrjagin-Thom construction ρ_e as a (base point preserving) homotopy between it's restrictions

$$\rho_e^0 : \big(\mathbb{R}^{n+k} \times \{0\}\big) \cup \infty \to \big(\mathbb{R}^{n+k} \times \{0\}\big) \cup \infty / \big((\mathbb{R}^{n+k} \times \{0\}) - \eta_e^0\big) \to MO(k)$$

and

$$\rho_e^1 : \big(\mathbb{R}^{n+k} \times \{1\}\big) \cup \infty \to \big(\mathbb{R}^{n+k} \times \{0\}\big) \cup \infty / \big((\mathbb{R}^{n+k} \times \{1\}) - \eta_e^1\big) \to MO(k)$$

But these maps are Pontrjagin-Thom constructions for the embeddings of the boundary components $e_{|_{M^n}} : M^n \hookrightarrow \mathbb{R}^{n+k} \times \{0\}$ and $e_{|_{N^n}} : M^n \hookrightarrow \mathbb{R}^{n+k} \times \{1\}$. Thus for k-sufficiently large these represent α_{M^n} and α_{N^n} respectively. Part (1) of this proposition now follows.

To prove Part (2) of the proposition, assume we have an embedding of the disjoint union

$$e_{M^n} = e_1 \sqcup e_2 : M_1^n \sqcup M_2^n \hookrightarrow \mathbb{R}^{n+k}.$$

Clearly the images of the components are disjoint. We may assume that the tubular neighborhood $\eta_e = \eta_1 \sqcup \eta_2$ where η_i, $i = 1, 2$, are tubular neighborhoods of the embeddings e_1 and e_2 respectively, and $\eta_1 \cap \eta_2 = \emptyset$. Furthermore, by translating one of these tubular neighborhoods if necessary, we can assume there are disjoint open balls, B_1^{n+k} and B_2^{n+k} in $\mathbb{R}^{n+k}$ containing the tubular neighborhoods $\eta_!$ and η_2 respectively.

Therefore the Thom collapse map

$$\pi_e : \mathbb{R}^{n+k} \cup \infty \to S^{n+k} / \left(S^{n+k} - \eta_e\right) \cong T(\nu_e)$$

factors up to homotopy as the composition

$$\pi_e : \mathbb{R}^{n+k} \cup \infty \to (B_1^{n+k} \cup \infty) \vee (B_2^{n+k} \cup \infty) \to$$
$$(B_1^{n+k} \cup \infty) / \left((B_1^{n+k} \cup \infty) - \eta_{e_1}\right) \vee (B_2^{n+k} \cup \infty) / \left((B_2^{n+k} \cup \infty) - \eta_{e_2}\right) \cong$$
$$T(\nu_{e_1}) \vee T(\nu_{e_2}) \cong T(\nu_e)$$

Notice that the first map in this composition is homotopic to the pinch map $p : S^{n+k} \to S^{n+k} \vee S^{n+k}$, and the second map in this composition is homotopic to the wedge of the Thom collapse maps

$$\pi_{e_1} \vee \pi_{e_2} : S^{n+k} \vee S^{n+k} \to T(\nu_{e_1}) \vee T(\nu_{e_2}).$$

Thus the Pontrjagin-Thom construction α_e is homotopic to the composition

$$\alpha_e : S^{n+k} \xrightarrow{p} S^{n+k} \vee S^{n+k} \xrightarrow{\pi_{e_1} \vee \pi_{e_2}} T(\nu_{e_1}) \vee T(\nu_{e_2}) \to MO(k) \vee MO(k) \to MO(k)$$

where the last map in this composition is the fold map (which exists for any space, $X \vee X \to X$). But this composition represents the sums of the homotopy classes

$$\alpha_{e_1} + \alpha_{e_2} \in \pi_{n+k}(MO(k)).$$

Thus

$$\alpha_{M_1^n \sqcup M_2^n} = \alpha_{M_1^n} + \alpha_{M_2^n} \in \pi_n(\mathbb{MO}).$$

□

We now have a well defined homomorphism

$$\alpha : \eta_n \to \pi_n(\mathbb{MO}).$$

In order to prove that it is an isomorphism, we exhibit an inverse homomorphism, $\rho : \pi_n(\mathbb{MO}) \to \eta_n$. We describe it's construction, but our description will not contain full details. It would be a valuable exercise for the reader to fill in the details, or (s)he may consult one of many references that give complete proofs, for example Thom's original paper [94] or the book by R. Stong [93].

Let $\theta \in \pi_n(\mathbb{MO})$ be represented by a basepoint preserving map $f_\theta : S^{n+k} \to MO(k)$. Here we are using the infinite Grassmannian $Gr_k(\mathbb{R}^\infty)$ to represent the classifying space $BO(k)$, and $MO(k)$ is the Thom space of the canonical bundle $\gamma_k \to Gr_k(\mathbb{R}^\infty)$. By the compactness of S^{n+k}, for sufficiently large $N > 0$, f_θ factors through a map, which by abuse of notation we also call

$$f_\theta : S^{n+k} \to T(\gamma_{k,N})$$

where $T(\gamma_{k,N})$ is the Thom space of the canonical bundle $\gamma_{k,N} \to Gr_k(\mathbb{R}^N)$, which we take to be the one-point compactification

$$T(\gamma_{k,N}) = \gamma_{k,N} \cup \infty.$$

Since f_θ must be a basepoint preserving map, we can take the basepoints of both $S^{n+k} = \mathbb{R}^{n+k} \cup \infty$ and $\gamma_{k,N} \cup \infty$ to be ∞. Notice that the total space of $\gamma_{k,N} \subset \gamma_{k,N} \cup \infty = T(\gamma_{k,N})$ is an open subspace which is a smooth manifold. Notice also that the inverse image $f_\theta^{-1}(\gamma_{k,N}) \subset S^{n+k}$ is an open submanifold. We may then assume that the restriction of f_θ to that inverse image,

$$f_\theta : f_\theta^{-1}(\gamma_{k,N}) \to \gamma_{k,N}$$

is a smooth map which is transverse to the zero section $Gr_k(\mathbb{R}^N) \hookrightarrow \gamma_{k,N}$. Notice that this zero section is a codimension k-submanifold of $\gamma_{k,N}$, and so its inverse image, $f_\theta^{-1}(Gr_k(\mathbb{R}^N)) \subset \mathbb{R}^{n+k} \subset S^{n+k}$ is a closed, codimension k-submanifold of $\mathbb{R}^{n+k} \subset S^{n+k}$. We call this n-dimensional manifold

$$M_\theta^n \subset \mathbb{R}^{n+k}.$$

We will define $\rho(\theta) = [M_\theta^n] \in \eta_n$.

Of course we need to show that $\rho : \pi_n(\mathbb{MO}) \to \eta_n$ is well-defined. But first we observe that if $e : M^n \hookrightarrow \mathbb{R}^{n+k} \subset S^{n+k}$ represents a class in the cobordism group η_n, then the Pontrjagin-Thom construction,

$$\alpha_{M^n} : S^{n+k} \xrightarrow{\pi_e} T(\nu_{M^n}) \xrightarrow{T(\nu_e)} T(\gamma_{k,n+k})$$

has the property that $\alpha_{M^n}^{-1}(M^n) \subset S^{n+k}$ is equal to $M^n \subset \mathbb{R}^{n+k}$. This is because, since $T(\nu_e) : T(\nu_{M^n}) \to T(\gamma_{k,n+k})$ is induced by a map of vector bundles which induces a vector space isomorphism along each fiber, the inverse image of the zero section of $T(\gamma_{k,n+k})$ is the zero section of $T(\nu_{M^n})$. That is, $T(\nu_{M^n})^{-1}(Gr_k(\mathbb{R}^{n+k})) = M^n$. Also, clearly the inverse image under the Thom collapse map π_e of M^n is $M^n \subset S^{n+k}$.

This observation tells us that

$$\rho \circ \alpha = identity_{\eta_n}. \tag{11.6}$$

Similarly if $\theta \in \pi_n(\mathbb{MO})$ is represented by $f_\theta : S^{n+k} \to T(\gamma_{k,N})$ as above, and $e : M^n \hookrightarrow \mathbb{R}^{n+k}$ is $f_\theta^{-1}(Gr_k(\mathbb{R}^N))$, then the Pontrjagin - Thom construction

$$\alpha_e : S^{n+k} \to T(\nu_e) \to T(\gamma_{k,N})$$

is clearly homotopic to f_θ. Thus $\alpha \circ \rho = identity_{\pi_n(\mathbb{MO})}$.

Thus once we know that the map $\rho : \pi_n(\mathbb{MO}) \to \eta_n$ is well-defined, we would know that it is an inverse to the Pontrjagin-Thom construction $\alpha : \eta_n \to \pi_n(\mathbb{MO})$ thus proving that α is an isomorphism.

The main step in showing that ρ is well defined, is to show that if f^0_θ and $f^1_\theta : S^{n+k} \to T(\gamma_{k,N})$ are homotopic maps, transverse to $Gr_k(\mathbb{R}^N)$, then their inverse images, $M^n_0 = (f^0_\theta)^{-1}(Gr_k(\mathbb{R}^N))$ and $M^n_1 = (f^1_\theta)^{-1}(Gr_k(\mathbb{R}^N))$ are cobordant. To see this, suppose

$$F_\theta : S^n \times [0,1] \to T(\gamma_{k,N})$$

be a homotopy between f^0_θ and f^1_θ. Again, assuming F_θ is transverse to the zero section $Gr_k(\mathbb{R}^N)$, its inverse image would be a $(n+1)$-dimensional manifold with boundary, W^{n+1} embedded in $\mathbb{R}^{n+k} \times [0,1]$, that would be a cobordism between $M^n_0 \subset \mathbb{R}^{n+k} \times \{0\}$ and $M^n_1 \subset \mathbb{R}^{n+k} \times \{1\}$. We leave it to the reader to fill in the details of this sketch of the proof that $\rho : \pi_n(\mathbb{MO}) \to \eta_n$ is well-defined. Once done, this will complete the proof of the Pontrjagin-Thom Theorem (11.1). □

Notice that Theorem 11.1 gives an isomorphism between each homotopy group $\pi_n(\mathbb{MO})$ and the corresponding cobordism group η_n. But recall that since $\mathbb{MO}$ is a homotopy commutative ring spectrum, its homotopy groups, $\pi_*(\mathbb{MO})$ for a graded commutative ring. Similarly the cobordism groups $\{\eta_n,\, n \geq 0\}$. fit together to give a graded ring

$$\eta_* = \bigoplus_{n=0}^{\infty} \eta_n$$

where the product structure is represented by the cartesian product of manifolds.

Exercise: Show that the cartesian product of manifolds induces a well-defined product structure on η_*, That is, show that if M_1 is cobordant to M_1', and M_2 is cobordant to M_2', then $M_1 \times M_2$ is cobordant to $M_1' \times M_2'$.

Proposition 11.3. *The Pontrjagin-Thom construction*

$$\alpha : \eta_* \to \pi_*(\mathbb{MO})$$

is an isomorphism of graded rings.

Proof. . Suppose $e_1 : M^n_1 \hookrightarrow \mathbb{R}^{n+k}$ and $e_2 : N^m_2 \hookrightarrow \mathbb{R}^{m+s}$ are smooth embeddings. Consider the Thom collapse maps,

$$\rho_1 : S^{n+k} = \mathbb{R}^{n+k} \cup \infty \to \mathbb{R}^{n+k}/(\mathbb{R}^{n+k} - \eta_{e_1}) = T(\nu_{e_1}) \quad \text{and}$$
$$\rho_2 : S^{m+s} = \mathbb{R}^{m+s} \cup \infty \to \mathbb{R}^{m+s}/(\mathbb{R}^{m+s} - \eta_{e_1}) = T(\nu_{e_2})$$

The Thom collapse map for the product $e_1 \times e_2 : M^n_1 \times M^m_2 \hookrightarrow \mathbb{R}^{n+k} \times \mathbb{R}^{m+s}$ makes the following diagram commute:

$$\begin{array}{ccccc}
(\mathbb{R}^{n+k}\times\mathbb{R}^{m+s})\cup\infty & \xrightarrow{\rho_{e_1\times e_2}} & (\mathbb{R}^{n+k}\times\mathbb{R}^{m+s})/((\mathbb{R}^{n+k}\times\mathbb{R}^{m+s})-(\eta_{e_1}\times\eta_{e_2})) & \xrightarrow{=} & T(\nu_{e_1\times e_2}) \\
=\downarrow & & \downarrow= & & \downarrow= \\
(\mathbb{R}^{n+k}\cup\infty)\wedge(\mathbb{R}^{m+s}\cup\infty) & \xrightarrow[\rho_{e_1}\wedge\rho_{e_2}]{} & \mathbb{R}^{n+k}/(\mathbb{R}^{n+k}-\eta_{e_1})\wedge\mathbb{R}^{m+s}/(\mathbb{R}^{m+s}-\eta_{e_2}) & \xrightarrow[=]{} & T(\nu_{e_1})\wedge T(\nu_{e_2})
\end{array}$$

Notice also that the classifying map of the normal bundle of $e_1 \times e_2$, $\nu_{e_1\times e_2}$ is given by the composition

$$M_1^n \times M_2^m \xrightarrow{\nu_{e_1}\times\nu_{e_2}} BO(k)\times BO(s) \xrightarrow{\mu} BO(k+s)$$

where μ represents the Whitney sum map. Therefore the following diagram of Thom spaces commutes:

$$\begin{array}{ccccc}
T(\nu_{e_1})\wedge T(\nu_{e_2}) & \xrightarrow{t\nu_{e_1}\wedge t\nu_{e_2}} & MO(k)\wedge MO(s) & \xrightarrow{t\mu} & MO(k+s) \\
=\downarrow & & & & \downarrow= \\
T(\nu_{e_1\times e_2}) & & \longrightarrow & & MO(k+s)
\end{array}$$

In this diagram when $\nu : X \to BO(n)$ is a classifying map, then $t\nu : T(\nu) \to MO(n)$ represents the induced map of Thom spaces.

Putting these two diagrams together means we have a commutative diagram of Pontrjagin-Thom constructions:

$$\begin{array}{ccccc}
\alpha_{M_1\times M_2} : S^{n+m+k+s} & \xrightarrow{\rho_{e_1\times e_2}} & T(\nu_{e_1\times e_2}) & \longrightarrow & MO(k+s) \\
=\downarrow & & \downarrow= & & \downarrow= \\
\alpha_{M_1}\cdot\alpha_{M_2} : S^{n+k}\wedge S^{m+s} & \xrightarrow[\rho_{e_1}\wedge\rho_{e_2}]{} & T(\nu_{e_1})\wedge T(\nu_{e_2}) & \longrightarrow & MO(k+s)
\end{array}$$

That is, the map $\alpha : \eta_* \to \pi_*(\mathbb{MO})$ satisfies

$$\alpha([M_1 \times M_2]) = \alpha([M_1]) \cdot \alpha([M_2])$$

thus proving that α is a ring homomorphism. Combining this with Theorem 11.1 implies that the Pontrjagin-Thom map α is an isomorphism of graded rings. □

11.2 Unoriented cobordism: Thom's calculation

Thom also did a complete calculation of these graded rings.

Theorem 11.4. *[94]*

$$\eta_* \cong \mathbb{Z}_2[b_2, b_4, b_5, \cdots, b_r, \cdots : r \neq 2^k - 1].$$

In other words, η_* is a polynomial algebra over the field $\mathbb{Z}/2$ with one generator b_r of dimension $r > 0$ so long as r is not of the form $2^k - 1$ for any integer $k > 0$.

In fact Thom gave a complete description of the homotopy type of the spectrum $\mathbb{MO}$.

Theorem 11.5. *[94] The spectrum $\mathbb{MO}$ has the homotopy type of a wedge of Eilenberg-MacLane spectra,*

$$\mathbb{MO} \simeq \bigvee_{\omega \in I} \Sigma^{|\omega|} \mathbb{HZ}/2$$

where the indexing set I consists of all monomials in $\mathbb{Z}/2[b_2, b_4, \cdots, b_r \cdots, : r \neq 2^k - 1]$. The notation $|\omega|$ refers to the dimension of the monomial $b_\omega \in \mathbb{Z}/2[b_2, b_4, \cdots, b_r \cdots, : r \neq 2^k - 1]$.

The main step in proving both Theorems 11.4 and 11.5 is to compute the cohomology $H^*(\mathbb{MO}; \mathbb{Z}/2)$ as a module over the Steenrod algebra $\mathcal{A}_2$.

Proposition 11.6. *$H^*(\mathbb{MO}; \mathbb{Z}/2)$ is a free module over $\mathcal{A}_2$.*

Proof. We begin this proof with a basic algebraic lemma below about *Hopf algebras* and coalgebras. Recall the following definition:

Definition 11.2. *A (graded) Hopf algebra $\mathcal{B}$ over a field k is both a unital, associative algebra and a counital coassociative coalgbra over k such that the coproduct map,*

$$\psi : \mathcal{B} \to \mathcal{B} \otimes \mathcal{B}$$

is a map of algebras.

We observe that the Steenrod algebra $\mathcal{A}_2$ is a Hopf algebra over $\mathbb{Z}/2$, where the coproduct is induced by the map on generators,

$$\psi(Sq^k) = \sum_{i=0}^{k} Sq^i \otimes Sq^{k-i}. \tag{11.7}$$

Exercise. Show that with the coproduct as defined above, $\mathcal{A}_2$ is a Hopf algebra. That is, show that the coproduct ψ is a map of algebras over $\mathbb{Z}/2$ defining a coalgebra structure on $\mathcal{A}_2$.

Hint. Let $\bar{\mathcal{A}}_2$ be the free algebra over $\mathbb{Z}/2$ generated by $Sq^i : i > 0$. There is a natural surjective map $\pi : \bar{\mathcal{A}}_2 \to \mathcal{A}_2$ sending Sq^i to Sq^i that has kernel generated by the Adem relations. The map ψ defined above defines an algebra homomorphism $\bar{\psi} : \bar{\mathcal{A}}_2 \to \mathcal{A}_2 \otimes \mathcal{A}_2$. You have to show that $\bar{\psi}$ vanishes on $ker\, \pi$. If you have trouble carrying this out, see the argument at the beginning of Chapter II of [91].

We make two more observations about the Hopf algebra structure of the Steenrod algebra, $\mathcal{A}_2$.

- $\mathcal{A}_2$ is a *connected* Hopf algebra. Recall that a Hopf algebra $\mathcal{A}$ over a field k is *connected* if it is connected as a coalgebra. This means that $\mathcal{A}$ has no nonzero terms of negative grading, and the counit map $\epsilon : \mathcal{A} \to k$ is an isomorphism in degree zero.

  Exercise. Show that in a connected Hopf algebra over k, the coproduct map satisfies
  $$\psi(a) = a \otimes 1 + 1 \otimes a + \sum_i a_i' \otimes a_i''$$
  where the degrees of all the terms a_i' and a_i'' in this summation are all positive. Notice that in the case of $\mathcal{A}_2$, this follows immediately from the definition of ψ on the generators Sq^k.

- As a coalgebra, $\mathcal{A}_2$ is cocommutative. That is, the following diagram commutes:
  $$\begin{array}{ccc} \mathcal{A}_2 & \xrightarrow{\psi} & \mathcal{A}_2 \otimes \mathcal{A}_2 \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle \tau} \\ \mathcal{A}_2 & \xrightarrow{\psi} & \mathcal{A}_2 \otimes \mathcal{A}_2 \end{array}$$
  where $\tau(a \times b) = b \otimes a$. This follows from the symmetry of the Cartan product formula upon which the coproduct ψ is based (11.7).

We now state and prove the basic algebraic lemma about Hopf algebras that we will need. Our proof is adapted from [99].

Lemma 11.7. *Let $\mathcal{A}$ be a connected Hopf algebra over a field k. Let $\mathcal{P}$ be a connected coalgebra over k which is a left $\mathcal{A}$-module and such that its coproduct map $\Delta : \mathcal{P} \to \mathcal{P} \otimes \mathcal{P}$ is a map of $\mathcal{A}$-modules. Let $u \in \mathcal{P}$ be the unique class of degree zero mapping to $1 \in k$ under the counit $\epsilon : \mathcal{P} \to k$. Consider the map*

$$\begin{aligned} \mu : \mathcal{A} &\to \mathcal{P} \\ a &\to a \cdot u. \end{aligned}$$

If the map μ is injective, then $\mathcal{P}$ is a free $\mathcal{A}$-module.

Proof. Let $\mathcal{A}^+$ be the submodule of $\mathcal{A}$ consisting of elements of positive degree.

Let $Q = \mathcal{P}/\mathcal{A}^+\mathcal{P}$. Consider a splitting splitting of k-vector spaces $\iota : Q \to \mathcal{P}$ of the natural projection map $\pi : \mathcal{P} \to Q$. Define

$$\begin{aligned} \phi : \mathcal{A} \otimes Q &\to \mathcal{P} \\ a \otimes q &\to a \cdot \iota(q). \end{aligned}$$

Clearly ϕ is a map of $\mathcal{A}$-modules. To prove the lemma we show that ϕ is an isomorphism.

We first show that ϕ is surjective. Notice that in degree zero ϕ is the identity map. Inductively assume that ϕ is surjective in all degrees less than k. Let $\alpha \in \mathcal{P}$ have degree k.

$$\begin{aligned} \pi(\alpha - \phi(1 \otimes \pi(\alpha)) &= \pi(\alpha - \iota(\pi(\alpha))) \\ &= \pi(\alpha) - \pi(\iota(\pi(\alpha))) \\ &= \pi(\alpha) - \pi(\alpha) \\ &= 0 \end{aligned}$$

So one can write

$$\alpha - \phi(1 \otimes \pi(\alpha)) = \sum a_i \alpha_i$$

where $a_i \in \mathcal{A}^+$ and $\alpha_i \in \mathcal{P}$. Notice that all of the α_i's have degree less than the degree of α, which is k. So by the inductive hypothesis we can find $x_i \in \mathcal{A} \otimes Q$ with $\phi(x_i) = \alpha_i$. This implies that

$$\alpha = \phi\left(1 \otimes \pi(\alpha) + \sum a_i x_i\right)$$

which proves surjectivity. To see that ϕ is injective, consider the sequence of $\mathcal{A}$-module maps:

$$\mathcal{A} \otimes Q \xrightarrow{1 \otimes \iota} \mathcal{A} \otimes \mathcal{P} \to \mathcal{P} \xrightarrow{\Delta} \mathcal{P} \otimes \mathcal{P} \xrightarrow{1 \otimes \pi} \mathcal{P} \otimes Q.$$

By tracing through these maps, one sees that the image of a class $a \otimes q$ is of the form $a \cdot u \otimes q$ plus elements of different bidegrees. So since the map

$$\begin{aligned} \mu : \mathcal{A} &\to \mathcal{P} \\ a &\to a \cdot u. \end{aligned}$$

is injective, then this composition is injective. But since $\phi : \mathcal{A} \otimes Q \to \mathcal{P}$ is the composition of the first two maps, it also is injective. This establishes that $\phi : \mathcal{A} \otimes Q \to \mathcal{P}$ is an isomorphism of $\mathcal{A}$-modules and hence $\mathcal{P}$ is a free $\mathcal{A}$-module. □

We want to make use of Lemma 11.7, by applying it to $H^*(\mathbb{MO}; \mathbb{Z}/2)$. We first observe that since $\mathbb{MO}$ is an associative, homotopy commutative ring spectrum, the multiplication map

$$\mu : \mathbb{MO} \wedge \mathbb{MO} \to \mathbb{MO}$$

induces a commutative algebra structure on it's homology, $\mu_* : H_*(\mathbb{MO};\mathbb{Z}/2) \otimes H_*(\mathbb{MO};\mathbb{Z}/2) \to H_*(\mathbb{MO};\mathbb{Z}/2)$, and a cocommutative coalgebra structure on it's cohomology

$$\mu^* : H^*(\mathbb{MO};\mathbb{Z}/2) \to H^*(\mathbb{MO};\mathbb{Z}/2) \otimes H^*(\mathbb{MO};\mathbb{Z}/2).$$

Notice furthermore that the comultiplication map μ^* is a map of $\mathcal{A}_2$-modules, since it is induced by a map of spectra. Furthermore $H^*(\mathbb{MO};\mathbb{Z}/2)$ is obviously a connected coalgebra because $\mathbb{MO}$ is a connected spectrum. Also, since $H^0(\mathbb{MO};\mathbb{Z}/2) = \mathbb{Z}/2$ and is generated by the Thom class $u \in H^0(\mathbb{MO};\mathbb{Z}/2)$, then by Lemma 11.7, in order to prove Proposition 11.6 it suffices to prove that the map

$$\begin{aligned} \phi : \mathcal{A}_2 &\to H^*(\mathbb{MO};\mathbb{Z}/2) \\ a &\to a \cdot u \end{aligned} \tag{11.8}$$

is injective.

To do this it suffices to work on the space level, to show that the map

$$\begin{aligned} \phi : \mathcal{A}_2 &\to H^*(MO(k);\mathbb{Z}/2) \\ a &\to a \cdot u_k \end{aligned}$$

is injective for $deg\, a \leq \rho(k)$ where $\rho : \mathbb{Z} \to \mathbb{Z}$ is some strictly increasing function of k.

Now consider the multiplication map $\mu_k : MO(1) \wedge \cdots \wedge MO(1) \to MO(k)$ where there are k-copies of the Thom space $MO(1)$ in this wedge product. This map is the induced map on Thom spaces of the product map on classifying spaces

$$BO(1) \times \cdots \times BO(1) \to BO(k).$$

The induced map in cohomology,

$$\mu_k^* : H^*(MO(k);\mathbb{Z}/2) \to H^*(MO(1) \wedge \cdots \wedge MO(1);\mathbb{Z}/2) \cong \tilde{H}^*(MO(1);\mathbb{Z}/2)^{\otimes k}$$

preserves Thom classes, so it suffices to show that the map

$$\begin{aligned} \phi : \mathcal{A}_2 &\to H^*(MO(1) \wedge \cdots \wedge MO(1);\mathbb{Z}/2) \cong \tilde{H}^*(MO(1);\mathbb{Z}/2)^{\otimes k} \\ a &\to a \cdot u_k \end{aligned} \tag{11.9}$$

is injective for $deg\, a \leq \rho(k)$ where $\rho(k)$ is an increasing function.

Now consider the homotopy type of the Thom space $MO(1)$. By definition,

$$MO(1) = D(\gamma^1)/S(\gamma^1)$$

where $\gamma^1 \to BO(1) = \mathbb{RP}^\infty$ is the universal line bundle and $D(\gamma^1)$ and $S(\gamma^1)$ are the associated unit disk and sphere bundles respectively. Clearly $D(\gamma^1)$ has the base space $\mathbb{RP}^\infty$ as a neighborhood deformation retract, so it is homotopy equivalent to $\mathbb{RP}^\infty$. On the other hand

$$S(\gamma^1) = \{(L, u) : L \subset \mathbb{R}^\infty \text{is a one dimensional subspace,} \\ \text{and}\, u \in L\, \text{has}\, \|u\| = 1\}.$$

But this is just the infinite dimensional sphere $S^\infty \subset \mathbb{R}^\infty$, which is contractible. Therefore we have a homotopy equivalence

$$MO(1) = D(\gamma^1)/S(\gamma^1) \simeq D(\gamma(1)) \simeq \mathbb{RP}^\infty. \tag{11.10}$$

With respect to this homotopy equivalence, the Thom class $u_1 \in H^1(MO(1); \mathbb{Z}/2)$ corresponds to the generator $a_1 \in H^1(\mathbb{RP}^\infty; \mathbb{Z}/2) = \mathbb{Z}/2$. Therefore under the product map

$$\mu_k^* : H^k(MO(k); \mathbb{Z}/2) \to H^k(MO(1)^{\wedge k}; \mathbb{Z}/2) \cong H^k((\mathbb{RP}^\infty)^{\wedge k}; \mathbb{Z}/2)$$

the Thom class u_k maps to $a_1^{\otimes k}$.

We therefore can complete the proof of Proposition 11.7 by proving the following.

Lemma 11.8. *The map*

$$\phi_k : \mathcal{A}_2 \to H^*(\mathbb{RP}^\infty; \mathbb{Z}/2)^{\otimes k} \\ \alpha \to \alpha \cdot (a_1 \otimes \cdots \otimes a_1)$$

is injective for deg $\alpha \leq k$.

This lemma follows because we know explicitly how the Steenrod algebra acts on $H^*(\mathbb{RP}^\infty; \mathbb{Z}/2)$ (see Proposition 10.51 above), the Cartan product formula (which tells us how $\mathcal{A}_2$ acts on $H^*(\mathbb{RP}^\infty; \mathbb{Z}/2)^{\otimes k}$) and induction on n. This argument is carried out in the proof of Proposition 3.2 in [91]. □

We can now turn back to the proof of Theorem 11.5.

Proof. In our proof in the last chapter of Theorem 10.61, if $\mathbb{E}$ is a spectrum whose mod 2 cohomology is a free module over $\mathcal{A}_2$ with basis $\mathcal{B}$, then there is a map

$$\phi : \mathbb{E} \to \bigvee_{b_\omega \in \mathcal{B}} \Sigma^{|\omega|} \mathbb{HZ}/2 \tag{11.11}$$

that induces an isomorphism in cohomology with $\mathbb{Z}/2$-coefficients. From Proposition 11.6 we can let $\mathbb{E} = \mathbb{MO}$. In order to know that ϕ is a weak homotopy equivalence, we can appeal to Theorem 10.61 once we know that the cohomology of $\mathbb{MO}$ is zero with $\mathbb{Z}/p$-coeffiencients for p an odd prime, and with rational coefficients. For this, the first thing to recall is since, by Thom's Theorem 11.1, the homotopy groups $\pi_*(\mathbb{MO}) \cong \eta_*$ is a vector space over $\mathbb{Z}/2$. As was observed after the statement of Theorem 11.1, this is because any cobordism class represented by a manifold M^n is 2-torsion, since twice this class $2[M^n]$ is represented by the disjoint union $M^n \sqcup M^n$ with is the boundary of $M^n \times I$, and therefore is zero as a cobordism class.

Next, in order to apply Theorem 10.61 to $\mathbb{MO}$, we need to prove the following lemma.

Lemma 11.9. *Let $\mathbb{E}$ be a spectrum such that $\pi_*(\mathbb{E})$ is finitely generated 2-torsion. That is,*

$$\pi_*(\mathbb{E}) \otimes \mathbb{Z}/p = 0 \quad \text{for } p \text{ any odd prime, and } \pi_*(\mathbb{E}) \otimes \mathbb{Q} = 0.$$

Then, the same is true of homology, That is,

$$H_*(\mathbb{E}; \mathbb{Z}/p) = 0 \quad \text{for } p \text{ any odd prime, and } H_*(\mathbb{E}; \mathbb{Q}) = 0.$$

Proof. We first discuss $H_*(\mathbb{E}; \mathbb{Z}/p)$, where p is an odd prime. Consider the self map of the sphere spectrum $\times p : \mathbb{S} \to \mathbb{S}$ representing $p \in \mathbb{Z} = \pi_0(\mathbb{S})$. Taking the smash produce with the spectrum $\mathbb{E}$ gives us a self map,

$$\times p : \mathbb{E} \to \mathbb{E}$$

defined to be

$$\mathbb{E} \wedge \mathbb{S} \xrightarrow{1 \wedge (\times p)} \mathbb{E} \wedge \mathbb{S}.$$

Notice that the map $\times p : \mathbb{E} \to \mathbb{E}$ induces multiplication by the prime p in homotopy groups, since, by definition, it does so on the sphere spectrum. That means, since $\pi_*(\mathbb{E})$ is 2-torsion, then

$$(\times p)_* : \pi_*(\mathbb{E}) \to \pi_*(\mathbb{E})$$

is an isomorphism. Therefore $\times p : \mathbb{E} \to \mathbb{E}$ is a weak homotopy equivalence. So if $\mathbb{X}$ is a spectrum representing a generalized homology theory, then

$$\times p : \mathbb{E} \wedge \mathbb{X} \xrightarrow{(\times p) \wedge 1} \mathbb{E} \wedge \mathbb{X}$$

is a weak homotopy equivalence that represents multiplication by the prime p in homotopy groups. But when $\mathbb{X}$ is the Eilenberg-MacLane spectrum $H\mathbb{Z}/p$, this map in homotopy groups is given by

$$(\times p)_* : H_*(\mathbb{E}; \mathbb{Z}/p) \xrightarrow{\cong} H_*(\mathbb{E}; \mathbb{Z}/p).$$

Since $\tilde{H}_*(\mathbb{E}; \mathbb{Z}/p)$ is a $\mathbb{Z}/p$- vector space, multiplication by p must be zero, so we must conclude that $\tilde{H}_*(\mathbb{E}; \mathbb{Z}/p) = 0$.

We now turn our attention to $H_*(\mathbb{E}; \mathbb{Q})$. By Proposition 10.22, we know that

$$H_*(\mathbb{E}; \mathbb{Q}) \cong \pi_*(\mathbb{E}) \otimes \mathbb{Q} = 0$$

since each $\pi_q(\mathbb{E})$ is assumed to be a finitely generated abelian 2-torsion group. □

□

Thus we know that $\mathbb{E} = \mathbb{MO}$ satisfies the hypotheses of Theorem 10.61, and so we may conclude that there is a weak homotopy equivalence

$$\mathbb{MO} \simeq \bigvee_{\omega \in \mathcal{B}} \Sigma^{|\omega|} \mathbb{HZ}/2, \tag{11.12}$$

where $\mathcal{B}$ forms a basis for $H^*(\mathbb{MO})$ as a module over the Steenrod algebra, $\mathcal{A}_2$. Notice that the homotopy groups of the right hand side of this equivalence, and hence of $\mathbb{MO}$, is the $\mathbb{Z}/2$-vector space spanned by $\mathcal{B}$. That is, the basis generating $H^*(\mathbb{MO})$ as an $\mathcal{A}_2$-module can be identified with a basis for its homotopy groups. Comparing the homology of both sides of this equivalence we have the following:

Corollary 11.10. *We again assume all (co)homology is taken with $\mathbb{Z}/2$-coefficients. There is an isomorphism*

$$\begin{aligned} H_*(\mathbb{MO}) &\cong \pi_*(\mathbb{MO}) \otimes H_*(\mathbb{HZ}/2) \\ &\cong \pi_*(\mathbb{MO}) \otimes \mathcal{A}_2^* \end{aligned}$$

where $\mathcal{A}_2^*$ is the dual of the Steenrod algebra.

Remark. By the dual of the Steenrod algebra $\mathcal{A}_2^*$ we mean the *graded* dual. that is,

$$\mathcal{A}_2^* = \bigoplus_{d=0}^{\infty} (\mathcal{A}_2)_d^*$$

where $(\mathcal{A}_2)_d^* = Hom(\mathcal{A}_d, \mathbb{Z}/2)$ with $(\mathcal{A}_2)_d$ being the degree d component of $\mathcal{A}_2$.

Before we compute the cobordism ring $\eta_* \cong \pi_*(\mathbb{MO})$, we draw some immediate geometric conclusions from what we've shown so far.

Corollary 11.11. *Two closed n-manifolds M^n and N^n are cobordant if and only if the images of their fundamental classes under their stable normal bundle homomorphisms are equal. That is, if $\nu_M : M^n \to BO$ and $\nu_N : N^n \to BO$ are the stable normal bundle maps for M^n and N^n respectively, then these manifolds are cobordant if and only if*

$$(\nu_M)_*([M^n]) = (\nu_N)_*([N^n]) \in H_n(BO; \mathbb{Z}/2).$$

Proof. By the Pontrjagin-Thom theorem, we need to know when the classes

$$\alpha_{M^n} \quad \text{and} \quad \alpha_{N^n} \in \pi_n(\mathbb{MO})$$

are equal. But as one sees from Corollary 11.10, the Hurewicz homomorphism

$$h : \pi_n(\mathbb{MO}) \to H_n(\mathbb{MO}; \mathbb{Z}/2)$$

is injective, so this is equivalent to knowing that $h(\alpha_{M^n}) = h(\alpha_{N^n}) \in H_n(\mathbb{MO};\mathbb{Z}/2)$. But by the Thom isomorphism theorem, this is equivalent to knowing that

$$u \cap h(\alpha_{M^n}) = u \cap h(\alpha_{N^n}) \in H_n(BO;\mathbb{Z}/2).$$

Here $u \in H^0(\mathbb{MO};\mathbb{Z}/2)$ is the Thom class. The corollary then follows from the following straightforward exercise.

Exercise. For any closed manifold M^n,

$$u \cap h(\alpha_{M^n}) = (\nu_{M^n})_*[M^n] \in H_n(BO;\mathbb{Z}/2).$$

□

The result of Corollary 11.11 is often stated in a different way. Recall that $H^*(BO;\mathbb{Z}/2) \cong \mathbb{Z}/2[w_1, \cdots, w_i, \cdots]$ where w_i is the i^{th} Stiefel-Whitney class.

Definition 11.3. *Let M^n be a closed n- manifold, and $f : M^n \to BO$, be a map, which we think of as classifying a stable vector bundle over M^n. We define "Stiefel-Whitney numbers" of f as follows. Let $p(w)$ be a polynomial in the Stiefel-Whitney classes, that is an element of $H^*(BO;\mathbb{Z}/2)$. $p(w)$ determines a "Stiefel-Whitney number" by the rule*

$$\langle f^*(p(w));\, [M^n]\rangle \in \mathbb{Z}/2.$$

Note. This evaluation map is to be interpreted as follows. The polynomial $p(w)$ is a sum of monomials of varying dimensions. The evaluation $\langle f^*(p(w));\, [M^n]\rangle = \langle p(w), \alpha\rangle$ on an n-dimensional homology class α is the sum of the evaluations of its monomials on $f_*(\alpha)$. A monomial having dimension other than n has, by convention, zero evaluation on an n-dimensional homology class.

If $\tau_{M^n} : M^n \to BO$ classifies the stable tangent bundle of M^n we call it's Stiefel-Whitney numbers the "*tangential Stiefel-Whitney numbers*" of M^n, or sometimes just the Stiefel-Whitney numbers of M^n. Similarly, if $\nu_{M^n} : M^n \to BO$ classifies the stable normal bundle, we call it's Stiefel-Whitney numbers, the "*normal Stiefel Whitney numbers*" of M^n.

Notice that the tangential and normal Stiefel-Whitney numbers of a manifold are invariants of the manifold. We can now interpret Corollary 11.11 as follows:

Corollary 11.12. *Two manifolds are cobordant if and only if they have the same normal Stiefel-Whitney numbers.*

The inverse relation between the stable tangent and stable normal bundle maps of a manifold will allow us to quickly prove the following.

Corollary 11.13. *Two manifolds are cobordant if and only if they have the same tangential Stiefel-Whitney numbers.*

Proof. Recall that for any space X the fact that BO is an infinite-loop space implies that the set of homotopy classes, $[X, BO]$ is an abelian group (equal to it's reduced K-theory). Let $-\iota : BO \to BO$ represent the homotopy class in $[BO, BO]$ that is inverse to the class represented by the identity map. Since the sum of the tangent bundle of a manifold with the normal bundle of any embedding of the manifold in Euclidean space is a trivial bundle, that means that the classes in $[M^n, BO]$ represented by τ_{M^n} and ν_{M^n} are inverse to each other in this group structure. In other words, the composition

$$M^n \xrightarrow{\tau_{M^n}} BO \xrightarrow{-\iota} BO$$

is homotopic to ν_{M^n}, and similarly the composition

$$M^n \xrightarrow{\nu_{M^n}} BO \xrightarrow{-\iota} BO$$

is homotopic to τ_{M^n}. Because $(-\iota) \circ (-\iota) \simeq id : BO \to BO$, $-\iota : BO \to BO$ is a homotopy equivalence.

Let $p(w)$ be a polynomial in the Stiefel-Whitney classes. Then

$$(\tau_{M^n})^*(p(w)) = (\nu_{M^n})^*((-\iota)^*(p(w)) \quad \text{and} \quad (\nu_{M^n})^*(p(w)) = (\tau_{M^n})^*((-\iota)^*(p(w)).$$

So the tangential Stiefel-Whitney number determined by $p(w)$ is the normal Stiefel-Whitney number determined by $(-\iota)^*(p(w)))$, and vice-versa. Since $(-\iota)^*$ is an isomorphism, we can conclude that two manifolds have the same normal Stiefel-Whitney numbers if and only if they have the same tangential Stiefel-Whitney numbers. The corollary now follows from Corollary 11.12. □

Corollary 11.14. *A manifold that can be* **stably framed,** *i.e a manifold M^n whose stable normal bundle map $\nu_{M^n} : M^n \to BO$ is null homotopic, is the boundary of an $(n+1)$-dimensional manifold.*

Proof. If M^n is framed manifold, all of its Stiefel-Whitney numbers are zero. The sphere S^n is stably frameable, since the standard embedding in $\mathbb{R}^{n+1}$ has a trivial normal bundle. So by Corollary 11.12 M^n is cobordant to S^n, which is null-cobordant since it is the boundary of D^{n+1}. Therefore M^n is null-cobordant. □

We remark that the fact that the Stiefel-Whitney numbers of a manifold are cobordism invariants is not very difficult, as we will see below. What was difficult, and was a major achievement of Thom, is that the Stiefel-Whitney numbers of a manifold are a *complete* cobordism invariant as stated in Corollaries 11.12 and 11.13.

We now give an elementary proof of the following fact.

Proposition 11.15. *. If M^n is a closed manifold that is the boundary of W^{n+1}, then all of the tangential Stiefel-Whitney numbers of M^n are zero.*

Proof. Pick a metric on W^{n+1}. Then there is a unique outward normal vector field along $\partial W^{n+1} = M^n$, spanning a trivial line bundle ϵ^1. Therefore, the restriction of the tangent bundle to its boundary

$$T(W^{n+1})_{|M^n} = T(M) \oplus \epsilon^1.$$

Hence, the Stiefel-Whitney classes of M^n are the restriction of Stiefl-Whitney classes of W^{n+1}. By the long exact sequence,

$$\cdots \to H^n(W^{n+1};\mathbb{Z}/2)) \to H^n(\partial W^{n+1};\mathbb{Z}/2) \xrightarrow{\delta} H^{n+1}(W,\partial W);\mathbb{Z}/2) \to \cdots$$

this implies that $\delta(w) = 0$ for every tangential Stiefel-Whitney class w. The natural map $\partial : H_{n+1}(W,\partial W;\mathbb{Z}/2) \to H_n(\partial W;\mathbb{Z}/2)$ takes the fundamental class $[W,\partial W]$ to the fundamental class $[\partial W] = [M^n]$.

Therefore if $p(w)$ is any polynomial in the Stiefel-Whitney classes,

$$\langle (\tau_{M^n})^* p(w), [M^n]\rangle = \langle \delta((\tau_{M^n})^* p(w)), [W^{n+}, \partial W^{n+1}]\rangle = 0.$$

□

Exercise. Show that this proposition implies that if M_1 and M_2 are cobordant manifolds, they have the same tangential Stiefel-Whitney numbers.

We now turn our attention back to computing the cobordism ring $\eta_* \cong \pi_*(\mathbb{MO})$.

Since we know the homology $H_*(\mathbb{MO})$ we will be able to calculate $\pi_*(\mathbb{MO}) = \eta_*$, using knowledge of $\mathcal{A}_2^*$ and their relation given by Corollary 11.10. Fortunately $\mathcal{A}_2^*$ is well-understood by work of Milnor [75], which we now quickly describe.

Recall from Theorem 10.53 that the Steenrod algebra $\mathcal{A}_2$ has an additive basis over $\mathbb{Z}/2$ consisting of cohomology operations Sq^I, where $I = (i_1, \cdots, i_k)$ is an admissible sequence. Recall this means that $i_j \geq 2i_{j+1}$ for all $j = 1, \cdots, k-1$. Recall also from the exercise following Definition 11.2 that $\mathcal{A}_2$ is a Hopf algebra, whose coproduct $\Delta : \mathcal{A}_2 \to \mathcal{A}_2 \otimes \mathcal{A}_2$ is determined by the Cartan formuia,

$$\Delta(Sq^k) = \sum_{j=0}^{k} Sq^j \otimes Sq^{k-j}.$$

The symmetry of the Cartan formula implies that the coalgebra structure of $\mathcal{A}_2$ is cocommutative. This implies the dual $\mathcal{A}_2^*$ is a commutative algebra. The following is the result of Milnor's calculation of $\mathcal{A}_2^*$.

Theorem 11.16. *(Milnor [75]). Let $\{(Sq^I)^* : \quad I \text{ is admissible}\}$ be the additive basis of $\mathcal{A}_2^*$ dual to the basis of admissible sequences of $\mathcal{A}_2$. Let I_k be the admissible sequence $I_k = (2^{k-1}, 2^{k-2}, \cdots, 2, 1)$, and let $\xi_k = (Sq^{I_k})^* \in \mathcal{A}_2^*$. Then as an algebra, $\mathcal{A}_2^*$ is the polynomial algebra on the ξ_j's,*

$$\mathcal{A}_2^* \cong \mathbb{Z}/2[\xi_1, \xi_2, \cdots, \xi_j, \cdots]$$

We refer the reader to Milnor's original paper [75] or the book by Steenrod and Epstein [91] for nice proofs of this theorem. We will use this result and Corollary 11.10 to compute the cobordism ring $\eta_* \cong \pi_*(\mathbb{MO})$.

To understand $\pi_*(\mathbb{MO})$ as a graded $\mathbb{Z}/2$ vector space, it suffices to compute the dimension of $\pi_n(\mathbb{MO})$ for each n. For this we use a little combinatorics as is done in [99].

Recall that a *partition* of a positive integer n is an unordered sequence $(i_i, \cdots, i_k)$ of positive integers whose sum equals n. Let $p(n)$ be the number of partitions of n. Notice that in the polynomial ring $\mathbb{Z}/2[e_1, e_2, \cdots, e_k, \cdots]$ where $|e_i| = i$ (recall from Theorem 10.36 that this is $H_*(\mathbb{MO}; \mathbb{Z}/2)$), a monomial in degree n determines, and is determined by, a partition of n. Let $p(n)$ denote the number of partitions of n. This is then the dimension of $H_n(\mathbb{MO})$. (Here, as above, when we don't specify coefficients in (co)homology we mean $\mathbb{Z}/2$-coefficients.

Now by Theorem 11.16, $\mathcal{A}_2^* \cong \mathbb{Z}/2[\xi_j, \, j \geq 1]$, where the degree $|\xi_j| = 2^j - 1$. A monomial in this ring of degree n is also a partition, but a very special one. Namely it is a *"dyadic partition"*, meaning a partition $(i_i, \cdots, i_k)$ where each i_j is of the form $2^m - 1$ for some m. We write $p_d(n)$ to be the number of dyadic partitions of n. By convention we let $p_d(0) = 1$. Notice that $p_d(n)$ is the dimension of $\mathcal{A}_2^*$ in degree n.

Finally we say that a partition $(i_i, \cdots, i_k)$ of n is *nondyadic* if none of the i_j's are of the form $2^m - 1$ for any m. We let $p_{nd}(n)$ be the number of nondyadic partitions of n, with the convention that $p_{nd}(0) = 1$. The following gives a calculation of the cobordism groups:

Theorem 11.17. *The cobordism group of n-dimensional closed manifolds, $\eta_n \cong \pi_n(\mathbb{MO})$ is the $\mathbb{Z}/2$-vector space of dimension $p_{nd}(n)$.*

Proof. The main step in the proof of this theorem is the following fact from combinatorics.

Lemma 11.18. *For every positive integer n,*

$$p(n) = \sum_{i=0}^{n} p_d(i) p_{nd}(i).$$

Furthermore, if $f(n)$ is an function defined for nonnegative integers n satisfying

$$p(n) = \sum_{i=0}^{n} p_d(i) f(n-i)$$

then $f(m) = p_{nd}(m)$ for all $m \geq 0$.

Proof. If $I = (i_1, \cdots, i_k)$ and $J = (j_1, \cdots, j_r)$ are partitions of n and m respectively, write $IJ = (i_1, \cdots, i_k, j_1, \cdots, j_r)$. This is a partition of $n+m$. Notice that if I is any partition of n, the we can uniquely write

$$I = I_d I_{nd}$$

where I_d is the dyadic partition obtained by taking all the entries of I of the form $2^k - 1$ for some k, and I_{nd} is the nondyadic partition obtained by taking the remaining entries of I (i.e those not of the form $2^k - 1$ for any k). Conversely, for every i such that $0 \leq i \leq n$, if I_d is a dyadic partition of i and I_{nd} is a nondyadic partition of $n - i$, the $I_d I_{nd}$ is a partition of n. Notice that there are $p_d(i)p_{nd}(n-i)$ ways of making such a partition of n for each i. This verifies the formula $p(n) = \sum_{i=0}^{n} p_d(i)f(n-i)$. The second statement is proved by an easy induction on n. □

We now complete the proof of Theorem 11.17. By Corollary 11.10, we know that

$$H_*(\mathbb{MO}) \cong \pi_*(\mathbb{MO}) \otimes \mathcal{A}_2^*. \tag{11.13}$$

We then have

$$dim\, H_n(\mathbb{MO}) = \sum_{i=0}^{n} dim\, \pi_i(\mathbb{MO}) \cdot dim\, (\mathcal{A}_2^*)_{n-i}.$$

As observed above, $dim\, H_n(\mathbb{MO}) = p(n)$ and $dim\, (\mathcal{A}_2^*)_k = p_d(k)$. If we let $f(m) = dim\, \pi_m(\mathbb{MO})$ then the result follows from Lemma 11.18. □

We can now draw an immediate geometric consequence of Theorem 11.17, which would be very difficult to prove without Thom-Pontrjagin theory,

Corollary 11.19. *Every closed 3-dimensional manifold is the boundary of a 4-dimensional manifold.*

Proof. By Theorem 11.17 η_3 is a $\mathbb{Z}/2$-vector space of dimension $p_{nd}(3)$. But there are no nondyadic partitions of 3, so $p_{nd}(3) = 0$. □

We now observe that we can restate Theorem 11.17 in the following way.

Corollary 11.20. *Consider the polynomial algebra $\mathbb{Z}/2[b_2, b_4, \cdots, b_i, \cdots]$ such that $|b_i| = i$ and i is not of the from $2^k - 1$ for any k. Then there is a graded $\mathbb{Z}/2$ vector space isomorphism between this algebra and $\pi_*(\mathbb{MO}) \cong \eta_*$.*

Proof. Notice that the monomials of degree n in this polynomial algebra are in bijective correspondence with nondyadic partitions of n, The result then follows from Theorem 11.17. □

Our goal is to strengthen this corollary to show that there is an isomorphism of algebras between the cobordism ring and this polynomial algebra. This would establish Theorem 11.4.

Proof. Recall from Theorem 5.20 that the splitting principle gives us an alternative description of $H^*(BO(n))$ as the ring of symmetric polynomials $\mathbb{Z}/2[\sigma_1, \cdots, \sigma_n]$, where σ_i is the i^{th} elementary symmetric polynomial in m-variables, say $x_1, \cdots, x_m$, all of which have degree one. Here we are choosing $m > n$ so that the elementary symmetric polynomials are algebraically independent.

Definition 11.4. *We say that two monomials in $x_1, \dots, x_m$ are equivalent if there is a permutation of $x_1, \dots, x_m$ that takes one to the other. Define $\sum x_1^{a_1} \cdots x_r^{a_r}$ to be the sum of all monomials in $x_1, \dots, x_m$ which are equivalent to $x_1^{a_1} \cdots x_r^{a_r}$.*

Exercise. (See Lemma 16.1 of [76]) Show that an additive basis for $\mathcal{S}^k$, the group of homogeneous symmetric polynomials of degree k in $x_1, \dots, x_m$ is given by the polynomials $\sum x_1^{a_1} \cdots x_r^{a_r}$, where $(a_1, \dots, a_r)$ range through all partitions of k of length $r \leq m$.

Now let $I = (i_1, \cdots i_r)$ be a partition of n, and let $s_I = s_{(i_1, \cdots, i_r)}$ be the unique polynomial satisfying

$$s_I(\sigma_1, \cdots \sigma_n) = \sum x_1^{i_1} \cdots x_r^{i_r}.$$

For $m \geq n$ the $p(n)$ polynomials $s_I(\sigma_1, \cdots \sigma_n)$ are linearly independent and form a basis of $\mathcal{S}^n$. See Milnor and Stasheff's book [76] for a more complete discussion of these symmetric polynomials.

Given a vector bundle ξ over a closed n-manifold M^n, by recalling that the i^{th} Stiefel-Whitney class $w_i \in H^*(B0)$ can be identified with the i^{th} elementary symmetric polynomial σ_i, then I is a partition of n we may write

$$s_I(w(\xi)) = s(w_1(\xi), \dots, w_n(\xi)) \in H^n(M).$$

(We recall that unless otherwise stated, all (co)homology is taken with $\mathbb{Z}/2$-coefficients.).

Now since the symmetric polynomials s_I form an additive basis for the ring of symmetric polynomials, which, by the splitting principle is isomorphic to $H^*(BO)$, then by Corollary 11.13 the cobordism type of any closed n-manifold M^n is completely determined by the collection of numbers

$$S_I[M^n] = \langle (s_I(\tau_{M^n}), [M^n] \rangle \in \mathbb{Z}/2,$$

where τ_{M^n} is the tangent bundle and I ranges over all partitions of n. A quick calculation using the Cartan product rule for Stiefel-Whitney classes that is done in [99], verifies the following:

Lemma 11.21. *Let ξ and ζ be any two vector bundles over a closed n-manifold M^n. Then for any partition I*

$$s_I(w(\xi \oplus \zeta)) = \sum_{I_1 I_2 = I} s_{I_1}(w(\xi) s_{I_2}(w(\zeta)),$$

and if N^m is another closed manifold,

$$S_I(M^n \times N^m) = \sum_{I_1 I_2 = I} S_{I_1}[M^n] S_{I_2}[N^m].$$

Recall the fact from Theorem 5.22 that the total Stiefel-Whitney class of the projective space is given by

$$w(\tau_{\mathbb{RP}^n}) = (1+a)^{n+1}$$

where $a \in Hq(\mathbb{RP}^n)$ is the nonzero class. This will immediately imply the following:

Lemma 11.22. *Consider the length-one partition (n) of n. Then*

$$S_{(n)}[\mathbb{RP}^n] = n + 1 \in \mathbb{Z}/2.$$

As we will see below, these projective spaces can be taken to be generators of the cobordism ring η_* when n is even. To construct other generators, Thom considered hypersurfaces $H_{m,n}$ defined as follows.

Definition 11.5. *Let m and n be positive integers with $m \leq n$. Let $\mathbb{RP}^m$ have homogeneous coordinates $[x_0, \ldots, x_m]$ and $\mathbb{RP}^n$ have homogeneous coordinates $[y_0, \ldots, y_n]$. Let $H_{m,n} \subset \mathbb{RP}^m \times \mathbb{RP}^n$ be the subset defined by coordinates $([x_0, \ldots, x_m], [y_0, \ldots, y_n])$ satisfying the equation*

$$\sum_{i=0}^{m} x_i y_i = 0.$$

Exercise Show that $H_{m,n}$ is a smooth manifold of dimension $m + n - 1$.

Using Lemma 11.21 Weston gave a direct calculational proof of the following (see Proposition 11.4 of [99]).

Lemma 11.23.

$$S_{(m+n-1)}[H_{m,n}] = \binom{m+n}{m} \in \mathbb{Z}/2.$$

We are now ready to complete the proof of Theorem 11.4. Define manifolds B_n that we show will generate the cobordism ring, as follows.

Definition 11.6. *Let* $B_i = \begin{cases} \mathbb{RP}^i & \text{if } i \text{ is even} \\ H_{2^p, 2^{p+1}q} & \text{if } i = 2^p(2q+1) - 1 \text{ and not of the form } 2^m - 1 \end{cases}$

Let $b_i \in \eta_i$ be the cobordism class represented by the manifold B_i. We will show that η_* is the polynomial algebra generated by the classes $\{b_i \text{ such that } i \text{ is not of the form } 2^k - 1\}$.

First of all notice that

$$S_{(i)}[B_i] = 1$$

by Lemmas 11.22 and 11.23. Let $I = (i_1, \cdots, i_k)$ be a nondyadic partitiion of n. We define the n-manifold

$$M_I = M_{i_1} \times \cdots \times M_{i_k}.$$

We will show that the set $\{[M_I] : I \quad \text{is a nondyadic partition of} \, n\}$ is a $\mathbb{Z}/2$-vector space basis for the cobordism group η_n. For this we follow the argument in [99] (Theorem 13.4). Since by Theorem 11.17 we know the dimension of this vector space is $p_{nd}(n)$, which is the number of nondyadic partitions of n, we need only show that this set of cobordism classes is linearly independent.

To do this we put a partial ordering on the set of partitions of n using the notion of *"refinement"*. Let I' be another partition of n. We say that I' is a ' refinement of the partition $I = (i_1, \ldots, i_k)$ if we can write $I' = I_1 \cdots I_k$, where each I_j is a partition of the coordinate i_j of I. This gives a partial order to the set of partitions of n by saying that $I \leq I'$ is I' refines I. In particular it gives a partial ordering to the set of nondyadic partitions of n.

Let I and J be nondyadic partitions of n with $I = (i_1, \cdots, i_k)$. Then by Lemma 11.21, we have

$$S_J[M_I] = \sum_{I_i \cdots I_k = J} S_{I_1}([M_{I_1}]) \cdots S_{I_k}([M_{i_k}]). \tag{11.14}$$

So in particular if J does not refine I, $S_J[M_I]$ must be zero. Also this equation says that $S_I([M_I]) = 1$ since in this case there is exactly one choice of $I_1, \cdots I_k$ giving a nonzero contribution to this sum.

Now choose a total ordering of the partitions of n compatible by the partial ordering given by refinement. Then we can form a $(p_{nd}(n) \times p_{nd}(n))$- dimensional matrix whose rows and columns are indexed by nondyadic partitions of n according to our ordering, and the entry indexed by (I, J) is given by $S_J([M_I])$. Then these calculations tell us that this is a triangular matrix with one's along the diagonal. Therefore the $p_{nd}(n)$ columns of this matrix are linearly independent. Each column is indexed by a nondyadic partition J of n, and its I^{th} coordinate is $S_J([M_I])$. Again, as J varies, these columns are linearly independent. Now since the polynomials s_J are a basis of the symmetric functions in degree n which can be viewed as the cohomology group

$H^n(BO)$, this says that the tangential Stiefel-Whitney numbers of the M_I's also give linearly p_{nd} linearly independent vectors. By Corollary 11.13 this says that the M_I's are linearly independent vectors in η_n, which as mentioned above, implies that they form a basis for η_n. But the set of $[M_I]$'s in η_n constitutes the set of monomials in the b_j's of degree n, and therefore this describes $\eta_* = \oplus_n \eta_n$ as the polynomial algebra $\mathbb{Z}/2[b_i$: i is not of the form $2^m - 1]$. □

11.3 Almost complex cobordism: Milnor's calculation

11.4 Framed, Oriented, and Spin cobordism

11.5 Cobordism categories and the classifying space of diffeomorphisms of manifolds

12

Classical Morse Theory

In this chapter we discuss the traditional, "classical" approach to Morse theory. An approach based on moduli spaces of flows will be discussed in the next chapter. The best reference to this classical approach is Milnor's well known book [69]. We encourage the reader to study that book, not only for the details of the foundations of the subject, but also for applications that are still quite relevant more than 50 years after its publication.

12.1 The Hessian and the index of a critical point

Let M be a manifold, and $f : M \longrightarrow \mathbb{R}$ a C^2 function. As explained earlier, a point $p \in M$ is called a *critical point* of f if $df_p = 0$. $f : M \to \mathbb{R}$ is a **Morse function** if all of its critical points are nondegenerate. To understand what it means for a critical point $p \in M$ to be nondegenerate, we may work in a coordinate chart around p, with respect to which we may think of $f : \mathbb{R}^n \to \mathbb{R}$, with p corresponding to the origin in $\mathbb{R}^n$. In such coordinates we can think of the derivative as a map

$$\begin{aligned} Df : \mathbb{R}^n &\to (\mathbb{R}^n)^* \\ x &\to df_x. \end{aligned}$$

A critical point is then a zero of Df, and $0 \in \mathbb{R}^n$ is a nondegenerate critical point precisely if it is a *regular point* of Df. Notice that this means that as $v \in \mathbb{R}^n$ varies in a small neighborhood around the origin, then $Df(x) = df_x$ takes on every value of $(\mathbb{R}^n)^*$ exactly once. Note also that $0 \in \mathbb{R}^n$ being a nondegenerate critical point is equivalent to the linear map

$$D(Df)_0 : \mathbb{R}^n \to (\mathbb{R}^n)^*$$

being an isomorphism. This in turn is equivalent to the $n \times n$ Hessian matrix,

$$\mathrm{Hess}_0\, f = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(0)\right)$$

is nonsingular.

We now make this into a formal definition. Let $p \in M$ be a critical point o

$f : M \to \mathbb{R}$, and let $(U, \phi : U \longrightarrow \mathbb{R}^n)$ be a coordinate chart around p, so that $\phi(p) = 0$. Write ϕ as $(x_1, \ldots, x_n)$. Write tangent vectors v and w in T_pM as $(v_1, \ldots, v_n)$ and $(w_1, \ldots, w_n)$, respectively (specifically, $d\phi_p(v) = (v_1, \ldots, v_n)$ and similarly for w).

Definition 12.1. *Using the coordinate chart* (U, ϕ)*, The Hessian of* f *at* p*, is the quadratic form* $\mathrm{Hess}_p\, f$ *defined by the formula*

$$\mathrm{Hess}_p(f)(v) = \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} v_i v_j$$

Proposition 12.1. *When* p *is a critical point for* $f : M \longrightarrow \mathbb{R}$*, the Hessian at* p *is independent of the coordinate chart.*

Proof. One can do this directly by a straightforward calculation which we leave to the reader. But more generally, with respect to local coordinates, we may consider an open set $V \subset \mathbb{R}^n$ containing $0 \in \mathbb{R}^n$, and a C^2-map $g : V \to \mathbb{R}$ having 0 as a critical point. Let $h : V \xrightarrow{\cong} U$ be a C^2 diffeomorphism of open sets in $\mathbb{R}^n$ taking $0 \in \mathbb{R}^n$ to itself. Then the reader should verify that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{\mathrm{Hess}_0(gh)} & \mathbb{R} \\ {\scriptstyle Dh_0}\downarrow & & \downarrow{\scriptstyle =} \\ \mathbb{R}^n & \xrightarrow[Hess_0 g]{} & \mathbb{R}. \end{array}$$

This gives an invariance of the Hessian under local diffeomorphisms, which is to say, an invariance of the Hessian under changes of coordinate charts around a critical point. □

Remark. If p is not a critical point of f, then the Hessian at p is not well-defined, in that using the above notation, it would depend on the coordinate chart. However there are ways to extend the Hessian to all of M: by patching together coordinate charts and using partitions of unity; by choosing a metric on M, then using the Levi–Civita connection corresponding to this metric to take the covariant derivative of *df* at p, and so on. But these approaches all require extra data (namely the choice of metric or connection). In these notes, however, we will primarily be concerned with the Hessian at critical points.

12.2 Morse Functions

Definition 12.2. *If $p \in M$ is a critical point for a C^2 function $f : M \longrightarrow \mathbb{R}$, then we call p* nondegenerate *if the quadratic form $\text{Hess}_p(f)$ is nonsingular. If all critical points of M are nondegenerate, we say that f is* Morse*.*

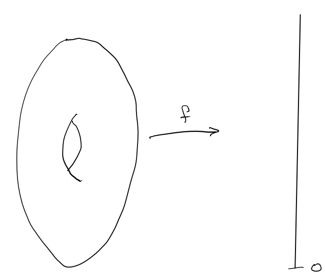


FIGURE 12.1
f is the "height function" given by projecting the torus onto the vertical line. This is probably the archetypical example of a Morse function.

We will show in Section 12.5 that every manifold M admits a Morse function, and in fact the set of Morse functions is dense in the set of smooth functions.

An important property of Morse functions on closed, Riemannian manifolds, is that they lead to a CW complex description of the manifold, with with a cell of dimension λ for each critical point of index λ of f. In this section, we prove this statement up to homotopy. That is, we construct a homotopy equivalence of the manifold to a CW complex of the kind just described. We follow the approach of Milnor [69] in this chapter.

Throughout this chapter, we will assume M is a closed manifold and $f : M \longrightarrow \mathbb{R}$ is a smooth Morse function. We will also consider the following manifolds (with boundary):

$$M^a = f^{-1}(-\infty, a] = \{x \in M \mid f(x) \leq a\}.$$

where a is any real number. If a is less than the minimum value of f, then M^a is the empty set. If a is larger than the maximum value of f, then M^a is M. The values of a in between will provide, up to homotopy, the necessary cell decomposition.

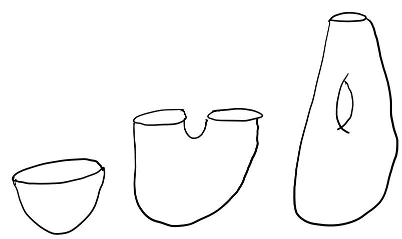


FIGURE 12.2
M^a for different values of a

There are a number of technical details, but the intuition is simple: Let M be a surface embedded in $\mathbb{R}^3$, and f be the vertical coordinate z. We initially let a be less than the minimum value of f so that $M^a = \emptyset$, and gradually increase a (see Figure 12.2). This is analogous to gradually filling the surface with water, so that M^a is the part of the surface that is under water. Now if a increases from a_1 to a_2 without passing through critical values, then M^{a_1} and M^{a_2} are diffeomorphic.

But if, by increasing from a_1 to a_2, we pass through one critical point, then at that point the water may do something more interesting. Up to homotopy,

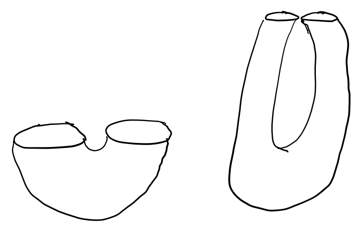


FIGURE 12.3
M^{a_1} and M^{a_2} are diffeomorphic if there are no critical values between a_1 and a_2.

this turns out to be an attaching of a cell of dimension λ, where λ is the index of the critical point (see Figure 12.4).

So as we pass critical points one by one, the manifold is created by successively attaching cells (up to homotopy type). This demonstrates that the manifold is homotopy equivalent to a CW complex of the type described above.

In this chapter we prove the details of the above intuition. First we prove that nothing happens to the homotopy type (and even to the diffeomorphism type) if there is no critical point between two levels, using the results of gradient flow lines from chapter 12.1. Then we show that if there is one critical point between the two levels, the homotopy type changes by adding a cell. We prove this via the Morse Lemma (Theorem 12.4), which studies the behavior of f near a critical point. We conclude by producing the homotopy equivalence between the manifold and the CW complex, and giving some interesting applications to topology.

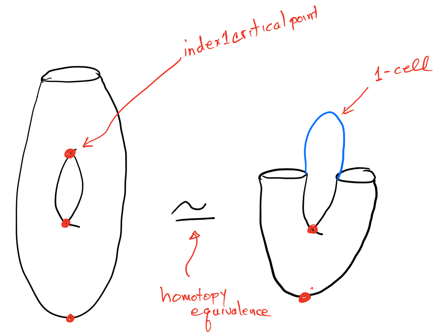


FIGURE 12.4
When there is one critical value between a_1 and a_2, M^{a_2} is homotopy equivalent to M^{a_1} with a cell attached.

Exercise:

Let M be a manifold and let $f : M \longrightarrow \mathbb{R}$ be a Morse function. Prove that $f^{-1}(\{a\})$, the boundary of M^a, is a manifold if a is a regular value of f.

12.3 The Regular Interval Theorem

We first show that if we increase M^a from M^{a_1} to M^{a_2}, and there are no critical values between a_1 and a_2, then M^{a_1} and M^{a_2} are diffeomorphic.

The main point is the following theorem:

Theorem 12.2 (Regular interval theorem)**.** *Let $f : M \longrightarrow [a, b]$ be a smooth map on a compact Riemannian manifold with boundary. Suppose that f has no critical points and that $f(\partial M) = \{a, b\}$. Then there is a diffeomorphism*

$$F : f^{-1}(a) \times [a, b] \longrightarrow M$$

making the following diagram commute:

$$\begin{array}{ccc} f^{-1}(a) \times [a,b] & \xrightarrow{F} & M \\ {\scriptstyle proj.}\downarrow & & \downarrow{\scriptstyle f} \\ [a,b] & \xrightarrow[=]{} & [a,b]. \end{array}$$

In particular all the level surfaces are diffeomorphic.

In the proof of this theorem we will make use of the **gradient vector field** $\nabla(f)$**.** of the function $f : M \to \mathbb{R}$, when M has a Riemannian metric. The definition of $\nabla(f)$ depends on the metric in the following way. Recall that a Riemannian metric g defines a nonsingular, symmetric biinear pairing on the tangent bundle,

$$< , >_g : TM \times TM \to \mathbb{R}.$$

Equivalently, by taking the adjoint of this pairing we may think of the metric g as defining an isomorphism of the tangent bundle with the cotangent bundle,

$$g : TM \xrightarrow{\cong} T^*M.$$

The differential df is a section of the cotangent bundle, $df(x) \in T^*_x M$ for every $x \in M$, and its definition *does not* depend on the metric. The *gradient vector field* $\nabla_x(f) \in T_x M$ is defined to be the section of TM determined by df, using the metric g. Said more explicitly, the gradient is the unique vector field (section of TM) that satisfies

$$< \nabla_x f \, , v >_g = df(x)(v) \tag{12.1}$$

for every $x \in M$ and $v \in T_x M$. We notice that the zeros of the gradient $\nabla(f)$ are the same as the zeros of the differential df and are exactly the critical points of $f : M \to \mathbb{R}$.

With this definition we are now ready to prove this theorem.

Proof. Since f has no critical points we may consider the vector field

$$X(x) = \frac{\nabla_x(f)}{|\nabla_x(f)|^2}.$$

Let $\eta_x(t)$ be a curve through x satisfying

$$\frac{d}{dt}\eta_x(t) = X(\eta_x(t))$$

and $f(\eta_x(t)) = t$.

Let I be a maximal interval on which η_x is defined. We wish to show that $I = [a, b]$. First, since M is compact, $f(\eta_x(I)) = I$ is bounded.

Let $d = \sup(I)$. Then by the compactness of M, there is a point $x \in M$ that is a limit point of $\eta_x(d - 1/n)$. Since $\eta_x'(t) = X(\eta_x(t))$ is bounded, this limit point is unique, and $\lim_{t\to d^-} \eta_x(t) = x$. We can extend η_x to d by making $\eta_x(d) = x$.

Now $\lim_{t\to d} \eta_x'(t) = \lim_{t\to d} X(\eta_x(t)) \to X(\eta_x(d))$, and let v be this limit. We will now show that $\eta_x'(d) = v$. In particular, we will show that for every $\epsilon > 0$, there exists a $\delta > 0$ so that for all h with $0 < h < \delta$,

$$\left|\frac{\eta_x(d) - \eta_x(d-h)}{h} - v\right| < \epsilon.$$

Note that a coordinate chart is chosen near $\eta_x(d)$ to allow the subtraction here.

So let $\epsilon > 0$ be given. By the definition of v, there exists a δ_1 so that for all w with $0 < h < \delta_1$,

$$|\eta_x'(d-h) - v| < \epsilon$$

By the fundamental theorem of calculus,

$$\begin{aligned}
\eta_x(d-h) - \eta_x(d) &= \int_{d-h}^{d} \eta_x'(t)\,dt \\
\eta_x(d-h) - \eta_x(d) + vh &= \int_{d-h}^{d} (\eta_x'(t) - v)\,dt \\
|\eta_x(d-h) - \eta_x(d) + vh| &\le \int_{d-h}^{d} |\eta_x'(t) - v|\,dt \\
&\le \int_{d-h}^{d} \epsilon\,dt \\
&\le \epsilon h \\
\left|\frac{\eta_x(d-h) - \eta_x(d)}{h} + v\right| &\le \epsilon \\
\left|\frac{\eta_x(d-h) - \eta_x(d)}{-h} - v\right| &\le \epsilon
\end{aligned}$$

Therefore $\eta'_x(d) = v$, and since $v = X(\eta_x(d))$, the flow equation is satisfied by η_x at d.

By maximality of I, $d \in I$. Similarly with $c = \inf(I)$, we see that $c \in I$. Therefore I is closed.

If $\eta_x(s) \notin \partial M$, then by the existence of solutions of ODEs, there is an interval $(s - \epsilon, s + \epsilon)$ around s on which η_x satisfies the differential equation $\eta'_x(t) = X(\eta_x(t))$. Therefore $\eta_x(c)$ and $\eta_x(d)$ are in ∂M. Thus $c = f(\eta_x(c))$ and $d = f(\eta_x(d))$ may be either a or b. Since the derivative of $f \circ \eta_x$ is one, we see that $c = a$ and $d = b$. Therefore $I = [a, b]$.

Since $x \in M$ was arbitrary, and $a \leq f(x) \leq b$, we see that $f(M) = [a, b]$. Furthermore, if $x \notin \partial M$, then by the existence of solutions to ODEs, as above, we have η_x defined in a small neighborhood of $t = f(x)$, so that $a < f(x) < b$. Therefore $f^{-1}(a)$ and $f^{-1}(b)$ are unions of boundary components.

Define a map

$$F : f^{-1}(a) \times [a, b] \longrightarrow M$$

by the formula

$$F(x, t) = \eta_x(t).$$

The differentiability of F follows from the same argument as in Theorem 13.2 to prove the differentiability of T, but with η_x instead of γ_x.

Define

$$G : M \longrightarrow f^{-1}(a) \times [a, b]$$

as

$$G(x) = (\eta_x(a), f(x)).$$

The differentiability of G follows in the same way as the differentiability of F. We claim that F and G are inverses. To prove this, note that the integral curves through x and $\eta_x(t)$ are the same, that $f(\eta_x(t)) = t$ and by uniqueness of solutions to ODEs, we have $F(G(x)) = x$ and $G(F(x, t)) = (x, t)$. This proves that F is a diffeomorphism. $\square$

Corollary 12.3. *Let M be a compact manifold, and $f : M \longrightarrow \mathbb{R}$ a smooth Morse function. Let $a < b$ and suppose that $f^{-1}[a, b] \subset M$ contains no critical points. Then M^a is diffeomorphic to M^b. Furthermore, M^a is a deformation retract of M^b.*

Proof. First we prove that M^a is a deformation retract of M^b. By the regular interval theorem (Theorem 12.2), there is a natural diffeomorphism F from $f^{-1}([a, b])$ to $f^{-1}(a) \times [a, b]$. Since $f^{-1}(a) \times \{a\}$ is a deformation retract of $f^{-1}(a) \times [a, b]$, we see that $f^{-1}(a)$ is a deformation retract of $f^{-1}([a, b])$. We can now paste this deformation retraction with the identity on M_a to obtain the deformation retracton from M_b to M_a.

To prove that M^a is diffeomorphic to M^b we apply the same principle, but we need to be more careful to preserve smoothness during the patching process.

Since the set of critical points of f is a closed subset of the compact set M

(and hence is compact), the set of critical values of f is compact. Therefore there are real numbers c and d with $c < d < a$ so that there are no critical values in $[c, b]$.

By Theorem 12.2 there is a natural diffeomorphism F from $f^{-1}([c, b])$ to $f^{-1}(c) \times [c, b]$, that maps $f^{-1}([c, a])$ diffeomorphically onto $f^{-1}(c) \times [c, a]$. There is also a diffeomorphism $H : f^{-1}(c) \times [c, b] \longrightarrow f^{-1}(c) \times [c, a]$, and we can insist that it be the identity on $f^{-1}(c) \times [c, d]$ (finding this function is an easy exercise in one-variable analysis, and in case you are interested, is listed as an exercise below). Thus

$$F^{-1} \circ H \circ F : f^{-1}([c, b]) \longrightarrow f^{-1}([c, a])$$

is a diffeomorphism that is the identity on $f^{-1}([c, d])$, and thus we can patch it together with the identity on M_d to create a diffeomorphism from M_b to M_a. □

This corollary says that the topology of the submanifolds M^a does not change with $a \in \mathbb{R}$ so long as a does not pass through a critical value.

Exercise Fill in the detail of the proof of Corollary 12.3 that finds a diffeomorphism $H : f^{-1}(c) \times [c, b] \longrightarrow f^{-1}(c) \times [c, a]$ that is the identity on $f^{-1}(c) \times [c, d]$.

12.4 Passing through a critical value

We now examine what happens to the topology of these submanifolds when one does pass through a critical value. For this, we will need to understand the function f in the neighborhood of a critical point. This is what the Morse lemma provides us:

Theorem 12.4 (Morse Lemma)**.** *Let p be a nondegenerate critical point of index λ of a smooth function $f : M \longrightarrow \mathbb{R}$, where M is an n-dimensional manifold. Then there is a local coordinate system $(x_1, \dots, x_n)$ in a neighborhood U of p with $x_i(p) = 0$ with respect to which*

$$f(x_1, \dots, x_n) = f(p) - \sum_{i=1}^{\lambda} x_i^2 + \sum_{j=\lambda+1}^{n} x_j^2.$$

The proof given here is essentially that in Milnor's famous book on Morse theory [69].

Proof. Since this is a local theorem we might as well assume that $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ with a critical point at the origin, $p = 0$. We may also assume without loss of generality that $f(0) = 0$. Given any coordinate system for $\mathbb{R}^n$ we can therefore write

$$f(x_1, \ldots, x_n) = \sum_{j=1}^{n} x_j g_j(x_1, \ldots, x_n)$$

for $(x_1, \ldots, x_n)$ in a neighborhood of the origin. In this expression we have

$$g_j(x_1, \ldots, x_n) = \int_0^1 \frac{\partial f}{\partial x_j}(tx_1, \ldots, tx_n)dt.$$

Now since 0 is a critical point of f, each $g_j(0) = 0$, and hence we may write it in the form

$$g_j(x_1, \ldots, x_n) = \sum_{i=0}^{n} x_i h_{i,j}(x_1, \ldots, x_n).$$

Let $\phi_{i,j} = (h_{i,j} + h_{j,i})/2$. Hence we can combine these equations and write

$$f(x_1, \ldots, x_n) = \sum_{i,j=1}^{n} x_i x_j \phi_{i,j}(x_1, \ldots, x_n)$$

where $(\phi_{i,j})$ is a symmetric matrix of functions. By doing a straightforward calculation one sees furthermore that the matrix

$$(\phi_{i,j}(0)) = \left(\frac{1}{2}\frac{\partial^2 f}{\partial x_i \partial x_j}(0)\right)$$

and hence by the nondegeneracy assumption is nonsingular. From linear algebra we know that symmetric matrices can be diagonalized. The Morse lemma will be proved by going through the diagonalization process with the representation of f as $\sum x_i x_j \phi_{i,j}$.

Assume inductively that there is a neighborhood U_k of the origin and coordinates $\{u_1, \ldots, u_n\}$ with respect to which

$$f = \pm(u_1)^2 \pm \cdots \pm (u_k)^2 + \sum_{i,j \geq k+1} u_i u_j \psi_{i,j}(u_1, \ldots, u_n)$$

where $(\psi_{i,j})$ is a symmetric, $n-k \times n-k$ matrix of functions. By a linear change in the last $n-k$ coordinates if necessary, we may assume that $\psi_{k+1,k+1}(0) \neq 0$.

Let

$$\sigma(u_1, \ldots, u_n) = \sqrt{|\psi_{k+1,k+1}(u_1, \ldots, u_n)|}$$

in perhaps a smaller neighborhood $V \subset U_k$ of the origin. Now define new coordinates

$$v_i = u_i \quad \text{for } i \neq k+1$$

and

$$v_{k+1}(u_1, \dots u_n) = \sigma(u_1, \dots, u_n) \left[u_{k+1} + \sum_{i=k+2}^{n} u_i \frac{\psi_{i,k+1}(u_1, \dots, u_n)}{\psi_{k+1,k+1}(u_1, \dots, u_n)} \right].$$

The v_i's give a coordinate system in a sufficiently small neighborhood U_{k+1} of the origin. Furthermore a direct calculation verifies that with respect to this coordinate system

$$f = \sum_{i=1}^{k+1} \pm (v_i)^2 + \sum_{i,j=k+2}^{n} v_i v_j \theta_{i,j}(v_1, \dots, v_n)$$

where $(\theta_{i,j})$ is a symmetric matrix of functions. This completes the inductive step. The only remaining point in the theorem is to observe that the number of negative signs occuring in the expression for f as a sum and difference of squares is equal to the number of negative eigenvalues (counted with multiplicity) of $Hess_0(f)$ which does not depend on the particular coordinate system used. □

Remark. The Morse Lemma describes the behavior of the function f near a critical point, but it does not describe the behavior of the gradient near the critical point. The reason for this is that the gradient vector field depends on the Riemannian metric, and if we use the coordinate system given by the Morse Lemma, we do not know how this metric behaves.

Corollary 12.5. *If M is a manifold and $f : M \longrightarrow \mathbb{R}$ is Morse, then the set of critical points of f is a discrete subset of M.*

Proof. Suppose there were a sequence of critical points x_n converging to some point $a \in M$. Since *df* is a continuous one-form on M, we know that a is a critical point of f. Then apply the Morse Lemma above to a, which gives a formula for f in a neighborhood of a. But there are no critical points in this neighborhood as can be seen directly by calculating *df* in these coordinates. This is a contradiction. □

Exercise.

Prove the converse of Exercise 12.2; that is, if M is a compact manifold and $f : M \longrightarrow \mathbb{R}$ is a Morse function, and if a is not a regular value of f, then $f^{-1}(\{a\})$ is not a manifold.

Definition 12.3. *Let $f : M \longrightarrow [a, b]$ be a Morse function on a compact manifold. We say that f is* admissible *if $\partial M = f^{-1}(a) \cup f^{-1}(b)$, where a and b are regular values. This implies that each of $f^{-1}(a)$ and $f^{-1}(b)$ are unions of connected components of ∂M.*

Theorem 12.6. *Let $f : M \longrightarrow \mathbb{R}$ be an admissible Morse function on a compact manifold. Suppose f has a unique critical point z of index λ. Say $f(z) = c$. Then there exists a λ - dimensional cell D^λ in the interior of M with $D^\lambda \cap f^{-1}(c) = \partial D^\lambda$, and there is a deformation retraction of M onto $f^{-1}(c) \cup D^\lambda$.*

Proof, following [44]. By replacing f by $f(x)-c$ we can assume that $f(z) = 0$. Notice that by the regular interval theorem Theorem 12.2 it is sufficient to prove the theorem for the restriction of f to the inverse image of any closed subinterval of $[a, b]$ around $c = 0$.

Let (ϕ, U) be an chart around z with respect to which the Morse lemma is satisfied. Write $\mathbb{R}^n = \mathbb{R}^\Lambda \times \mathbb{R}^{n-\Lambda}$. ϕ maps U diffeomorphically onto an open set $V \subset \mathbb{R}^\Lambda \times \mathbb{R}^{n-\Lambda}$, and

$$f \circ \phi^{-1}(x, y) = -|x|^2 + |y|^2.$$

Notice that $\phi(z) = (0, 0)$. Put $g(x, y) = -|x|^2 + |y|^2$.

We will use gradient flows, which depend on the metric on M. We choose a metric for M by pulling back the Euclidean metric on $\mathbb{R}^n$ by ϕ, and extending the metric arbitrarily to the rest of M. In this way, ϕ will be a local isometry, and

$$D\phi(u)(\nabla_u(f)) = \nabla_v(g),$$

for any $u \in U$ such that $\phi(u) = v \in V$.

Let $0 < \delta < 1$ be such that V contains $\Lambda = B^\Lambda(\delta) \times B^{n-\Lambda}(\delta)$ where

$$B^i(\delta) = \{x \in \mathbb{R}^i \mid \sum_{j=1}^{n} x_j^2 \leq \delta\}$$

is the closed coordinate ball around the origin of radius δ.

Let $\epsilon > 0$ be small enough that $\sqrt{4\epsilon} < \delta$, and let

$$c^\Lambda = B^\Lambda\left(\sqrt{\epsilon}\right) \times \{0\} \subset V$$

and we define

$$D^\Lambda = \phi^{-1}(c^\Lambda) \subset M.$$

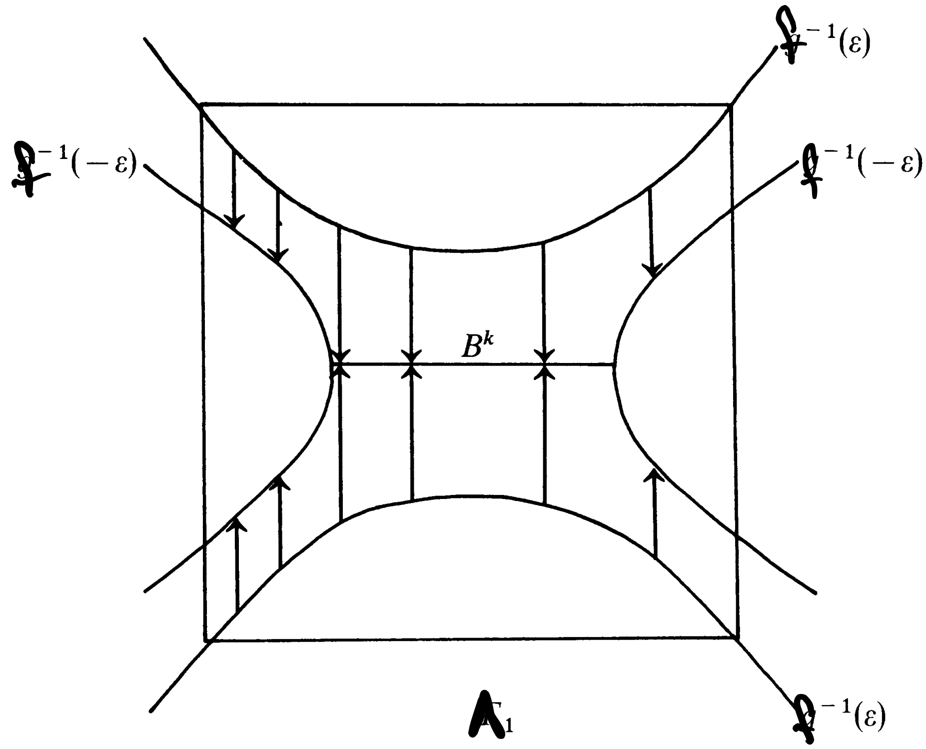
A deformation of $f^{-1}[-\epsilon, \epsilon]$ to $f^{-1}(\epsilon) \cup D^\Lambda$ is made by patching together two deformations. First consider the set

$$\Lambda_1 = B^\Lambda\left(\sqrt{\epsilon}\right) \times B^{n-\Lambda}\left(\sqrt{2\epsilon}\right).$$

Consider the following figure for the case $\Lambda = 1$, $n = 2$.

Note that inside Λ_1, $f(x, y) = -|x|^2 + |y|^2 > -\epsilon + |y|^2 > -\epsilon$. Furthermore, since $x \in B^\Lambda\left(\sqrt{\epsilon}\right)$, we have that $(x, 0) \in c^\Lambda$.

In $\Lambda_1 \cap g^{-1}[\epsilon, \epsilon]$ a deformation is obtained by moving (x, y) at constant speed along the interval joining (x, y) to the point $(x, 0) \in g^{-1}(-\epsilon) \cup B^\Lambda$, by $(x, (1-t)y)$. This deformation then induces a deformation of $\phi^{-1}(\Lambda_1)$.



Outside the set

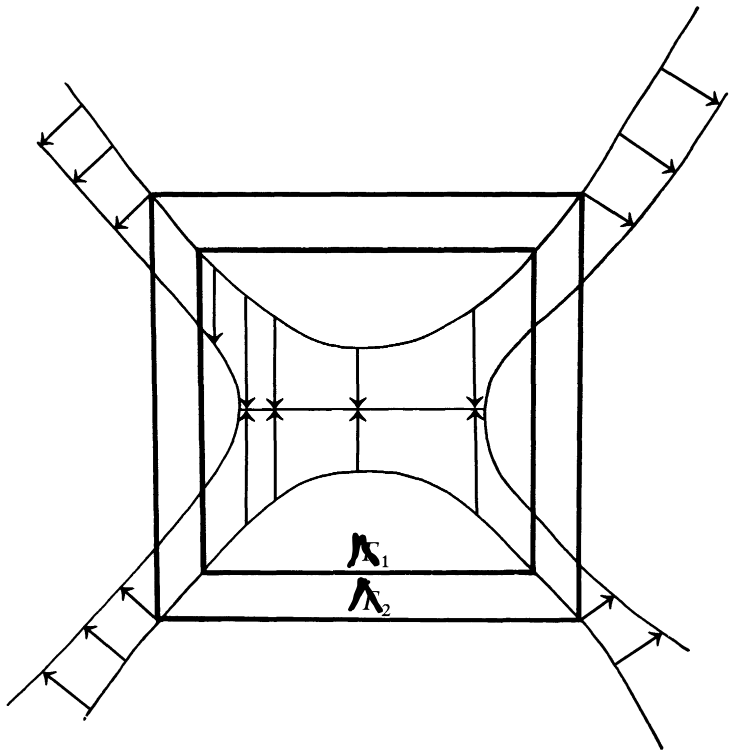
$$\Lambda_2 = B^{\Lambda}(\sqrt{2\epsilon}) \times B^{n-\Lambda}(\sqrt{3\epsilon})$$

the deformation moves each point along the vector field $-\nabla(g)$ so that it reaches $g^{-1}(-\epsilon)$ in unit time. (The speed of each point is chosen to equal the length of its path under the deformation.) See the following figure for a pictorial description of this deformation.

This deformation is transported to $U - \phi^{-1}(\Lambda_2)$ by ϕ, and is then extended over $M - \phi^{-1}(\Lambda_2)$ by following the gradient flow lines of f.

Now if such a flow enters V, we now show it may not enter Λ_2: Suppose we have a flow that enters V from the outside at time t. Then since the closure of Λ_2 is in V, there is a time arbitrarily close to t where the point is (x, y) which is not in Λ_2. Then at this time either $|x|^2 > 2\epsilon$ or $|y|^2 > 3\epsilon$. But if $|y|^2 > 3\epsilon$ then because for$g^{-1}([-\epsilon, \epsilon])$, we have $\epsilon > -|x|^2 + |y|^2 > -|x|^2 + 3\epsilon$ so that $|x|^2 > 2\epsilon$. Therefore, either way, $|x|^2 > 2\epsilon$. But for x non-zero, $|x|$ increases along flow lines. Therefore (x, y) will not be in Λ_2 for any later time until it leaves V (and by repeating the argument for future visits to V, it never enters Λ_2).

In $f^{-1}([-\epsilon, \epsilon]) - \phi^{-1}(\Lambda_2)$, then, the downward gradient flow is defined, and since we assume there are no other critical points than z, the methods of



the proof of Theorem 12.2 show that the flows defined there flow downward to $f^{-1}(-\epsilon)$.

On $f^{-1}([-\epsilon,\epsilon]) - \phi^{-1}(\Lambda_2)$, then, we can define the deformation to flow along the gradient flow with constant speed, with speed equal to the length of the flow line from the point to its destination on $f^{-1}(-\epsilon)$. In this way, after unit time, everything in $f^{-1}([-\epsilon,\epsilon]) - \phi^{-1}(\Lambda_2)$ is deformed into $f^{-1}(-\epsilon)$.

To extend the deformation to points of $\Lambda_2 - \Lambda_1$ it suffices to find a vector field on Λ which agrees with X in Λ_1 and with $-\nabla(g)$ in $\Lambda - \Lambda_2$. Such a vector field is

$$Y(x, y) = 2(\mu(x, y)x, -y)$$

where the map $\mu : \mathbb{R}^\Lambda \times \mathbb{R}^{n-\Lambda} \longrightarrow [0, 1]$ vanishes in Λ_1 and equals 1 outside Λ_2. The fact that each integral curve of Y which starts at a point of

$$(\Lambda_2 - \Lambda_1) \cap g^{-1}[-\epsilon, \epsilon]$$

must reach $g^{-1}(-\epsilon)$ because $|x|$ is nondecreasing along integral curves.

The global deformation of $f^{-1}[-\epsilon, \epsilon]$ into $f^{-1}(-\epsilon) \cup D^\Lambda$ is obtained by moving each point of Λ at constant speed along the flow line of Y until it reaches $g^{-1}(-\epsilon) \cup B^\Lambda$ in unit time and transporting this motion to M via ϕ;

while each point of $M - \phi^{-1}(\Lambda)$ moves at constant speed along the flow line of $\nabla(f)$ until it reaches $f^{-1}(-\epsilon)$ in unit time. Points on $f^{-1}(-\epsilon) \cup D^\Lambda$ stay fixed. □

12.5 Homotopy equivalence to a *CW* complex and the Morse inequalities

Theorem 12.7. *Let M be a closed manifold, and $f : M \longrightarrow \mathbb{R}$ a Morse function on M. Then M has the homotopy type of a CW complex, with one cell of dimension Λ for each critical point of index Λ.*

Proof. Without loss of generality, the critical points of f all have different values under f (if $f(p) = f(q)$ and p and q are critical points, then let $B_1 \subset B_2$ be balls around q small enough that in $B_2 - B_1$, we have $|\nabla f|$ bounded away from zero by some ϵ, and add a small bump function to f supported in B_2 and constant in B_1 whose gradient is bounded above by ϵ, and which does not raise the value of $f(q)$ high enough to reach another critical value of f).

Now let $a_0 < \cdots < a_k$ be a sequence of real numbers so that a_0 is less than the minimum value of f, a_k is greater than the maximum value of f, and between a_i and a_{i+1} there is exactly one critical point. By Theorem 12.6 we have a homotopy equivalence h_i between $M^{a_{i+1}}$ and $M^{a_i} \cup D^{\lambda_i}$ (where the union is via an attaching map as in a *CW* complex). By composing the h_i's, we obtain a homotopy equivalence from $M = M^{a_k}$ to a union of disks attached by *CW* attaching maps.

□

Corollary 12.8. *Given $f : M \longrightarrow \mathbb{R}$ as above there is a chain complex referred to as the* Morse–Smale complex

$$\ldots \longrightarrow C_\lambda \xrightarrow{\partial_\Lambda} C_{\lambda-1} \longrightarrow \ldots \xrightarrow{\partial_1} C_0 \tag{12.2}$$

whose homology is $H_*(M;\mathbb{Z})$, where C_λ is the free abelian group generated by the critical points of f of index Λ.

Proof. This is the cellular chain complex coming from the *CW* complex in Theorem 12.7. □

We can now prove some of the results promised in the introduction, that relate the topology of M to the numbers of critical points of f:

Corollary 12.9 (Morse's Theorem)**.** *Let $f : M \longrightarrow \mathbb{R}$ be a C^∞ function so that all of its critical points are nondegenerate. Then the Euler characteristic $\chi(M)$ can be computed by the following formula:*

$$\chi(M) = \sum (-1)^i c_i(f)$$

where $c_i(f)$ is the number of critical points of f having index i.

Proof. The Euler characteristic $\chi(M)$ can be computed as the alternating sum of the ranks of the chain groups of any CW decomposition of M. □

Corollary 12.10 (Weak Morse Inequalities)**.** *Let c_p be the number of critical points of index p and let β_p be the rank of the homology group $H_p(M)$. Then*

$$\beta_p \leq c_p.$$

Proof. The chain group $C_p \otimes \mathbb{R}$ generated by the c_p cells of dimension p is a vector space of dimension c_p. The group of cycles is of dimension at most c_p. After quotienting by the boundaries, we see that $H_p(M;\mathbb{R})$ is a vector space of dimension at most c_p. □

Corollary 12.11 (Strong Morse Inequalities)**.** *Let M, f, $c_i(f)$, and $b_i(M)$ be as above. Then for all natural numbers i,*

$$\sum_{k=0}^{i}(-1)^{i-k}c_i(f) \geq \sum_{k=0}^{i}(-1)^{i-k}b_i(M).$$

Proof. The proof is similar except we take a closer look at the boundaries. Tensoring the chains with $\mathbb{R}$, so that we write $V_k = C_k \otimes \mathbb{R}$, we get the following chain complex of vector spaces:

$$\ldots \longrightarrow V_i \xrightarrow{\partial_i} V_{i-1} \longrightarrow \ldots \xrightarrow{\partial_1} V_0$$

We write V_k as $Im(\partial_{k+1}) \oplus H_k(M;\mathbb{R}) \oplus (V_k/\ker(\partial_k))$ and note that $Im(\partial_{k+1})$ is of the same dimension as $V_{k+1}/\ker(\partial_{k+1})$. Thus if we define d_k to be the dimension of $V_k/\ker(\partial_k)$, we have

$$c_k = d_{k+1} + b_k + d_k$$

and applying the alternating sum above we get

$$\sum_{k=0}^{i}(-1)^{i-k}c_i(f) = d_{i+1} + \sum_{k=0}^{i}(-1)^{i-k}b_i(M)$$

(where here we need that $d_0 = 0$). This proves the strong Morse inequalities. □

To see that the strong Morse inequalities prove the weak Morse inequalities, write down the strong Morse inequality for i and for $i+1$, and subtract the two inequalities. To see that the strong Morse inequalities imply Morse's theorem, apply the strong Morse inequality for i and for $i+1$ for i larger than the dimension of the manifold M, noting that $c_j = 0$ and $b_j = 0$ for all $j > \dim(M)$.

It is instructive to work out the following:

Exercise

Show that the strong Morse inequalities is "strictly stronger" than the weak Morse inequalities together with Morse's theorem. More specifically, show that given the $n+1$-tuple of natural numbers $(b_0, \ldots, b_n)$, we can find another $n+1$-tuple of natural numbers $(c_0, \ldots, c_n)$ so that these numbers satisfy the weak Morse inequality and the Morse theorem but not the strong Morse inequalties.

A typical application of these result is to use homology calculations to deduce critical point data. For example we have the following.

Application

Every Morse function on the complex projective space

$$f : \mathbb{CP}^n \longrightarrow \mathbb{R}$$

has at least one critical point in every even dimension $\leq 2n$.

The following is a historically important application of Morse theory, due to Reeb, that follows from the techniques we have mentioned so far.

Application

Let M^n be a closed manifold admitting a Morse function

$$f : M \longrightarrow \mathbb{R}$$

with only two critical points. Then M is homeomorphic to the sphere S^n.

Remark This theorem does *not* imply that M is diffeomorphic to S^n. In [73] Milnor found an example of a manifold that is homeomorphic, but not diffeomorphic to S^7. Indeed he proved that there are 28 distinct differentiable structures on S^7! Milnor actually used this fact to prove that the manifolds he constructed were homeomorphic to S^7.

Proof of Theorem 12.5. Let S and N be the critical points. By the compactness of M we may assume that S is a minimum and N is a maximum. (Think of them as the eventual south and north poles of the sphere.) Let $f(S) = t_0$ and $f(N) = t_1$. By the Morse lemma there are coordinates $(x_1, \ldots, x_n)$ in a neighborhood U_+ of N with respect to which f has the form

$$-x_1^2 + \cdots + -x_n^2 + t_1.$$

Therefore there is a $b < t_1$ so that if we let $D_+ = f^{-1}[b, t_1]$ then there is a diffeomorphism

$$D_+ \cong D^n$$

with $\partial D_+ = f^{-1}(b) \cong S^{n-1}$. Repeating this process with the minimum point P we obtain a point $a > t_0$ and a diffeomorphism of the space $D_- = f^{-1}[t_1, a]$,

$$D_- \cong D^n$$

with $\partial D_- = f^{-1}(a) \cong S^{n-1}$. By Theorem 12.2 we have that

$$f^{-1}[a, b] \cong f^{-1}(a) \times [a, b] \cong S^{n-1} \times [a, b].$$

Hence we have a decomposition of the manifold

$$\begin{aligned} M = f^{-1}[t_0, t_1] &= f^{-1}[t_0, a] \cup f^{-1}[a, b] \cup f^{-1}[b, t_1] \\ &\cong D^n \cup S^{n-1} \times [a, b] \cup D^n \end{aligned}$$

where the attaching maps are along homeomorphisms of S^{n-1}. We leave it as an exercise to now construct a homeomorphism from this manifold to S^n.

□

Exercise

Finish the proof of Theorem 12.5 by showing that the resulting space

$$D^n \cup S^{n-1} \times [a, b] \cup D^n$$

is homeomorphic to S^n. Hint: Start by embedding one D^n into S^n, then embed $S^{n-1} \times [a, b]$ into S^n to match the first embedding, then to put the last D^n in, you must think of D^n as the cone on S^{n-1}. This last part is why the proof does not prove that this is diffeomorphic to S^n.

In general, there are many applications of this work to the problem of classifying manifolds of dimensions 5 and higher, leading to the h-cobordism theorem and the s-cobordism theorem, and surgery theory. There are many books that describe these developments of the 1960s and 1970s, the old classics being Milnor's book on the h-cobordism theorem, [70], Wall's book on surgery theory [97], and Browder's book [14].

We now show that the set of Morse functions is open and dense in the set of smooth functions. In particular, every manifold M admits a Morse function $f : M \longrightarrow \mathbb{R}$. In the proof, we will use the transversality theorem, done in Chapter 8.

Theorem 12.12. *Let M be a compact n-manifold. Let $r \geq 2$. The set of C^r Morse functions from M to $\mathbb{R}$ is dense in $C^r(M, \mathbb{R})$.*

Proof. . We refer the reader to [44] for a complete proof. However we describe the proof of a related fact that is a key component of the proof of this theorem. Consider the exterior derivative map

$$d : C^\infty(M; \mathbb{R}) \to \Omega^1(M) = \Gamma_M(T^*M)$$

where $\Gamma_M(T^*M)$ denotes the space of smooth sections of the cotangent bundle.

Let $\zeta \subset T^*M$ be the zero section of T^*M. Inside $\Gamma_M(T^*M)$ we have the space of sections that are transverse to the zero section, which we denote by $\pitchfork (M, T^*M; \zeta) \subset \Gamma_M(T^*M)$. We observe that the space of Morse functions is simply the inverse image under d of $\pitchfork (M, T^*M; \zeta)$. Furthermore, by the transversality theorem (Corollary 8.9), we can conclude that $\pitchfork (M, T^*M; \zeta) \subset \Gamma_M(T^*M)$ a dense subspace.

To see this characterization of Morse functions, observe that df being transverse to the zero section means that whenever $df_p = 0$ (i.e p is a critical point), then $D(df)_p(T_pM) \oplus T_p\zeta(M)(= T_pM) = T_{df_p}T^*M$. But one can easily check that this condition is equivalent to Hess f_p being nonsingular. □

Exercises

(1) Let $M \subset \mathbb{R}^L$ be a closed, smooth submanifold. For each $v \in S^{L-1}$ let $f_v : M \to \mathbb{R}$ be the map $f_v(x) =< v, x >$. (This is essentially orthogonal projection into the line through v.) Show that the set of $v \in S^{L-1}$ such that f_v is a Morse function is open and dense.

(2) Let $M \subset \mathbb{R}^L$ be a closed, smooth submanifold.. Show that the set of points $u \in \mathbb{R}^L$ such that the map $x \to |x - u|^2$ is a Morse function on M, is open and dense.

Remark. The functions described in exercise (1) are called "height functions". The functions described in exercise (2) are "distance functions". These are both very common and highly useful examples of Morse functions.

(3). Recall that $\mathbb{RP}^n = \{(x_1, \cdots x_{n+1}) \in S^n \subset \mathbb{R}^{n+1}\}/\sim$ where $(x_1, \cdots x_{n+1}) \sim -(x_1, \cdots x_{n+1})$. We denote an equivalence class using square brackets $[x_1, \cdots x_{n+1}] \in \mathbb{RP}^n$.

Define a smooth function

$$f : \mathbb{RP}^n \to \mathbb{R}$$

by

$$f([x_1, \cdots x_{n+1}]) = \sum_{k=1}^{n+1} k x_k^2$$

(a) Show that the critical points of f are $u_1, \cdots u_{n+1}$, where $u_i = [0, \cdots 0, 1, 0, \cdots 0]$, where the 1 occurs in the i^{th} coordinate.

(**Hint.** First construct charts U_i, $i = 1, \cdots, n+1$, where $U_i = \{[x_1, \cdots x_{n+1}] : x_i \neq 0\}$, by proving that there are diffeomorphisms $\psi_i : U_i \cong B_1^n$, where B_i^n the unit open ball around the origin in $\mathbb{R}^n$. ψ_i given by

$$\psi_i[x_1, \cdots x_{n+1}] = (x_1, \cdots x_{i-1}, x_{i+1}, \cdots, x_n).$$

Then compute the differential of the composition

$$B_i^n \xrightarrow{\psi_i^{-1}} U_i \subset \mathbb{RP}^n \xrightarrow{f} \mathbb{R}.$$

Use this to show that the only critical point of f in U_i is u_i.

(b). Compute the index of each critical point.

(c). Show that $f : \mathbb{RP}^n \to \mathbb{R}$ is a Morse function.

(d). Using parts (a) - (c) to show that the Euler characteristic of $\mathbb{RP}^n$ is 0 if n is odd and 1 if n is even.

(e) Prove that if n is even, $\mathbb{RP}^n$ does not admit a nowhere zero vector field.

13

Spaces of Gradient Flows

13.1 The gradient flow equation

Let M be a manifold, g a Riemannian metric on M, and $f : M \longrightarrow \mathbb{R}$ be a Morse function. A *(gradient) flow line* is a curve

$$\gamma : (a, b) \longrightarrow M$$

that satisfies the differential equation

$$\frac{d\gamma}{dt}(s) + \nabla_{\gamma(s)}(f) = 0 \tag{13.1}$$

for all $s \in (a, b)$. Here $\nabla(f)$ is the gradient vector field as defined in (12.1). If we imagine a particle that travels along γ, with t describing time, the particle travels in the path of steepest descent, with velocity given by the gradient.

Recall from the discussion in the previous chapter, that the gradient vector field $\nabla(f)$, and therefore the gradient flow equation depends on the Riemannian metric g in the following way:

$$< \nabla_x f \, , v >_g = df(x)(v)$$

where $< \, , >_g : T_xM \times T_xM \to \mathbb{R}$ is the nonsingular, symmetric bilinear form on the tangent space at $x \in M$ defined by the metric g. The typical gradient seen in undergraduate calculus classes occurs on $\mathbb{R}^n$ with the standard Euclidean metric.

Exercises

(1). Verify that if $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a differentiable function on $\mathbb{R}^n$, and if we use the Euclidean metric on $\mathbb{R}^n$, then

$$\nabla(f) = \frac{\partial f}{\partial x_1} e_1 + \cdots + \frac{\partial f}{\partial x_n} e_n.$$

(2). Let f be as in the previous exercise, but suppose the metric is given by an arbitrary symmetric matrix g (that is, $< e_i, \, e_j >_g = g_{ij}$).

Find the formula for $\nabla(f)$ in terms of f and g.

Remark. Notice that the property of $p \in M$ being a critical point of f does not depend on the metric. As a bilinear form, the Hessian of f at a critical point $p \in M$ does not depend on the metric either. Therefore the concepts of p being a non-degenerate critical point, and the index of a critical point do not depend on a choice of metric.

Example. If a is a critical point of f, then the constant curve $\gamma(t) = a$ satisfies the flow equations, so γ is a flow line. Conversely, by the uniqueness of solutions of ordinary differential equations, if any flow line contains a critical point $a \in M$, then it must be the constant curve at a.

Example Let $M = \mathbb{R}^2$ with the Euclidean metric, and let $f(x, y) = x^2 + y^2$. Then we can solve the gradient flow equations:

$$\frac{dx}{dt} = -2x$$
$$\frac{dy}{dt} = -2y$$

and therefore the gradient flow lines are $(x(t), y(t)) = (ae^{-2t}, be^{-2t})$ for some fixed a and b. For any such flow line, y/x is a constant, so each flow parameterizes an open line in the plane emanating from the origin. See figure 13.1.

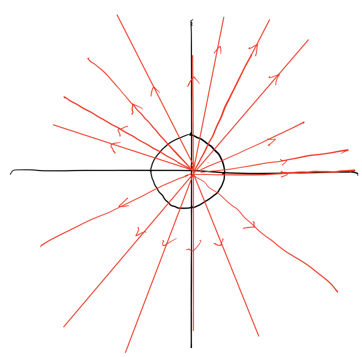


FIGURE 13.1
Flow lines for $f(x, y) = x^2 + y^2$

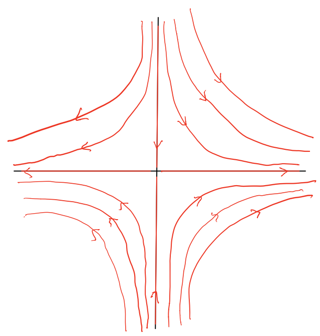


FIGURE 13.2
Flow lines for $f(x, y) = x^2 - y^2$

Example Let $M = \mathbb{R}^2$ with the Euclidean metric, and let $f(x, y) = x^2 - y^2$. Then it turns out that the gradient flow lines are $(x, y) = (ae^{2t}, be^{-2t})$ for some fixed a and b. For any such flow line, xy is a constant, so the gradient flow lines are hyperbolas of the form $xy = c$. See figure 13.1.

Example Let $M = S^2 \subset \mathbb{R}^3$ with the standard round metric, and let $f(x, y, z) = z$ (the so-called "height function" defined by the embedding of S^2 into $\mathbb{R}^3$). Then there are two critical points: one minimum at $(0, 0, -1)$, and one maximum at $(0, 0, 1)$. The flow lines are "lines of longitude". See figure 13.1.

Example Let T^2 be the torus in $\mathbb{R}^3$, embedded as follows:

$$(\theta, \phi) \longrightarrow (b\cos(\phi), (a + b\sin(\phi))\cos(\theta), (a + b\sin(\phi))\sin(\theta)$$

where $0 < b < a$. The picture looks like a donut standing on its edge, as in figure 13.4. Again, take for f the "height function" z. Then there are four critical points: $(\theta, \phi) = (\pm\pi/2, \pm\pi/2)$, as you can check. The index for $(\pi/2, \pi/2)$ is 2, the index for $(\pi/2, -\pi/2)$ and $(-\pi/2, \pi/2)$ is 1, and the index for $(-\pi/2, -\pi/2)$ is 0.

There are two natural choices for a metric on T^2: either the metric induced from the embedding from $\mathbb{R}^3$, or the flat metric defined by $ds^2 = d\theta^2 + d\phi^2$. Although pictorially it may help to ponder the resulting gradient flow lines

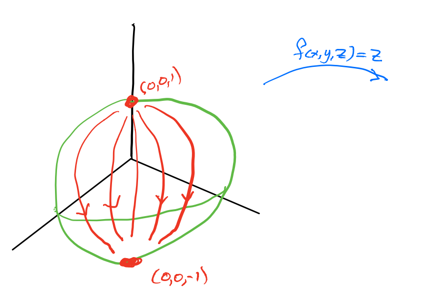


FIGURE 13.3
Flow lines for the height function on S^2

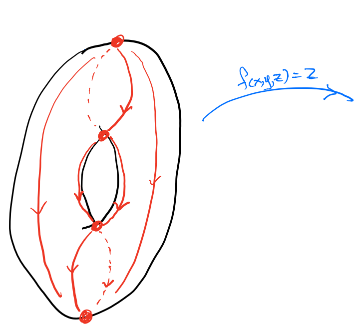


FIGURE 13.4
Flow lines for the height function on the torus

from the metric induced by $\mathbb{R}^3$ (these are the actual flows of steepest descent on a physical donut), it is easier to calculate the flow lines when the flat metric is used. The flow lines can be described explicitly, or else you can verify that there are flows with $\theta = \pm\pi/2$ for which θ is constant, and flows with $\phi = \pm\pi/2$ for which ϕ is constant. These flows give rise to two flows from the index 2 critical point to one of the index 1 critical points, two flows from one index 1 critical point to the other, and two flows from the lower index 1 critical point to the index 0 critical point. The other flows are in a one-parameter family of flows which go from the index 2 critical point to the index 0 critical point.

Exercise Work out the details of the above examples. Find the closed form solutions to the gradient flow equations and find which critical points they connect to.

Lemma 13.1. *A smooth function $f : M \longrightarrow \mathbb{R}$ is nonincreasing along flow lines. f is strictly decreasing along any flow line which does not contain a critical point.*

Proof. Let $\gamma : (a,b) \longrightarrow M$ be a flow line. Consider the composition $f \circ \gamma : (a,b) \longrightarrow \mathbb{R}$. Its derivative is given by

$$\begin{aligned}\frac{d}{dt} f(\gamma(t)) &= \langle \nabla_{\gamma(t)}(f), \frac{d\gamma(t)}{dt} \rangle \\ &= \langle \nabla_{\gamma(t)}(f), -\nabla_{\gamma(t)}(f) \rangle \\ &= -\left|\nabla_{\gamma(t)}(f)\right|^2 \leq 0.\end{aligned}$$

The only way this can be zero is if $\gamma(t)$ is on a critical point of f. In particular, if $\gamma(t)$ does not contain in its image a critical point of f, then $f(\gamma(t))$ is strictly decreasing.

□

Remark In the above proof, we showed

$$\frac{d}{dt} f(\gamma(t)) = -\left|\nabla_{\gamma(t)}(f)\right|^2.$$

We can also show

$$\begin{aligned}\frac{d}{dt} f(\gamma(t)) &= \langle \nabla_{\gamma(t)}(f), \frac{d\gamma(t)}{dt} \rangle \\ &= \langle -\frac{d\gamma(t)}{dt}, \frac{d\gamma(t)}{dt} \rangle \\ &= -\left|\frac{d\gamma(t)}{dt}\right|^2 \leq 0.\end{aligned}$$

and this would also prove that $f(\gamma(t))$ is nonincreasing.

Remark Now if $\gamma(t)$ *does* contain a critical point p, then by Example 13.1 the flow must be a constant flow, and $f(\gamma(t))$ is constant on this flow.

Thus there are two kinds of flow lines: constant flows that stay at a critical point, and flows that descend for all t, and do not contain a critical point.

Theorem 13.2. *Suppose that M is a closed, smooth manifold, and $f : M \to \mathbb{R}$ a smooth map. Then given any $x \in M$ there is a unique flow line defined on entire real line*

$$\gamma_x : \mathbb{R} \longrightarrow M$$

that satisfies the initial condition

$$\gamma_x(0) = x.$$

Furthermore the limits

$$\lim_{t \to -\infty} \gamma_x(t) \quad \textit{and} \quad \lim_{t \to +\infty} \gamma_x(t)$$

converge to critical points of f. These are referred to as the starting *and* ending points *of the flow γ_x.*

The flow map

$$T : M \times \mathbb{R} \longrightarrow M$$

defined by $T(x, t) = \gamma_x(t)$ is smooth.

Proof. Let $x \in M$. By the existence and uniqueness of solutions to ordinary differential equations, there is an $\epsilon > 0$ and a unique path

$$\gamma_x : (-\epsilon, \epsilon) \longrightarrow M$$

satisfying the flow equation

$$\frac{d\gamma_x(t)}{dt} + \nabla_{\gamma_x(t)}(f) = 0$$

for all $|t| < \epsilon$, and the initial condition $\gamma_x(0) = x$. By the compactness of M we can choose a uniform ϵ for all $x \in M$. Notice therefore that for $|t| < \epsilon$ we can define a self map of M,

$$\gamma_t : M \longrightarrow M$$

by the formula $\gamma_t(x) = \gamma_x(t)$. Notice that $\gamma_0 = id$, the identity map. By uniqueness it is clear that

$$\gamma_{t+s} = \gamma_t \circ \gamma_s$$

providing that $|t|$, $|s|$, $|t + s| < \epsilon$. Among other things this implies that each γ_t is a *diffeomorphism* of M because $\gamma_t^{-1} = \gamma_{-t}$.

Now suppose that $|t| \geq \epsilon$. Write $t = k(\epsilon/2) + r$ where $k \in \mathbb{Z}$ and $|r| < \epsilon/2$. If $k \geq 0$ we define

$$\gamma_t = \gamma_{\frac{\epsilon}{2}} \circ \gamma_{\frac{\epsilon}{2}} \circ \ldots \gamma_{\frac{\epsilon}{2}} \circ \gamma_r$$

where the map $\gamma_{\frac{\epsilon}{2}}$ is repeated k times. If $k < 0$ then replace $\gamma_{\frac{\epsilon}{2}}$ by $\gamma_{\frac{-\epsilon}{2}}$. Thus for every $t \in \mathbb{R}$ we have a map $\gamma_t : M \longrightarrow M$ satisfying $\gamma_t \circ \gamma_s = \gamma_{t+s}$, and hence each γ_t is a diffeomorphism.

The curves

$$\gamma_x : \mathbb{R} \longrightarrow M$$

defined by $\gamma_x(t) = \gamma_t(x)$ clearly satisfy the flow equations and the initial condition $\gamma_x(0) = x$. This means that the gradient flow equations can be solved for all $t \in \mathbb{R}$, and in particular, we will from now on require that gradient flow lines be defined as functions $\gamma : \mathbb{R} \longrightarrow M$ instead of being defined only on an open interval.

Now let γ be a flow line. Consider the composition $f \circ \gamma : \mathbb{R} \longrightarrow \mathbb{R}$. By the Fundamental Theorem of Calculus, if $a < b$, then

$$(f \circ \gamma)(b) - (f \circ \gamma)(a) = \int_a^b \frac{d}{dt}(f \circ \gamma)(t)\, dt.$$

Since M is compact $f \circ \gamma$ has bounded image, so the left side is bounded. By Lemma 13.1, $\frac{d}{dt}(f \circ \gamma) < 0$. Therefore

$$\lim_{t \to \pm\infty} \frac{d}{dt}(f \circ \gamma)(t) = 0.$$

By the proof of Lemma 13.1 we know that

$$0 = \lim_{t \to \pm\infty} \frac{d}{dt} f(\gamma(t)) = \lim_{t \to \pm\infty} -\left|\nabla_{\gamma(t)}(f)\right|^2.$$

Let U be any union of small disjoint open balls around the critical points. By the compactness of M, $M - U$ is compact, so $|\nabla_x(f)|^2$ has a minimum value on $M - U$. Since $M - U$ has no critical points, this minimum value is strictly positive. But since the above limit is zero, we know that for sufficiently large $|t|$, $\gamma(t) \in U$. Since the balls are disjoint and $\gamma(t)$ is continuous, there is a critical point p so that for any open ball around p, $\gamma(t)$ is in that ball for sufficiently large t. Therefore $\lim_{t\to\infty} \gamma(t)$ exists and is equal to p; similarly, $\lim_{t\to-\infty} \gamma(t)$ exists and is equal to a critical point.

The differentiability of the flow map $T(x, t) = \gamma_x(t)$ with respect to t follows because $\gamma_x(t)$ satisfies the differential equation. The differentiability of T with respect to x follows from Peano's theorem (the differentiable dependence of solutions to ODEs with respect to initial conditions). This is proved in Hartman's book on ODEs [40] in chapter V, Theorem 3.1. □

Let $\gamma(t)$ be a non-constant gradient flow line from p to q. Then by Lemma 13.1, we know that $h(t) = f(\gamma(t))$ is strictly decreasing, and in particular, is a diffeomorphism from $\mathbb{R}$ to the open interval $(f(q), f(p))$. We can therefore consider the smooth curve $\eta(t) = \gamma(h^{-1}(t))$ from $(f(q), f(p))$ to M. Then it is easy to check that $f(\eta(t)) = t$. So γ and η have the same image, but the parameter in η represents height (that is, the value of f).

Exercise Prove that $f(\eta(t)) = t$ as claimed above.

We can also extend η to a continuous map from the closed interval $[f(q), f(p)]$ to M by defining $\eta(f(q)) = q$ and $\eta(f(p)) = p$.

Exercise Prove that the extension of η to the closed interval $[f(q), f(p)]$ is continuous.

Definition 13.1. *If $\gamma(t)$ is a non-constant gradient flow line for f, and $h(t) = f(\gamma(t))$, then*

$$\eta(t) = \gamma(h^{-1}(t)) : [f(q), f(p)] \longrightarrow \mathbb{R}$$

is the height-reparameterization *of γ, and such a curve is a* height-parameterized gradient flow *of f.*

Remark This reparameterization of γ is a direction-reversing one, since h is strictly decreasing. This is to be expected since $f(\gamma(t))$ is decreasing but $f(\eta(t)) = t$ is increasing.

We now differentiate η.

Exercise Prove

$$\frac{d}{dt}\eta(t) = \frac{\nabla_{\eta(t)}(f)}{\left|\nabla_{\eta(t)}(f)\right|^2}$$

Therefore, $\eta(t)$ is the solution to another differential equation which may be described as follows:

Lemma 13.3. *Away from the critical points of f, we may consider the vector field*

$$X(x) = \frac{\nabla_x(f)}{|\nabla_x(f)|^2}.$$

Then a curve $\zeta : (s_1, s_2) \longrightarrow M$ that satisfies

$$\frac{d}{dt}\zeta(t) = X(\zeta(t))$$

is a height-reparameterized flow line.

Proof. We insist that (s_1, s_2) be maximal. We then can show that $\frac{d}{dt}f(\zeta(t)) = 1$ as usual (do this now if you wish). Pick a number $s \in (s_1, s_2)$, and consider the gradient flow line $\gamma(t)$ so that $\gamma(0) = \zeta(s)$. We do the height-reparameterization to γ to get a height-reparameterized curve η. Now η satisfies the same differential equation as ζ, and $\eta(f(\zeta(s))) = \zeta(s)$, so we translate the domain as follows: $\eta_0(t) = \eta(t + f(\zeta(s)) - s)$ satisfies the same differential equation as ζ and $\eta_0(s) = \zeta(s)$ so by the uniqueness of solutions to ODEs, $\eta_0 = \zeta$.

Therefore solutions to $\frac{d}{dt}\zeta(t) = X(\zeta(t))$ are precisely those that are height-parameterized flows. □

Therefore $X(x)$ and $\nabla(f(x))$ have the same integral curves, although with different parameterizations.

13.2 Stable and unstable manifolds

As before, for any point $x \in M$, let $\gamma_x(t)$ be the flow line through x, i.e. it satisfies the differential equation

$$\frac{d}{dt}\gamma = -\nabla_\gamma(f)$$

with the initial condition $\gamma(0) = x$. We know by Theorem 13.2 that $\gamma_x(t)$ tends to critical points of f as $t \to \pm\infty$. So for any critical point a of f we define the *stable manifold* $W^s(a)$ and the *unstable manifold* $W^u(a)$ as follows:

Definition 13.2. *Let M be a manifold, and f a smooth function on M. Let a be a critical point for f. We define the two subsets of M:*

$$W^s(a) = \{x \in M : \lim_{t \to +\infty} \gamma_x(t) = a\}$$
$$W^u(a) = \{x \in M : \lim_{t \to -\infty} \gamma_x(t) = a\}.$$

and call $W^s(a)$ the stable manifold *of a and $W^u(a)$ the* unstable manifold *of a.*

In other words, $W^s(a)$ is the set of points on M that flow down to a, and $W^u(a)$ is the set of points on M flow out from a. The use of the term "manifold" is justified by the stable manifold theorem:

Theorem 13.4 (Stable Manifold Theorem)**.** *Let M be an n-dimensional manifold, and $f : M \longrightarrow \mathbb{R}$ a Morse function. Let a be a critical point of f of index λ. Then $W^u(a)$ and $W^s(a)$ are smooth submanifolds diffeomorphic to the open disks D^λ and $D^{n-\lambda}$, respectively.*

This will be proved in Section 13.2 below for a large class of metrics (though it is in general true for all metrics).

Proposition 13.5. *If M is a compact manifold with Riemannian metric g, and $f : M \longrightarrow \mathbb{R}$ is a Morse function, then*

$$M = \bigcup_a W^u(a)$$

is a partition of M into disjoint sets, where the union is taken over all critical points a of f.

Proof. The fact that the union of the $W^u(a)$ is M comes from the fact that every point of M lies on a flow line γ, and we can always find $\lim_{t \to -\infty} \gamma(t)$.

The fact that the $W^u(a)$ and $W^u(b)$ are disjoint when $a \neq b$ is due to the fact that γ is unique. □

Exercise Find the unstable manifolds for each critical point in Example 13.1.

Exercise Find the unstable manifolds for each critical point in Example 13.4.

From these exercises you can see that this decomposition of M makes M look like a CW complex, with one cell of dimension λ for each critical point of index λ. The torus example is problematic because an edge gets attached to the middle of another edge, but consider the following fix:

Consider the torus in $\mathbb{R}^3$ as before, but with a slight perturbation. That is, tilt the torus by pulling it down so it is not quite vertical. Then consider

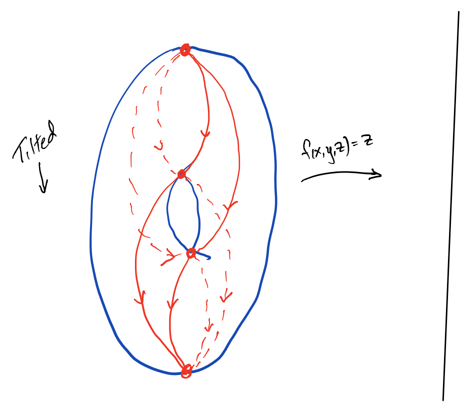


FIGURE 13.5
Flow lines for the height function on the "tilted torus"

the height function $f(x, y, z) = z$. The following is a picture of the resulting flow lines.

The point is that with this example, we have a decomposition of M into cells, with a cell of dimension λ for each critical point of index λ. These are essentially the cells D^λ in Theorem 12.6.

The disks appearing in this result and those appearing in Theorem 12.6 are related in the following way. Suppose that $[t_0, t_1] \subset \mathbb{R}$ has the property that $f^{-1}([t_0, t_1]) \subset M$ has precisely one critical point a of index λ with $f(a) = c \in (t_0, t_1)$. Then by Theorem 12.6 there is a disk $D^\lambda \subset M^{t_1}$ and a homotopy equivalence

$$M^{t_1} \simeq M^{t_0} \cup D^\lambda.$$

Now note that $W^u(a) \cap f^{-1}([t_0, t_1])$ is, under a Euclidean metric defined by the Morse coordinate chart, equal to the D^λ mentioned in the proof of Theorem 12.6.

In Chapter 11, we proved the Morse Lemma (Theorem 12.4), which says that locally, around any nondegenerate critical point, we can choose a coordinate chart so that

$$f(x_1, \ldots, x_n) = f(p) - \sum_{i=1}^{\lambda} x_i^2 + \sum_{j=\lambda+1}^{n} x_j^2. \tag{13.2}$$

In other words, we have a local explicit formula for f around a critical point, no matter what f is, as long as the critical point is non-degenerate.

What does the gradient vector field look like around such a critical point?

Based on the above equation (13.2), you might expect the gradient to be this:

$$\nabla(f) = (-2x_1, \ldots, -2x_\lambda, 2x_{\lambda+1}, \ldots, 2x_n) \tag{13.3}$$

But because the metric is not prescribed, it is possible (even likely) that the gradient vector field is *not* this at all. Recall that the gradient is obtained by $< v, \nabla(f)) >= df(v)$ and therefore depends on the metric (see the discussion in Chapter 11, and in particular where the gradient was defined 12.1).

Since we are dealing with gradient vector fields, and their corresponding flow lines, it would make sense for us to want to use a metric so that there are local coordinates where (13.3) is true. This is especially the case, since if equation (13.3) is true, then the gradient flow equation

$$\frac{d}{dt}\gamma(t) = -\nabla_{\gamma(t)}(f)$$

would take the form (if we write $\gamma(t) = (x_1(t), \ldots, x_n(t))$):

$$\begin{aligned}
\dot{x}_1 &= 2x_1 \\
&\vdots \\
\dot{x}_\lambda &= 2x_\lambda \\
\dot{x}_{\lambda+1} &= -2x_{\lambda+1} \\
&\vdots \\
\dot{x}_n &= -2x_n
\end{aligned}$$

which is easily solved.

If the metric is anything else, we might still hope to diagonalize this system of differential equations, choosing coordinates $(x_1, \ldots, x_n)$ so that $\dot{\gamma}(t) = -\nabla_{\gamma(t)}(f)$ looks like

$$\dot{x}_i = c_i x_i \tag{13.4}$$

for some non-zero real constants $c_1, \ldots, c_n$. Then the c_i would be negatives of the eigenvalues of the Hessian of f at the critical point, and the corresponding eigenvectors would be the standard basis vectors $\partial/\partial x_i$ in this coordinate chart.

Unfortunately, it is in general impossible to choose coordinates so that (13.4) holds, as the following exercises show:

Exercise Solve the system of differential equations (13.4).

Exercise Solve the system of differential equations

$$\dot{x} = 2x \tag{13.5}$$
$$\dot{y} = -y \tag{13.6}$$
$$\dot{z} = z + xy \tag{13.7}$$
$$\tag{13.8}$$

and show that there is no change of coordinates that transform it into the form (13.4).

Exercise Let $f(x, y, z) = x^2 - y^2 + z^2$. Find a metric $g(x, y, z)$ on a neighborhood of $(0, 0, 0) \in \mathbb{R}^3$ so that the gradient flow equations near the origin are as in equation (13.8). Hence prove that it is in general impossible to choose coordinates so that the gradient flow equations look like equation (13.4) in a neighborhood of the critical point. Note that the metric must be symmetric and positive definite in the neighborhood.

Note that in this exercise, what goes wrong is a kind of "resonance" phenomenon that occurs in ordinary differential equations when two eigenvalues are the same. By analogy, we would expect this kind of problem to be rare, and we might hope that for most situations, we can choose coordinates to put the gradient flow equations in the standard form of equation (13.4), but to address this will take us rather far afield (see [40]).

Instead, we choose to follow Hutchings [52] to modify the given metric, in neighborhoods of the critical points, to the standard metric so that equation (13.2) gives rise to the gradient flow equations in equation (13.4).

This motivates the following definition, due to Hutchings [52]:

Definition 13.3. *Let M be a manifold and f be a Morse function. A metric is said to be* nice *if there exist coordinate neighborhoods around each critical point of f so that for each such neighborhood there are non-zero real numbers $c_1, \ldots, c_n$ so that the gradient flow equations are*

$$\dot{x}_i = c_i x_i,$$

as in (13.4).

Proposition 13.6. *Let M be a compact manifold and f a Morse function. There exists a nice metric on (M, f). In fact, these are dense in the L^2 space of metrics.*

Proof. Let g_0 be any smooth metric on M. Consider the set of critical points of f. Apply the Morse lemma (Lemma 12.4), to find nonoverlapping coordinate neighborhoods of each critical point of f in M, each with coordinates $x_1, \ldots, x_n$ so that the Morse function in each neighborhood is

$$f(x_1, \ldots, x_n) = f(p) - \sum_{i=1}^{\lambda} x_i^2 + \sum_{j=\lambda+1}^{n} x_j^2.$$

For each critical point a of f, let U_a be the coordinate neighborhood given by the Morse lemma, let B_1 be a coordinate ball around a that is completely inside U_a, and let B_2 be another coordinate ball around a of smaller radius than B_1. (By *coordinate ball* I mean the set whose coordinates $(x_1, \ldots, x_n)$ satisfy $x_1^2 + \cdots + x_n^2 < r$ for some r.)

Let $\phi : U_a \longrightarrow \mathbb{R}$ be a smooth function so that ϕ is 1 on B_2 and 0 outside B_1. Let g_E be the standard Euclidean metric with respect to the $x_1, \ldots, x_n$ coordinates. Define g to be

$$g = g_0(x)(1 - \phi(x)) + g_E(x)\phi(x).$$

Since the set of symmetric positive definite bilinear forms is a convex set, this convex linear combination of the two metrics will be a metric on U_a. Extend g by setting it equal to g_0 on the rest of M. Then g is a metric for which a is nice.

Now proceed inductively through the other critical points of M. This creates a metric g so that there is a coordinate neighborhood metric ball B around each critical point where both f and the metric are in a standard form. Then the gradient flow equation

$$\frac{d\gamma}{dt} = -\nabla_\gamma(f)$$

looks like equation (13.4).

By taking B_2 smaller and smaller, we see that the difference between g and g_0 is supported on an arbitrarily small set, and by the boundedness of the metric on M, we know that this difference is arbitrarily small in L^2. $\square$

We now prove the Stable manifold theorem for nice metrics:

Theorem 13.7 (Stable Manifold Theorem)**.** *Let M be an n-dimensional manifold, with nice metric g, and $f : M \longrightarrow \mathbb{R}$ a Morse function. Let a be a critical point of f of index λ. Then $W^u(a)$ and $W^s(a)$ are smooth submanifolds diffeomorphic to the open disks D^λ and $D^{n-\lambda}$, respectively.*

Remark This theorem is actually true for all metrics (not necessarily "nice"), but to prove this would take too long and we don't need it in this generality. Curious readers can see [40] for the proof.

Proof of the Stable Manifold Theorem. If g is a nice metric, then there is a coordinate neighborhood B around each critical point where the gradient flow equations are

$$\frac{d\gamma_i}{dt} = c_i\gamma_i(t)$$

where $\gamma_i(t)$ is the i-th coordinate of γ. Note that the c_i are the negatives of eigenvalues of the Hessian, corresponding to the directions given by the standard basis in the coordinate chart. Reorder the coordinates so that the first λ eigenvalues are the negative ones (so that the first λ values of c_i are positive).

Then explicitly,

$$\gamma_i(t) = \begin{cases} \gamma_i(0)e^{|c_i|t}, & i \leq \lambda \\ \gamma_i(0)e^{-|c_i|t}, & i > \lambda \end{cases} \tag{13.9}$$

inside B.

We prove the theorem for $W^s(a)$. The proof for $W^u(a)$ is exactly analogous, and besides, it follows from the $W^s(a)$ case, applied to the function $-f$. We will first prove that $W^s(a)$ is smooth in a small neighborhood of a.

Let W_0 be the subset of B consisting of those points where $x_1 = x_2 = \cdots = x_\lambda = 0$. Then from the explicit solution (13.9), we see that $W_0 \subset W^s(a)$.

Now W_0 is an open disk of dimension $n - \lambda$ centered on a, and hence is a manifold, and is furthermore a submanifold of M.

Recall from Theorem 13.2 that the flow map defined as

$$T : M \times \mathbb{R} \longrightarrow M$$

$$T(x,t) = \gamma_x(t)$$

is smooth. Apply this flow backward in time by some time t: define $W_t = T(W_0, -t)$. This will be diffeomorphic to W_0 and a subset of $W^s(a)$. As t goes to infinity, we span a larger and larger subset of $W^s(a)$.

Let $x \in W^s(a)$, and γ the corresponding gradient flow line with $\gamma(0) = x$. Since $\lim_{t\to\infty} \gamma(t) = a$, we know that for some $t_0 > 0$, $\gamma(t) \in B$ for all $t \geq t_0$. I will now show that $\gamma(t_0) \in W_0$.

Suppose $\gamma(t_0) \notin W_0$. The translated flow $\eta(t) = \gamma(t+t_0)$ is a gradient flow line, with the property that $\eta(0) \notin W_0$, and $\eta(t) \in B$ for all $t > 0$. Then for some coordinate $i > \lambda$, $\eta_i(0) \neq 0$. By the explicit solution (13.9), $\eta_i(t)$ will grow indefinitely, so that eventually η (and hence γ) leaves the coordinate ball B. This is a contradiction. Therefore, $\gamma(t_0) \in W_0$.

Since every element of $W^s(a)$, when flowed forward, eventually is in W_0, we know that $\cup_t W_t = W^s(a)$.

Let $\psi : [0, 1) \longrightarrow \mathbb{R}$ be a smooth monotonic function with $\psi(0) = 0$ and $\lim_{t\to 1} \psi(t) = +\infty$. Using $|x|$ as $\sqrt{x_1^2 + \cdots + x_n^2}$, and r_0 as the radius of the coordinate ball B, we see that $T(x, \psi(|x|/r_0))$ maps W_0 diffeomorphically onto $W^s(a)$. Recall that W_0 is a submanifold of M which is a disk of dimension $n - \lambda$. Therefore, $W^s(a)$ is a submanifold of M and diffeomorphic to $D^{n-\lambda}$. □

Exercise Prove the Stable Manifold Theorem (Theorem 13.7) for the unstable manifold $W^u(a)$, without applying the theorem to stable manifolds of $-f$. Instead, carefully go through the proof for $W^s(a)$ and write out the corresponding proof that would work for $W^u(a)$.

Proposition 13.8. *The tangent space of $W^s(a)$ at a is the positive eigenspace of the Hessian of f at a. Similarly, the tangent space of $W^u(a)$ at a is the negative eigenspace of the Hessian of f at a.*

Proof. Again, for the sake of our proof we are assuming the metric is nice, but this is unnecessary. The result holds in general.

Now $W^s(a)$ is a smooth submanifold of M, so its tangent space at a is well-defined. Define W_0 as in the previous proof, as

$$\{(x_1, \ldots, x_n) \mid x_1 = \cdots = x_\lambda = 0\}.$$

The tangent space to W_0 is therefore the span of $\partial/\partial x_i$ for $i = \lambda + 1$ to n. This is the positive eigenspace of the Hessian.

On the other hand $W_0 \subset W^s(a)$, and since they are of the same dimension, W_0 is an open neighborhood of a in $W^s(a)$. Therefore W_0 and $W^s(a)$ have the same tangent space at a.

The proof for $W^u(a)$ can be done similarly, or if you wish, you may use the result for $W^s(a)$ on $-f$. □

Let a be a critical point of f. Let us consider the function f restricted to $W^u(a)$. Since $W^u(a)$ is defined to be the set of points which in some sense lie "below" a on gradient flow lines, we expect a to be a maximum of f on $W^u(a)$, and level sets to be spheres around a.

Theorem 13.9. *Let (M, g) be a Riemannian manifold and $f : M \longrightarrow \mathbb{R}$ a Morse function. Let a be a critical point of f. Let $h : W^u(a) \longrightarrow \mathbb{R}$ be the restriction of f to $W^u(a)$. Then a is the unique critical point of h, and it is the absolute maximum. If $\epsilon > 0$ is small enough, and $f(a) - \epsilon < c < f(a)$, then $h^{-1}(c)$ is diffeomorphic to a $\lambda - 1$ dimensional sphere in $W^u(a)$ around a.*

Similarly, let $j : W^s(a) \longrightarrow \mathbb{R}$ be the restriction of f to $W^s(a)$. Then a is the unique critical point of j, and it is the absolute minimum. If $\epsilon > 0$ is small enough, and $f(a) < c < f(a) + \epsilon$, then $j^{-1}(c)$ is diffeomorphic to a $n - \lambda - 1$ dimensional sphere in $W^s(a)$ around a.

Proof. We will prove this for $W^u(a)$, and the result for $W^s(a)$ is the same using $-f$ instead of f.

Let $x \in W^u(a)$, and $x \neq a$. Let $\gamma(t)$ be the unique gradient flow line with $\gamma(0) = x$. Since $x \in W^u(a)$, we have that $\lim_{t\to-\infty} \gamma(t) = a$.

According to Lemma 13.1, $f(\gamma(t))$ is strictly decreasing. By the continuity of f, $\lim_{t\to-\infty} f(\gamma(t)) = f(a)$. So $f(a) > f(x)$. Therefore, a is the absolute maximum of h.

Now, $\gamma(t) \in W^u(a)$ for all t, so $\gamma'(0) \in T_x W^u(a)$. Since $f(\gamma(t))$ is strictly decreasing, $\gamma'(0) \neq 0$ (if it were, $\frac{d}{dt} f(\gamma(t)) = \nabla(f) \cdot \gamma'(0)$ would be zero). By the gradient flow equation $\gamma'(t) = -\nabla_{\gamma(t)}(f)$, the $-\nabla_x(f) \neq 0$. Therefore, x is not a critical point of h. Since x was arbitrary, except for not equalling a, there are no critical points of h except for a.

Now we consider the Hessian of h at a. Find a coordinate chart of M around a so that $W^u(a)$ is given by the equations $x_{\lambda+1} = \cdots = x_n = 0$. By the invariance of the Hessian under coordinate change (Proposition 12.1), the Hessian of f can be computed in such a coordinate chart. Since $T_a W^u(a)$ is the negative eigenspace of the Hessian of f (Proposition 13.8) we conclude that the matrix

$$\left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{ij}$$

is negative definite. Since $W^u(a)$ is given by setting $x_{\lambda+1}, \ldots, x_n$ to be constant (in fact, zero), we see that for $i, j \leq \lambda$, this matrix is the same as

$$\left(\frac{\partial^2 h}{\partial x_i \partial x_j} \right)_{ij}.$$

Therefore the Hessian of h at a is negative definite. In particular, a is a non-degenerate critical point of h, and h is Morse.

We now consider the preimages $h^{-1}(c)$.

For this, we use the Morse Lemma (Theorem 12.4) applied to h on the manifold $W^u(a)$. The Morse Lemma states that there exist a coordinate neighborhood U around a with coordinates $x_1, \dots, x_\lambda$ on $W^u(a)$ so that

$$h(x_1, \dots, x_\lambda) = f(a) - x_1^2 - \dots - x_\lambda^2.$$

Let $\epsilon > 0$ be given so that the ball

$$B = \{(x_1, \dots, x_\lambda) | x_1^2 + \dots + x_\lambda^2 < \epsilon\}$$

is contained in U. Within this ball it is clear that the preimages $h^{-1}(c)$ (when $f(a) - \epsilon < c < f(a)$) are coordinate spheres around a. We will now verify that there are no other parts to $h^{-1}(c)$ which are outside B.

Suppose $x \in W^u(a)$, and $x \notin B$. As earlier in the proof, let $\gamma(t)$ be the gradient flow with $\gamma(0) = x$. As before, $\lim_{t \to -\infty} \gamma(t) = a$. But B is an open set around U. Therefore, for some $t < 0$, $\gamma(t) \in B$. Since $x = \gamma(0)$ is not in B, the generalized Jordan curve theorem says that there exists some $T < 0$ for which $\gamma(T)$ is on the boundary of B. Since $f(\gamma(t))$ is strictly decreasing,

$$f(x) = f(\gamma(0)) < f(\gamma(T)) = f(a) - \epsilon.$$

So $f(x) < f(a) - \epsilon$. Therefore, if $f(a) - \epsilon < c < f(a)$, then $h^{-1}(c)$ is a susbet of B, and is therefore the coordinate spheres we found earlier. □

13.3 The Morse–Smale condition

An important, generic condition of a Morse function on a Riemannian manifold, is that the unstable and stable manifolds of the various critical points intersect transversally. This is called the *Morse-Smale transversality*condition, which we study in this subsection.

Definition 13.4. *Suppose $f : M \longrightarrow \mathbb{R}$ is a Morse function on a Riemannian manifold M, that satisfies the extra condition that for any two critical points a and b the unstable and stable manifolds $W^u(a)$ and $W^s(b)$ intersect transversally. This is the* Morse–Smale condition*, and if f satisfies this condition, we call f a* Morse–Smale function*.*

Smale [87] showed that Morse–Smale functions exist. More specifically, given a metric g and function $f : M \longrightarrow \mathbb{R}$, there exists another metric g' and another function $f' : M \longrightarrow \mathbb{R}$ so that f' is Morse–Smale with respect to g'. His proof also demonstrates that f and f' and g and g' can be made arbitrarily close to each other. Hence the set of configurations of functions and metrics so that the functions are Morse–Smale with respect to that metric is dense.

Actually, more is true: if f is Morse, then for an open, dense set of metrics g, f is Morse–Smale. This can be proved using the same techniques that are used in the proofs in Smale's paper. We will sketch out a proof at the end of this chapter that the set of such metrics is dense. In the meantime we will first study some properties of Morse–Smale functions.

Exercise Suppose f is Morse (not necessarily Morse–Smale) and suppose b is a critical point of f. Do $W^u(b)$ and $W^s(b)$ always intersect transversally?

The main purpose of the Morse–Smale condition is that it allows us to see how stable and unstable manifolds of different critical points intersect. For every pair of critical points a and b, let

$$W(a, b) = W^u(a) \cap W^s(b).$$

$W(a, b)$ is the space of all points in M that lie on flow lines starting from a and ending at b.

Proposition 13.10. *Let (M, g) be a Riemannian manifold of dimension n, let $f : M \longrightarrow \mathbb{R}$ be Morse–Smale, and a and b be two critical points of f. Then $W(a, b)$ is a smooth manifold of dimension $index(a) - index(b)$.*

Proof. If f is Morse–Smale, then $W^u(a)$ and $W^s(b)$ intersect transversally. Therefore the intersection $W^u(a) \cap W^s(b) = W(a, b)$ is a manifold of dimension $\dim(W^u(a)) + \dim(W^s(b)) - n = index(a) + (n - index(b)) - n = index(a) - index(b)$. □

Corollary 13.11. *Let $f : M \longrightarrow \mathbb{R}$ be a Morse–Smale function, and let a and b be two distinct critical points of f. If $index(a) \leq index(b)$, then $W(a, b) = \emptyset$.*

Proof. If $index(a) < index(b)$, then the previous proposition shows that $W(a, b)$ is a manifold of negative dimension, so it must be empty.

If $index(a) = index(b)$, then similarly $W(a, b)$ must be a manifold of dimension 0, but since the gradient flow acts freely on elements of $W(a, b)$, the dimension of $W(a, b)$ must be at least one. Therefore it must be empty. □

Definition 13.5. *We refer to the number*

$$index(a) - index(b)$$

as the relative index *of a and b.*

Exercise Suppose a and b are critical points of f and $a \neq b$. Are a and b in $W(a, b)$? If there are other critical points of f, is it possible that these are in $W(a, b)$? Now consider the case $a = b$. What is $W(a, b)$?

Let $f : M \to \mathbb{R}$ be a Morse-Smale function on the smooth, closed Riemannian manifold M. Let a and b be critical points. The fundamental object of study will not usually be $W(a,b)$, which is the space of points lying on flow lines between a and b, but rather a space of flow lines themselves. Notice there is an action of the group $\mathbb{R}$ on $W(a,b)$ by the following. Let $x \in W(a,b)$, and let $\gamma_x : \mathbb{R} \to M$ be the unique gradient flow line satisfying the initial condition

$$\gamma_x(0) = x.$$

Then the action of $\mathbb{R}$ is just the flow. More specifically,

$$\mathbb{R} \times W(a,b) \to W(a,b) \tag{13.10}$$

$$t, x \to \gamma_x(t). \tag{13.11}$$

Notice that this action is free, and we can study the orbit space $W(a,b)/\mathbb{R}$. Notice that two points x and y in $W(a,b)$ are in the same orbit space under this $\mathbb{R}$-action if and only if they lie on the same flow line. Therefore a point in the orbit space $W(a,b)/\mathbb{R}$ can be viewed as simply a flow line. We therefore make the following definition.

Definition 13.6. *Define the "Moduli Space of flow lines" $\mathcal{M}(a,b)$ to be the orbit space,*

$$\mathcal{M}(a,b) = W(a,b)/\mathbb{R}.$$

For good intuition and for practical considerations it is useful to instead pick out a representative of each $\mathbb{R}$ orbit in $W(a,b)$. One way to do this is to select a real number t between $f(a)$ and $f(b)$ and pick the representative in $f^{-1}(t)$. This is the approach will allow us an alternate, but equivalent definition of the moduli space $\mathcal{M}(a.b)$.

Definition 13.7. *Pick a value $t \in \mathbb{R}$ between $f(a)$ and $f(b)$, and let $W(a,b)^t$ to be the set $W(a,b) \cap f^{-1}(t)$.*

Proposition 13.12. *If a and b are distinct critical points of f, then $W(a,b)^t$ is a smooth submanifold of M.*

Proof. First, we see that $f|_{W(a,b)} : W(a,b) \longrightarrow \mathbb{R}$ is transverse to the point $\{t\} \subset \mathbb{R}$. This is because for any point $x \in W(a,b)$ so that $f(x) = t$, $\nabla_x(f)$ is not zero, and so neither is $df_x(\nabla_x(f)) = \|\nabla_x(f)\|^2$. Therefore $t \in \mathbb{R}$ is a regular value of $ff|_{W(a,b)}$, and hence by the Regular Value Theorem, $\left(f|_{W(a,b)}\right)^{-1}(\{t\}) = W(a,b)^t$ is a smooth submanifold of $W(a,b)$ of codimension one. □

Theorem 13.13. *Let a and b be distinct critical points of f. The function*

$$\phi : W(a,b)^t \times \mathbb{R} \longrightarrow W(a,b)$$

defined by

$$\phi(p, s) = T_s(p)$$

is a diffeomorphism.

Proof. We begin by proving ϕ is onto. Let $x \in W(a, b)$. Let γ be the flow line that has $\gamma(0) = x$. Since $\lim_{t\to\infty} f(\gamma(t)) = f(b)$ and $\lim_{t\to-\infty} f(\gamma(t)) = f(a)$, by continuity we have that for some s, $f(\gamma(-s)) = t$. Then $\gamma(-s) = p$ and $T_s(p) = x$.

Now to show ϕ is one-to-one, suppose $x = \phi(p_1, s_1) = \phi(p_2, s_2)$. Then $T_{-s_1}(x) = p_1$ and $T_{-s_2}(x) = p_2$, meaning that the unique flow line γ with $\gamma(0) = x$ also has $\gamma(s_1) = p_1$ and $\gamma(s_2) = p_2$. Since $f(p_1) = t = f(p_2)$, and $\frac{d}{ds} f(\gamma(s)) < 0$, it must be that $s_1 = s_2$ and therefore $p_1 = p_2$.

Therefore ϕ^{-1} is defined as a set map. To show that ϕ^{-1} is continuous, it is necessary to show that if U is an open neighborhood of $(p, s) \in W(a, b)^t \times \mathbb{R}$, then there exists an open neighborhood of $\phi(p, s)$ in $W(a, b)$ that is a subset of $\phi(U)$. It suffices to show this for open neighborhoods U of the form $B_p(\epsilon) \times (s - \epsilon, s + \epsilon)$. Since T_{-s} is a diffeomorphism of M that maps neighborhoods of $\phi(p, s)$ to neighborhoods of $\phi(p, 0)$, it suffices to prove this for $s = 0$.

So what we need to show is if $\epsilon > 0$ is sufficiently small, and $p \in W(a, b)^t$, then there exists a δ so that whenever $d(p, y) < \delta$, then writing $y = \phi(q, r)$ gives us $|r| < \epsilon$ and $d(p, q) < \epsilon$.

Since p is not a critical point, there is a δ_1 so that $B_p(2\delta_1)$ does not contain critical points. In this ball, $m = \inf |\nabla f|^2$ is strictly greater than zero and $\sup |\nabla f|$ is finite. If $\sup |\nabla f| > 1$, then let $M = \sup |\nabla f|$, but otherwise let $M = 1$. By continuity of f there is a δ_2 so that $|f(p) - f(B_p(\delta_2))| < m\epsilon/2M$. Choose δ to be smaller than $\min(\delta_1, \delta_2, \epsilon/2)$.

Now in the proof of Lemma 13.1, we saw that

$$\frac{d}{dt} f(\gamma(t)) = -\left|\nabla(f)\right|^2.$$

Integrating and using the fundamental theorem of calculus, we get

$$|f(\gamma(-r)) - f(\gamma(0))| \geq |r| \inf |\nabla f|^2$$

which leaves us with

$$|r|m = |r| \inf |\nabla f|^2 \leq |f(p) - f(y)| < m\epsilon/2M$$

so that $|r| < \epsilon/2M < \epsilon$.

Now,

$$\begin{aligned} d(q, y) &\leq \int |\gamma'(t)|\, dt \\ &= \int |\nabla(f)|\, dt \\ &\leq Mr < \epsilon/2. \end{aligned}$$

So by the Triangle inequality, $d(p,q) \leq d(p,y) + d(q,y) < \delta + \epsilon/2 < \epsilon$. Therefore ϕ^{-1} is continuous.

To prove ϕ^{-1} is smooth, we estimate $d\phi$ and show it is non-degenerate. Let $(p,s) \in W(a,b)^t \times \mathbb{R}$ and let $v_1, \ldots, v_k$ be a basis for the tangent space of $W(a,b)^t$ at p, and let $\partial/\partial t$ be the tangent vector to $\mathbb{R}$. Now if $d\phi$ is degenerate at (p,s), then $d\phi(v_1), \ldots, d\phi(v_k), d\phi(\partial/\partial t)$ would be linearly dependent. Now since $\phi|_{W(a,b)^t \times \{s\}}$ is just the flow map T_s, and this flow map is a diffeomorphism, we know that $d\phi(v_1), \ldots, d\phi(v_k)$ are linearly independent. Therefore any linear dependence would involve $d\phi(\partial/\partial t)$, so that

$$d\phi(\partial/\partial t) = \sum c_k d\phi(v_k)$$

for some real numbers c_k.

Now since $\phi(p,s) = T_s(p)$, $d\phi(\partial/\partial t)$ at (p,s) is $\frac{\partial}{\partial s} T_s(p) = \gamma'(s)$, where γ is the flow with $\gamma(0) = p$. Then if we compose with T_{-s},

$$\begin{aligned} dT_{-s} d\phi(\partial/\partial t) &= \sum c_k dT_{-s} d\phi(v_k) \\ dT_{-s} \gamma'(s) &= \sum c_k v_k \\ \gamma'(0) &= \sum c_k v_k. \end{aligned}$$

But we know $\gamma'(0)$ is transverse to $TW(a,b)^t$, which is a level set of f. Therefore, we have a contradiction, and $d\phi$ is non-degenerate. Therefore ϕ^{-1} is smooth. □

The following is an immediate consequence of this theorem.

Corollary 13.14. *Let $f : M \to \mathbb{R}$ be a Morse-Smale function on a closed Riemannian manifold M. Let $a, b \in M$ be critical points and $t \in \mathbb{R}$ be a number strictly between $f(a)$ and $f(b)$. Then the composition*

$$W^t(a,b) \hookrightarrow W(a,b) \xrightarrow{project} W(a,b)/\mathbb{R} = \mathcal{M}(a,b)$$

is a diffeomorphism.

We therefore may identify the moduli space of flows $\mathcal{M}(a,b)$ with the level space $W^t(a,b)$.

If we use the notation $+a$ to denote the function $+a : \mathbb{R} \longrightarrow \mathbb{R}$ with $+a(x) = x + a$, then the following diagram commutes:

$$\begin{array}{ccc} W(a,b)^t \times \mathbb{R} & \xrightarrow{\phi} & W(a,b) \\ {\scriptstyle (1,+s)}\downarrow & & {\scriptstyle T_s}\downarrow \\ W(a,b)^t \times \mathbb{R} & \xrightarrow{\phi} & W(a,b) \end{array}$$

We now sketch a proof that the set of metrics for which a Morse function is Morse–Smale is dense.

Theorem 13.15. *Let M be a manifold. Let $f : M \longrightarrow \mathbb{R}$ be a Morse function. For a dense set of metrics g, f is Morse–Smale.*

Proof. (Sketch of proof) We suppose a Riemannian metric g is given, and show that there exists a Riemannian metric g' arbitrarily close to g so that f is Morse–Smale with respect to g'. Recall that ∇_g refers to the gradient using the metric g.

We start by finding a vector field X close to $\nabla_g f$ that agrees with $\nabla_g f$ near the critical points of f but so that the unstable and stable manifolds are transverse (step 1). We then show that for some metric g' close to g, $X = \nabla_{g'}(f)$ (step 2).

Step 1: finding the vector field X

The details of this step are found in Smale's proof of Theorem A in the work just cited above ([87]).

Let the critical values of f be $c_1 < \cdots < c_k$. Choose $\epsilon > 0$ arbitrary, but small enough so that for each i, $c_{i+1} > c_i + 4\epsilon$, and in fact, small enough so that for each critical point p, Theorem 13.9 gives us that $W^s(p) \cap f^{-1}((-\infty, c])$ is a ball for all $f(p) < c < f(p) + 4\epsilon$.

We first let $X = \nabla g$. Then we proceed by induction on i, starting at c_1 and ending at c_k, at each stage altering X in $f^{-1}(c_i + \epsilon, c_i + 3\epsilon)$.

At stage i in the induction, we consider each critical point p so that $f(p) = c_i$. In a neighborhood of p, we consider

$$Q = f^{-1}(c_i + 2\epsilon) \cap W^s(p).$$

Since $-\nabla(f)$ is transverse to level sets of f, the gradient flow can be integrated in a small neighborhood of Q so that there is a coordinate z with $-m \leq z \leq m$ so that $\partial/\partial z$ is $-\nabla(f)$ and $z = 0$ coinciding with Q. Here m is chosen so that this keeps us in $f^{-1}(c_i + \epsilon, c_i + 3\epsilon)$. By the coordinate structure of f near p, a tubular neighborhood U of Q is a trivial λ-disk bundle. So if P is a λ dimensional disk of radius 1, then there is a diffeomorphism sending $[-m, m] \times P \times Q$ onto this tubular neighborhood of Q, so that the first coordinate is the coordinate z, and $0 \times 0 \times Q$ is mapped to Q by the identity function. From now on, we will identify U with $[-m, m] \times P \times Q$ in our notation.

Consider all critical points q with $f(q) > c_i$. Let

$$S = \cup_{q, f(q) > c_i, \nabla_q(f) = 0} (0 \times P \times Q) \cap W^s(q)$$

and let $g : S \longrightarrow P$ be the restriction of $\pi_P : [-m, m] \times P \times Q \longrightarrow 0 \times P \times 0$ to S. By Sard's theorem there exist $v \in P$ arbitrarily close to zero so that $2v$ is a regular value of g.

Now construct $\beta : [-m, m] \longrightarrow \mathbb{R}$ so that $\beta(z) \geq 0$, $\beta(z) = 0$ in a neighborhood of $\partial[-m, m]$, and $\int_0^{\pm m} \beta(z)\, dz = \pm|v|$. If v was chosen small enough, $\beta(z)$ and $|\beta'(z)|$ can be kept smaller than ϵ.

Let $P_0 \subset P$ be a λ-dimensional disk of radius $1/3$.

We also construct a smooth $\gamma : P \longrightarrow \mathbb{R}$ so that $0 \leq \gamma \leq 1$, $\gamma = 0$ in a neighborhood of P, $\gamma = 1$ on P_0, and $|\partial\gamma/\partial x_i| \leq 2$.

Let X' be the vector field on M that equals X outside U, and on $[-m, m] \times P \times Q$ let X' be given by

$$X' = -\frac{\partial}{\partial z} - \beta(z)\gamma(x)\frac{v}{|v|}.$$

We use the bounds on β and γ to ensure that $df(X') > 0$.

To see that the new stable and unstable manifolds $W'^s(p)$ and $W'^u(q)$ intersect transversally, we examine any point of intersection, and flow by X' until it is in $f^{-1}(c_i + 2\epsilon)$. It will then be at a point $\{0\} \times P \times Q \subset [-m, m] \times P \times Q$. The flow X' for time $\pm m$ carries $(0, x, y) \in [-m, m] \times P \times Q$ to $(\pm m, x \pm v, y)$, as can be seen by explicitly integrating out X'.

If q is any critical point with $f(q) > c_i$, then consider the new stable manifold $W'^s(q)$ of q under X'. It agrees with the old stable manifold $W^s(q)$ on $(m, 0, y)$, and after flowing by $-m$ we get to $(0, -v, y)$.

Also, the new unstable manifold $W'^u(p)$ agrees with the old unstable manifold $W^u(p)$ for $z = -m$, and flowing by X' for time m from here shows that $W'^u(p) \cap (0 \times P \times Q)$ is

$$\{(0, x + v, y) | (0, x, y) \in W^u(p)\}.$$

So their intersection is the set

$$\{(0, -v, y) | (0, 2v, y) \in W^u(p)\}$$

and since $2v$ is a regular value of g, this intersection is transverse.

We do this for all the critical points with critical value c_i, and these do not interfere with each other as long as ϵ is small enough that the neighborhoods U do not intersect.

We then proceed with larger and larger i, until we have constructed a new X'.

Step 2: finding the metric g'

Note that X is unchanged (it still equals $\nabla_g f$) near critical points of f. So near critical points of f we define g' to equal g. Outside these neighborhoods we define, at each point $x \in M$, a linear transformation A_x on $T_x M$ that is the identity on the kernel of df, and sends X to

$$\frac{\sqrt{df(X)}}{\|df\|_g}\nabla_g(f).$$

Since $df(X) > 0$, this is invertible, and if X is close to $\nabla_g(f)$, then A_x is close to the identity. Let $g'(v, w) = g(Av, Aw)$. Then g' is close to g.

Now if we write an arbitrary vector $w \in T_x(M)$ as $w = w_0 + aX$ where $df(w_0) = 0$, then it is a matter of computation to verify that $g'(X, w) = df(w)$. By definition of gradient, this means $X = \nabla_{g'}(f)$. □

Corollary 13.16. *Given a Morse function $f : M \longrightarrow \mathbb{R}$, there exists a metric g so that f is Morse–Smale.*

13.4 The moduli space of gradient flows $\mathcal{M}(a, b)$, its compactification, and the flow category of a Morse function

Throughout this section we assume that M is a C^∞ closed, Riemannian metric and that $f : M \to \mathbb{R}$ is a Morse function satisfying the Morse-Smale condition. As seen above, the Morse-Smale condition is generic.

13.4.1 The moduli space $\mathcal{M}(a, b)$

Let a and b be critical points of $f : M \to \mathbb{R}$. As seen above, the moduli space $\mathcal{M}(a, b)$ is a smooth manifold of dimension equal to one less than the relative index,

$$dim\mathcal{M}(a, b) = index(a) - index(b) - 1.$$

The points of $\mathcal{M}(a, b)$ are the gradient flow lines that start at a and end at b. Of course the gradient flow lines in $\mathcal{M}(a, b)$ don't really "start" at a or "end" at b, but rather they satisfy the initial conditions $\lim_{t\to-\infty} \gamma(t) = a$ and $\lim_{t\to+\infty} \gamma(t) = b$. This is a rather clumsy arrangement, especially if we want to "glue" flow lines. That is, if $\alpha \in \mathcal{M}(a, b)$ and $\beta \in \mathcal{M}(b, c)$, then we should be able to describe a ("piecewise") flow $\alpha \circ \beta$ which should "start" at a and "'end" at c. This is most easily done if we reparameterize these curves so that they be "height parameterized gradient flow lines", as defined in Definition 13.1.

13.4.2 The compactified moduli space of flows and the flow category

As above let M be a closed Riemannian manifold and let $f : M \to \mathbb{R}$ be a Morse function satisfying the Morse-Smale condition. Let $\nabla(f)$ be the gradient vector field of f. Consider a flow lines of f which is a curve $\gamma : \mathbb{R} \to M$ satisfying the differential equation

$$\frac{d\gamma}{dt} = -\nabla(f).$$

If γ is a flow-line then $\gamma(t)$ converges to critical points of f as $t \to \pm\infty$ and we define

$$s(\gamma) = \lim_{t\to-\infty} \gamma(t), \qquad e(\gamma) = \lim_{t\to\infty} \gamma(t).$$

Since f is strictly decreasing along flow lines it defines a diffeomorphism of the flow line $\gamma(t)$ with the open interval $(f(b), f(a))$ where $s(\gamma) = a$ and $e(\gamma) = b$. This reparameterises the flow-line as a smooth function

$$\omega : (f(b), f(a)) \to M$$

such that

$$f(\omega(t)) = t.$$

We can extend ω to a smooth function defined on $[f(b), f(a)]$ by setting $\omega(f(b)) = b$ and $\omega(f(a)) = a$. Then as seen above, this extended function satisfies the differential equation

$$\frac{d\omega}{dt} = -\frac{\nabla(f)}{\|\nabla(f)\|^2} \tag{13.12}$$

with boundary conditions

$$\omega(f(b)) = b, \qquad \omega(f(a)) = a. \tag{13.13}$$

It is a "height-parameterized" flow line.

We define $\bar{\mathcal{M}}(a, b)$ to be the space of all continuous curves in M which are smooth on the complement of the critical points of f and satisfy the differential equation (13.12) and boundary condition (13.13). Here, of course, we understand that ω satisfies (13.12) on the complement of the set of critical points of f. This space $\bar{\mathcal{M}}(a, b)$ is topologized as a subspace of the space $Map([f(b), f(a)], M)$, of all continuous maps with the compact open topology. Note that if ω is any solution of (13.12) and (13.13) then if we remove the points where $\omega(t)$ is a critical point of f each component of ω is geometrically a flow-line but it is parameterized so that $f(\omega(t)) = t$. Therefore by an abuse of terminology we refer to a curve in $\bar{\mathcal{M}}(a, b)$ as a **piecewise flow-line** from a to b.

It is rather straightforward to check that $\bar{\mathcal{M}}(a, b)$ is a compact space and it clearly contains $\mathcal{M}(a, b)$. Furthermore, by work of Smale in [87], since f is $\mathcal{M}(a, b)$ is in fact open and dense in $\bar{\mathcal{M}}(a, b)$ and so $\bar{\mathcal{M}}(a, b)$ is a " *compactification*" of the moduli space of flow lines $\bar{\mathcal{M}}(a, b)$.

There is an obvious associative, continuous composition law

$$\bar{\mathcal{M}}(a, b) \times \bar{\mathcal{M}}(b, c) \to \bar{\mathcal{M}}(a, c)$$

which is denoted by $\gamma_1 \circ \gamma_2$.

Following the work of the author, Jones, and Segal [24] are now ready to define the *"flow category"* of f, $\mathcal{C}_f$:

Definition 13.8. *The flow category* $\mathcal{C}_f$ is the topological category defined as follows:

- **The objects of $\mathcal{C}_f$:** The objects of $\mathcal{C}_f$ are the critical points of f.
- **The morphisms of $\mathcal{C}_f$:** If a and b are critical points of f then the morphisms from a to b are defined to be

$$\mathcal{C}_f(a, b) = \bar{\mathcal{M}}(a, b).$$

- **The composition law:** The composition law is defined by

$$\begin{aligned} \bar{\mathcal{M}}(a,b) \times \bar{\mathcal{M}}(b,c) &\longrightarrow \bar{\mathcal{M}}(a,c) \\ (\gamma_1, \gamma_2) &\longrightarrow \gamma_1 \circ \gamma_2. \end{aligned} \tag{13.14}$$

In fact $\mathcal{C}_f$ is a topological category in the sense that each of the sets $\mathcal{C}_f(a,b)$ comes equipped with a natural topology and the composition law

$$\bar{\mathcal{M}}(a,b) \times \bar{\mathcal{M}}(b,c) \longrightarrow \bar{\mathcal{M}}(a,c)$$

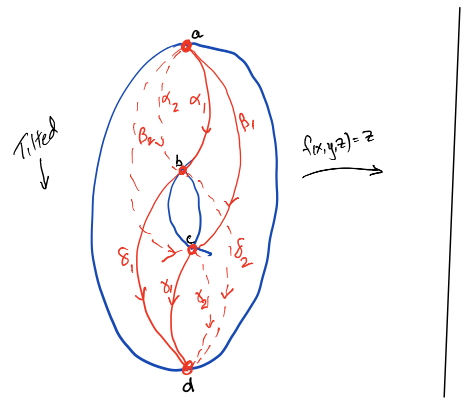
is continuous. The topological category $\mathcal{C}_f$ has a simplicial classifying space $B\mathcal{C}_f$. The main result of [24] is the following:

Theorem 13.17. *If M is a closed Riemannian manifold and $f : M \to \mathbb{R}$ is a Morse function satisfying the Morse-Smale condition, then there is a homeomorphism*

$$M \xrightarrow{\cong} B\mathcal{C}_f.$$

Moreover, even if f does not satisfy the Morse-Smale condition (but is still a Morse function), there is a homotopy equivalence, $M \simeq B\mathcal{C}_f$.

We now illustrate this theorem by considering the example of the height function on the "tilted torus". Recall that for this we view the torus as embedded in ordinary three-space, standing on one of its ends with the hole facing the reader, but tilted slightly toward the reader. We let f be the height function.



There are four critical points; a has index 2, b and c have index 1, and d has index 0. As the figure depicts, the moduli spaces $\mathcal{M}(a,b)$, $\mathcal{M}(a,c)$, $\mathcal{M}(b,d)$, and $\mathcal{M}(c,d)$ are all spaces consisting of two distinct points each. We will denote these flows by α_i, β_i, γ_i, and δ_i respectively. All points on the torus not lying on any of these flows is on a flow in $\mathcal{M}(a,d)$. This moduli space is one dimensional, and indeed is the disjoint union of four open intervals. Furthermore the compactification $\bar{\mathcal{M}}(a,d)$ is the disjoint union of four closed intervals.

Now consider the simplicial description in the classifying space $B\mathcal{C}_f$. The vertices correspond to the objects of the category $\mathcal{C}_f$, that is the critical points.

Thus there are four vertices. There is one one simplex (interval) for each morphism (flow line), glued to the vertices corresponding to the starting and endpoints of the flows. Notice that the points in $\bar{\mathcal{M}}(a,d)$ index a one parameter family of one simplices attached to the vertices labelled by a and d. Finally observe that there is a two-simplex for every pair of composable flows.

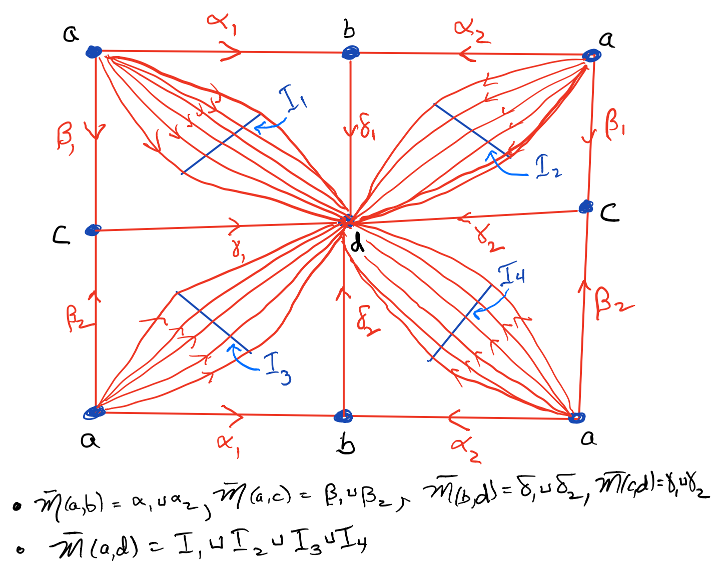


FIGURE 13.6
Simplicial decomposition of $B\mathcal{C}_f$, where f is the height function on the tilted torus

There are eight such pairs (coming from the four points in each of the product moduli spaces $\mathcal{M}(a,b) \times \mathcal{M}(b,d)$ and $\mathcal{M}(a,c) \times \mathcal{M}(c,d)$.) A two-simplex labelled by a pair of flows, say (α, β) will have its three faces identified with the one simplices labelled by α, β, and $\alpha \circ_1 \beta$ respectively. Notice that all higher dimensional simplices in the nerve $\mathcal{N}(\mathcal{C}_f)$ are degenerate and so do not contribute to the geometric realization. The figure depicts the resulting simplicial structure of the classifying space and illustrates Theorem 13.17 that this space is homeomorphic to the underlying manifold.

Remark. The manuscript [24] was never published, primarily because the proof of the main theorem relied on knowing that, assuming $f : M \to \mathbb{R}$ satisfies the Morse-Smale condition, then the compactified moduli spaces, $\bar{\mathcal{M}}(a,b)$ are manifolds with corners and that the corner structure is appropriately preserved under the composition of piecewise flow lines. At the time that manuscript was written, the authors thought that this was a "folk theorem". However upon further inspection, the authors realized that although experts in the community believed that this was true, there was no proof in the literature, and that the issues involved in proving this result were more complicated than the authors originally imagined. Therefore the manuscript was never submitted for publication. In any case, the required manifold with corners properties were eventually proved [81] [98], and the proof of Theorem 13.17 can now be completed using these results. A discussion of manifolds with corners and a sketch of such a proof will be given in the appendices.

A

Appendix: Manifolds with Corners

B

Appendix: Classifying Spaces and Morse theory

C

Appendix: Cohomology operations via Morse theory

D

Appendix: Floer homotopy theory

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