The Proof Theory of Classical and Constructive Inductive Definitions

A 40 year saga

Solomon Feferman

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The Problem

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- The problem: The need for an ordinally informative, conceptually clear, proof-theoretic reduction of classical theories of iterated inductive definitions to corresponding constructive systems.
- At Tübingen: Wolfram Pohlers, Wilfried Buchholz, both students of Kurt Schütte in Munich.

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- ID_α, $ID_\alpha(acc)$, $ID_\alpha(acc)^i$

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- W. Tait: Consistency of (Σ^1_2-AC) by abstract constructive cut-elimination methods applied to uncountably long derivations.

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- Sieg (1977): Formalization of Tait's argument to reduce $ID_{<\lambda}$ to $ID_{<\lambda}(acc)^i$

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- Both recapture ordinal analysis and constructive reduction for the ID_α and $ID_{<\lambda}$

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- Work on related theories of iterated fixed points (Feferman, Jäger, Strahm, ...)
- Work on monotone inductive definitions in a constructive setting (Takahashi, Rathjen, ...)

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- Blocked at a final crucial step.

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- The saga 1968-2008: Shifting interest from applications to subsystems of analysis to interest in theories of inductive definitions in their own right.

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- The methods: cut-elimination and functional interpretation

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- X is closed under Γ if $\Gamma(X) \subseteq X$
- Γ has a least fixed point I = the intersection of all subsets X of M which are closed under Γ
- So: (i) $\Gamma(I) \subseteq I$ and (ii) if $\Gamma(X) \subseteq X$ then $I \subseteq X$.
  Hence (iii) $\Gamma(I) = I$

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- if $\kappa = \text{card}(M)$ then there exists $\gamma < \kappa^+$ with $I_\gamma = I_{\gamma+1}$
- $I = I_\gamma$ for the least such γ (the closure ordinal of Γ)

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- Form L(P), P unary predicate symbol
- A(x, P) of L(P) in which P has only positive occurrences defines a monotone operator
  $\Gamma_A(X) = \{x \in N \mid A(x, X)\}$

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- The positivity condition: if A is reduced to negation normal form, i.e. is built up from atomic formulas or their negations by $\wedge$, $\vee$, $\forall$, and $\exists$, the atomic formulas $P(t)$ never occur negated.

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- $\Gamma_A(P) \subseteq P$ is expressed by the formula $\forall x(A(x, P) \rightarrow P(x))$ of L(P)

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- Let $A(x, P) = \forall y(R(x, y) \rightarrow P(y))$
- The least fixed point of Γ_A is the accessible part of the $<$ relation, i.e. its well-founded initial part.

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- O_1 is the smallest set satisfying: (i) $0 \in O_1$, and (ii) if e is an index of a total recursive function and for each $n \in N$, $\{e\}(n) \in O_1$ then $(1, e) \in O_1$.

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- $|a|$, for $a \in O_1$, is defined by: (i) $|0| = 0$, and (ii) $|(1, e)| = \sup\{ |\{e\}(n)| + 1 : n \in N\}$.

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- To define O_2, in addition to (i), (ii) now on O_2, take: (iii) if e is the index of a partial recursive function such that for each $a \in O_1$, $\{e\}(a) \in O_2$, then $(2, e) \in O_2$. Then take $|(2, e)| = \sup\{ |\{e\}(a)| + 1 : a \in O_1\}$.

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- This procedure can be iterated to form O_3, O_4, etc. It can also be extended into the transfinite, by taking the effective join at limits, e.g. $\langle n, m\rangle \in O_\omega \leftrightarrow m \in O_n$, and then continuing on.

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II. (Induction) $\forall x(A_1(x, F) \rightarrow F(x)) \rightarrow \forall x(O_1(x) \rightarrow F(x))$,

where $F(x)$ is any formula of L_1.

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where $F(x)$ is any formula of L_2.

- NB. Now we must also make sure to allow F to be any formula of L_2 in the induction axioms for both N and O_1.

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- Construct $ID_\alpha(O)$ and $ID_{<\alpha}(O)$ in general for any ordinal α for which we have a natural linear recursive ordering $<$ of N of order type α. For example, Cantor's ordinal ε_0
- In general, ID_1 is the extension of $ID_1(O)$ by predicates P_A for each arithmetic A(x, P) in which P has only positive occurrences, and by the associated closure and induction axioms, where now all induction axioms for N, O, and all the P_A's allow substitution instances by formulas F in the full language. Then ID_2 extends ID_1 and $ID_2(O)$ in the same way.

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- $ID_\alpha(acc)$ uses only A's that are of the form to give an accessibility inductive definition.
- $ID_\alpha(O) \subseteq ID_\alpha(acc) \subseteq ID_\alpha$

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- Core requirement: a proof of $\exists x\, A(x)$ must produce a witness t and a constructive proof of A(t).
- Thus, reject proof by contradiction,
  $\neg\neg\exists x\, A(x) \rightarrow \exists x\, A(x)$ or $\neg\forall x \neg A(x) \rightarrow \exists x\, A(x)$.

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- Intuitionistic logic (Arend Heyting): omit LEM from suitable forms of classical logic.

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- But the $ID_\alpha(O)^i$ and $ID_\alpha(acc)^i$ are generally accepted to be constructive.

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- The negative translation of PA in HA is conservative for (∨, ∃)-free formulas, because HA ⊦ A* ↔ A for A atomic.
- The negative translation does not necessarily work in general to reduce S to S^i, since atomic formulas need not be decidable in S^i. This is the case with the ID^i theories; so something else must be done to reduce S to S^i.

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- Translation is a special case of proof-theoretic reduction.

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- The trade-offs

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- One definition of $|S|$ in general: = the sup of the $|\prec|$ such that $S \vdash TI(\prec)$.
- A definition that works for the ID systems S (classical or intuitionistic): $|S| = \sup\{ |n| : S \vdash O_1(n) \}$

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- Schütte developed a recursive notation system based on the Veblen functions.

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- The Buchholz-Pohlers ordinal analysis: $|ID_\alpha| = |ID_\alpha(acc)^i| = |ID_\alpha(O)^i| = \varphi\varepsilon(\Omega_\alpha+1)0$.

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- So $PA \vdash \forall x \exists y\ R(x, y) \Rightarrow HA \vdash \forall x \neg\neg\exists y\ R(x, y) \Rightarrow$

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- $OR_1 + (I) \leq Q_0T_\Omega$ by the Diller-Nahm-Shoenfield variant of the Gödel functional interpretation.

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- They sketch extension of their work for finitely iterated ID_n's.

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- Are there reasonable theories of ID's over other sets M, e.g. the reals?

The End