From *PSA 1992*, vol. 2 (1993),
pp. 442–455 (with corrections)

Why a little bit goes a long way: Logical foundations of scientifically applicable mathematics[1]

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1. Introduction.

Does science justify any part of mathematics and, if so, what part? These questions are related to the so-called indispensability arguments propounded, among others, by Quine and Putnam. The general idea of the arguments has been formulated (for critical assessment) by Penelope Maddy in a recent article as follows:

> We have good reason to believe our best scientific theories, and mathematical entities are indispensable to those theories, so we have good reason to believe in mathematical entities. Mathematics is thus on an ontological par with natural science. Furthermore, the evidence that confirms scientific theories also confirms the required mathematics, so mathematics and science are on an epistemological par as well. (Maddy 1992, p. 78)

If one accepts the indispensability arguments, there still remain two critical questions:

Q1. Just which mathematical entities are indispensable to current scientific theories?, and
Q2. Just what principles concerning those entities are needed for the required mathematics?

Here we consider answers of an underlying character to these questions, i.e. from the point of view of the foundations of mathematics.[2] In this respect, both Quine and Putnam were led to accept set-theoretical notions and principles to some significant extent or other. However, neither one relied on any detailed examination of just what is needed for scientifically applicable mathematics in arriving at their positions. Nor did they seem to consider whether any of the alternative foundational schemes actively developed during this century – namely those of predicativism, constructivism, and finitism – ought to be preferred on philosophical grounds, particularly when natural science is given such primacy. On the face of it, scientific realism is at odds with the strong form of Platonic realism required to justify set theory through its assumption of the independent existence of abstract entities (such as sets of sets of sets... of unbounded infinite cardinality).

The failure to consider other foundational approaches no doubt stems from the common impression that – whatever their philosophical merits – these schemes are simply inadequate to meet the needs of everyday mathematics by being too restrictive and too foreign to practice. This impression needs to be corrected: there has been considerable logical work in recent years which has established in some detail the unexpected mathematical reach of each of these programs. Moreover, one result of the work in question is that surprisingly meager (in the proof-theoretical sense) predicatively justified systems suffice for the direct formalization of almost all, if not all, scientifically applicable mathematics. It is my main purpose here to describe this result (with the background leading to it) and, in its light, to re-examine the indispensability arguments.

In considering what mathematics is actually used in science it suffices to restrict attention to physics since, among all the sciences, that subject makes the heaviest use of mathematics and there is hardly any branch of mathematics, that has some scientific application, which is not applied there. It would be foolish to claim detailed knowledge of the vast body of mathematics that has been employed in mathematical physics. However, in general terms one can say that it makes primary use of mathematical analysis on Euclidean, complex, and Riemannian spaces, and of functional analysis on various Hilbert and Banach spaces. Any logical foundation for scientifically applicable mathematics should, at a minimum, cover all of 19th century mathematical analysis of (piece-wise) continuous functions on the former kind of spaces and should then go on to cover the theory of (Lebesgue) measurable functions and basic parts of 20th century functional analysis on the latter spaces.

We begin in the next section with a brief review of the set-theoretical foundations of analysis, followed in Section 3 by a discussion of the reasons for rejecting that framework on philosophical grounds. Section 4 then traces the historical development of predicative foundations of analysis. This leads in Section 5 to the description of a formal theory W of variable finite types with the following properties: (i) W is proof-theoretically reducible to the system PA of Peano Arithmetic (the starting system for predicativity), and (ii) almost all, if not all, of scientifically applicable mathematics, as described above, can be formalized directly in W. By way of comparison, Section 6 is devoted to a brief description of results concerning the mathematical reach of constructivist and finitist foundations. We return in Section 7 to a discussion of the significance of these results for the indispensability arguments, and conclude in Section 8 with some more speculative philosophical remarks.

2. Set-theoretical foundations of analysis.

The real number system or continuum **R** is the basic system for analysis; from it we define directly the complex numbers **C** and various Euclidean and non-Euclidean (Riemannian) spaces which figure in the applications. Then in functional analysis one makes use of certain spaces of functions satisfying continuity, measurability or integrability conditions (e.g. the L^p spaces). Concretely, in the case of functions of one variable, these are subspaces of the set $\mathbf{R} \twoheadrightarrow \mathbf{R}$ of all partial functions from **R** to **R**. Abstractly, given any spaces X and Y of a certain kind, one will form subspaces of the set $X \twoheadrightarrow Y$ of all partial functions from X to Y; functionals then act on one such space to another.

The reals are characterized set-theoretically as the unique ordered field satisfying the l.u.b. axiom or Dedekind's continuity axiom. A specific realization of these axioms may be constructed set-theoretically as the set D of all lower Dedekind sections in the rationals $\mathbf{Q}$. Let $\mathsf{P}(X)$ be the set of all subsets of X; then $\mathsf{D} \subseteq \mathsf{P}(\mathbf{Q})$. Moreover, $\mathsf{D} \sim \mathsf{P}(\mathbf{Q}) \sim \mathsf{P}(\mathbf{N})$ where $X \sim Y$ is the relation of equinumerosity (or set-theoretic equivalence) and $\mathbf{N} = \{0,1,2,...\}$ is the set of natural numbers. Hence the cardinality of the reals is $2^{\aleph_0}$. Alternatively, following Cantor, the reals may be represented as Cauchy sequences of rationals, which are members of $\mathbf{N} \rightarrow \mathbf{Q}$, under a suitable equivalence relation.

In set theory, functions are defined as sets of ordered pairs satisfying the many-one condition and pairs are in turn defined by a set construction. With the preceding, this leads to a representation of the real numbers in the cumulative hierarchy $\langle V_n(\mathbf{N}) \rangle_{n \in \omega}$ over **N**, where

(1) $$V_0(\mathbf{N}) = \mathbf{N},\ V_{n+1}(\mathbf{N}) = V_n(\mathbf{N}) \cup \mathsf{P}(V_n(\mathbf{N})) \text{ and } V_\omega(\mathbf{N}) = \bigcup_{n \in \omega} V_n(\mathbf{N}).$$

By the above, **R** is located in $V_\omega(\mathbf{N})$ essentially at the level $V_1(\mathbf{N})$, $\mathbf{R} \twoheadrightarrow \mathbf{R}$ at level $V_2(\mathbf{N})$, etc.; for any X, $Y \in V_\omega(\mathbf{N})$, $X \twoheadrightarrow Y$ is a subset of $\mathsf{P}(X \times Y)$ and hence is also in $V_\omega(\mathbf{N})$.

An alternative to the representation in the cumulative hierarchy over **N** is that in the pure cumulative hierarchy defined by

(2) $$V_0 = \varnothing,\ V_{\alpha+1} = V_\alpha \cup \mathsf{P}(V_\alpha) \text{ and } V_\lambda = \bigcup_{a < \lambda} V_\alpha \quad (\text{for } \lambda \text{ a limit ordinal}).$$

Using the von Neumann representation of ordinals, $\omega = \{0,1,2,\ldots\} \in V_{\omega+1}$ and thus $V_{\omega+\omega}$ includes $V_\omega(\mathbf{N})$ when we identify **N** with ω. $V_{\omega+\omega}$ is the "standard" model of the system ZC of Zermelo set theory with the Axiom of Choice. Set theorists now commonly accept the much stronger system ZFC of Zermelo-Fraenkel with Choice, whose standard model is V_κ for the first strongly inaccessible ordinal κ. Indeed, most working set theorists go far beyond ZFC in accepting various axioms of "large" transfinite cardinals beyond κ. One of the first to urge the plausibility of such extensions of ZFC was Gödel (1947/1964). In contrast, Quine's acceptance of some part of set theory is moderated by his version of the indispensability argument.

> So much of mathematics as is wanted for use in empirical science is for me on a par with the rest of science. Transfinite ramifications are on the same footing insofar as they come of a simplificatory rounding out, but anything further is on a par with uninterpreted systems. (Quine 1984, p. 788)

And, further:

> I recognize indenumerable infinities only because they are forced on me by the simplest known systematizations of more welcome matters. Magnitudes in excess of such demands, e.g. Beth_ω [the cardinal number of $V_\omega(\mathbf{N})$ and of $V_{\omega+\omega}$] or inaccessible numbers, I look upon only as mathematical recreation and without ontological rights. (Quine 1986, p. 400)

It seems from these quotations that Quine would readily accept Zermelo set theory because the power set operation P applied to $\mathbf{N}$ leads to $\mathbf{R}$, and then its application once more to $\mathbf{R} \to \mathbf{R}$ etc.; "simplificatory rounding out" thus suggests acceptance of the Power Set Axiom in general.

3. What's wrong with set-theoretical foundations?

Philosophically, set theory – even in its "moderate" form given by Zermelo's axioms – requires for its justification a strong form of Platonic realism. This is not without its defenders, most notably Gödel (1944 and 1947/1964) (cf. also (Maddy 1990)). For its critics, however, the following are highly problematic features of this philosophy:

(i) abstract entities are assumed to exist independently of any means of human definition or construction;
(ii) classical reasoning (leading to non-constructive existence results) is admitted, since the statements of set theory are supposed to be about such an independently existing reality and thus have a determinate truth value (true or false);
(iii) completed infinite totalities and, in particular, the totality of all subsets of any infinite set are assumed to exist;
(iv) in consequence of (iii) and the Axiom of Separation, impredicative definitions of sets are routinely admitted;
(v) the Axiom of Choice is assumed in order to carry through the Cantorian theory of transfinite cardinals.

The question of admissibility of impredicative definitions will be discussed in the next section. The Axiom of Choice has been the focus of much specific criticism, since choice sets are not in general definable; however, it is entirely in keeping with the other assumptions (and it would not be coherent to accept (i) – (iv) and reject the Axiom of Choice).

Quite surprisingly, Gödel at one time sided with the critics: in an unpublished lecture he delivered to a meeting of the Mathematical Association of America in 1933, after explaining the use of systems like ZFC for the axiomatic foundation of "all of mathematics", Gödel raised pointed objections to its features (ii), (iv) and (v) and went on to say

> The result of the preceding discussion is that our axioms [of set theory], if interpreted as meaningful statements, necessarily presuppose a kind of Platonism, which cannot satisfy any critical mind and which does not even produce the conviction that they are consistent. (Gödel 1933, p. 19)[3]

In the present context for assessing the indispensability argument(s), one should also note the ontological and epistemological anomalies in accepting set-theoretical foundations for mathematics, first of all because highly infinitary abstract objects are put on an ontological par with physical objects, and secondly because there is no observational knowledge of abstract objects (either directly or indirectly).[4]

4. The development of predicative foundations for analysis.[5]

4(a). Poincaré and Russell.

In seeking ways to avoid the set-theoretical paradoxes while pursuing the logicist program, Russell introduced the term *predicative* for properties which determine legitimate classes, but he had no settled criterion for telling which those are. Poincaré saw the root of the paradoxes in the existence of a vicious circle, namely in the use of definitions which purport to single out a member of a totality by reference to that totality. Such apparent definitions are called *impredicative*. There can be no objection to impredicative definitions when the totality in question is regarded as having a clear and determinate extent. For example, if one accepts the totality $\mathbf{N} = \{0, 1, 2, \ldots\}$ as definite, there can be no objection to definitions singling out some natural numbers as the least n satisfying a property $\varphi(n)$ when φ refers, by quantification, to the totality of objects in **N**. Poincaré regarded the natural number sequence and the principle of induction on it as an irreducible basis of our mathematical intuition, and he argued that the attempts of the logicists such as Frege and Russell to derive these from purely logical principles are fundamentally misguided and question-begging.[6] On the other hand, Poincaré thought all other mathematical notions should be introduced by proper definitions; in particular sets are to be defined only by reference to prior defined sets and notions, not by reference to any presumed totality of sets. The latter would thus constitute impredicative definitions; Poincaré banned their use under his *vicious-circle principle*. (For a useful recent analysis of Poincaré's philosophy of mathematics, see (Folina 1992).)

Russell adopted Poincaré's proscription of impredicative definitions in setting up the Ramified Theory of Types (RTT) for the *Principia Mathematica* (1910–1913). However, he did not follow Poincaré in taking the natural number system for granted, but aimed to construct that "logically" within RTT. Each set variable of RTT is ranked, and the Comprehension Axiom schema for existence of sets $\{x : \varphi(x)\}$ of a given rank is restricted to φ with all bound variables of a smaller rank; membership is restricted to successive ranks (the basic syntactic feature of typed theories of sets). But Russell then found that he was faced with a multiplicity of notions of natural number and could not even derive the simplest closure principles by induction using ranked formulas. Similarly, he was faced with an unworkable theory of the continuum, since one could only deal with real numbers of different ranks, for which the l.u.b. axiom would not hold in any one rank. For purely pragmatic reasons then, Russell introduced the so-called Axiom of Reducibility which asserts that every set is co-extensive with a set of lowest rank. This, in effect, completely compromised the predicative/impredicative distinction; Russell recognized the objections to this "axiom", but thought it could somehow be justified and in any case saw no alternative. Later Ramsey pointed out that if one dropped rankings but maintained the restriction of membership to successive types, thus yielding the Simple Theory of Types (STT), no obvious set-theoretical paradoxes would arise and some of the above problems would be avoided. STT allows impredicative definitions $\{x : \varphi(x)\}$ of sets of objects x in any given type n, with $\{x : \varphi(x)\}$ in the next type $n+1$. Contrary to Poincaré's view, impredicativity is thus not an essential ingredient of the paradoxes.

As with set theory, STT requires a Platonistic philosophy for its justification. Its "standard" model is given by

(1) $S_{n+1} = \mathsf{P}(S_n)$ for $n < \omega$,

where S_0 is a basic set of individuals. In order to derive arithmetic in STT one must assume that S_0 is infinite; hence it is taken for granted that one accepts completed infinite totalities, and that for any totality S one also has the totality $\mathsf{P}(S)$ of all subsets of S. Impredicativity already appears at type 1, since in forming the type 1 set $\{x : \varphi(x)\}$ for x of type 0, with bound variables in φ of type 1, one is implicitly assuming the existence of the totality of sets of type 1.

4(b). Weyl's approach.

The first steps, after Russell's aborted attempt, to see what part of analysis could be developed in strictly predicative terms were made by Weyl in his monograph *Das Kontinuum* (1918). Like Poincaré, Weyl accepted the natural numbers and the associated principles of proof by induction and definition by recursion as basic. This simplified his task in comparison with Russell's (modified) logicist program. The main question was how to develop a workable theory of real numbers, via an unramified yet predicatively acceptable theory of sets of natural numbers. His answer in essence was to restrict to arithmetical definitions $\{n: A(n)\}$ of such sets, i.e. where A contains no bound set variables but may contain free set variables and free or bound numerical variables. Relative arithmetic definitions determine functions $F(X) = \{n : A(n, X)\}$ under which the arithmetically definable sets are closed. Weyl's further principles have been analyzed in modern terms in (Feferman 1988) and a certain ambiguity was revealed, leading to two possible formal systems for his work. The first of these turns out to be a conservative extension of Peano Arithmetic, PA, and it suffices for all his applications.

Using a standard representation of the rational numbers in terms of the natural numbers, Weyl defined the reals as lower Dedekind sections in **Q**. And, though he was blocked from inferring the l.u.b. axiom for *sets* of reals because that requires (impredicative) quantification over "all" reals, he could obtain the l.u.b. for *sequences* of reals, since this only requires quantification over **N**: for a sequence of lower sections $\langle X_n \rangle_{n \in \mathbf{N}}$ in **Q**, we have $\bigcup_{n \in \mathbf{N}} X_n = \{x: (\exists n \in \mathbf{N})(x \in X_n)\}$ as the l.u.b. With continuous functions from reals to reals treated via approximating functions from **Q** to **Q**, Weyl was then able to sketch in his system a straightforward reconstruction of the whole of 19th c. analysis of (piece-wise) continuous functions of a real variable. He also suggested the possibility of extension to more general classes of functions, including the Lebesgue measurable functions, but gave no indications how this would be carried out.

4(c). Modern developments.

Weyl did no more work with his system after 1918[7], and his program lay dormant until the 1950s when the logical study of predicativity was taken up in a variety of ways by Lorenzen, Kleene, Kreisel, Grzegorczyck, Wang, Spector and others (the development is traced in (Feferman 1964) Part I). In particular, Kreisel proposed a characterization of predicativity in terms of a certain "autonomous" transfinite progression of ramified second-order systems whose exact extent was determined independently by Schütte and myself in 1964 to be given by a certain recursively described ordinal Γ_0 (cf.op.cit. for references). Peano Arithmetic PA is contained in the base system for this sequence of theories.

In order to develop analysis in a workable but still predicatively acceptable way, it was necessary to follow Weyl's lead in setting up suitable unramified systems which would be justified (at least indirectly) by reduction to the autonomous progression of ramified theories. This was achieved in my 1964 paper in a second-order unramified system, followed by some improved versions in later papers. However, higher types are called for in order to have greater ease of development of analysis. For this, (Feferman 1977) and, independently (Takeuti 1978) introduced certain systems of finite type which are conservative over PA by proof-theoretic arguments and, a fortiori, certainly predicatively justifiable. These permitted a direct development of 19th c. analysis and the beginnings of 20th c. analysis; but the latter called for an even more flexible and expressive formalism.

5. A flexible system W of variable finite types for the modern development of Weyl's program.

5(a). The formal system.

As indicated in Section 2 above, in order to represent the notions of modern analysis directly and develop analysis flexibly it is necessary to have not only the set **R** of real numbers but also for any set *S*, its definable subsets $X = \{x \in S : \varphi(x)\}$, and for any sets X and Y also the sets $X \times Y$ and $X \twoheadrightarrow Y$**,** where the latter is understood to be the set of all partial functions from X to Y. Now if functions were defined as many-one relations as in set theory, $X \twoheadrightarrow Y$ would simply be the set of all many-one $Z \subseteq X \times Y$, and this would call on $\mathsf{P}(X \times Y)$ for its definition; that route would, in effect, take us back to Zermelo's set theory. The crucial first step toward a flexible, predicatively justifiable system is to treat functions and classes as conceptually independent basic notions, i.e. with neither explained in terms of the other. A system taking this lead was worked out successively in (Feferman 1975, 1985, and 1988). We follow the last of these papers in describing the system W (so designated in honor of Weyl) presented there in detail.

The language of W is two-sorted, with individual variables *a, b, c, ..., x, y, z* ranging over a universe containing numbers, sets, functions, functionals, etc., and closed under pairing, while the variables *A, B, C, ...X, Y, Z* range over *classes* or (variable) *types*. Note the terminological shift here: *sets* will be reserved for subclasses of a given class *S* which have a characteristic function. There are constants for specific individuals and functions (indicated below); there are also binary operations $\langle x, y \rangle$ and $x(y)$, where the latter is interpreted as the value of x at y when x is a partial function and y is in its domain, otherwise undefined. *Individual terms s,t,...* are built up from individual variables by use of the given constants and operations by closure under the general process of explicit definition, including function definition $(\lambda x \in S).t(x)$. *Class terms S, T,...* are built up from class variables and the class constant **N** by closure under the operations: $S \times T$, $S \twoheadrightarrow T$ and $\{x \in S | \varphi(x)\}$ where φ is a *bounded predicative formula* (explained below). Formulas of W in general are built up from atomic formulas $s = t$, $t\downarrow$ and $t \in S$ by the propositional operations and quantification with respect to both individual and class variables; $t\downarrow$ is used to express that t is defined. The bounded predicative formulas admit no quantification over class variables, and individual quantifiers are restricted as: $(\forall x \in S)\varphi$ and $(\exists x \in S)\varphi$. By definition, $(S \to T) = \{z \in (S \twoheadrightarrow T) | (\forall x \in S) zx\downarrow\}$.

The individual and function constants are 0 and Sc (for *successor* on **N**), and μ (for the *non-constructive least number* or *search operator)*, P_1 and P_2 (for *projections* of pairing), D (for *definition by cases),* and $Rc_{\mathbf{N}}$ (for *definition by recursion on* **N)**. The *function axioms* of W follow these intended meanings together with usual axioms for explicit definition. Taking $1 = \mathrm{Sc}(0)$, we write $\mathsf{P}(S) = (S \to \{0,1\})$ for *the class of subsets* of S regarded as the class of characteristic functions on S. Then for $a \in \mathsf{P}(S)$ we write $x \in a$ for $a(x) = 0$. In particular, the axiom for μ is simply

$$(\mu) \quad \mu \in \mathsf{P}(\mathbf{N}) \to \mathbf{N} \wedge \forall a \in \mathsf{P}(\mathbf{N}) \, [(\exists n)(n \in a) \Rightarrow \mu(a) \in a]$$

The *class axioms* of W are the evident ones for $S \times T$, $S \twoheadrightarrow T$ and $\{x \in S | \varphi(x)\}$.

Induction on **N** is taken as so-called *set induction*:

$$(\mathrm{Ind}_{\mathrm{Set}}) \qquad \forall a \in \mathsf{P}(\mathbf{N})[0 \in a \wedge \forall n(n \in a \Rightarrow \mathrm{Sc}(n) \in a) \Rightarrow \forall n \in \mathbf{N}(\mathrm{n} \in a)].$$

A stronger natural principle to consider is *class induction:*

$$(\mathrm{Ind}_{\mathrm{Class}}) \qquad \forall X \subseteq \mathbf{N}[0 \in X \wedge \forall n(n \in X \Rightarrow \mathrm{Sc}(n) \in X) \Rightarrow \mathbf{N} \subseteq X];$$

this is still predicative , but it leads us beyond PA, so is not included in W.

Using the μ operator one shows in W that sets are closed under numerical quantification, since $\exists n(n \in a) \Leftrightarrow \mu(a) \in a \Leftrightarrow a(\mu(a)) = 0$. It follows that every arithmetical formula defines a subset of **N**; then by $\mathrm{Ind}_{\mathrm{Set}}$, we obtain the induction scheme for arithmetical formulas. The logic of W is assumed to be that of classical two-sorted predicate calculus. Hence the system PA of classical first-order arithmetic is contained in W. The main meta-theoretical result about W is the following:

Main Theorem *W is proof-theoretically reducible to* PA *and is a conservative extension of* PA.

The proof of this will appear in (Feferman and Jäger, forthcoming); for the proof of the corresponding result for a precursor of W see (Feferman 1985). It follows from this theorem that W rests on entirely predicative grounds, though it has much of the conceptual richness and flexibility of systems like Zermelo set theory.

5(b). Analysis in W.

Space does not begin to permit the demonstration that all (or practically all) of the necessary 19th and 20th c. analysis needed for scientific applications can be carried out directly in W; indications are given in (Feferman 1985 and 1988). To begin with, the class **R** of real numbers may be introduced as the class of Dedekind sections in $\mathsf{P}(\mathbf{Q})$ or alternatively as the class of Cauchy sequences in $(\mathbf{N} \to \mathbf{Q})$. By closure under numerical quantification, we obtain closure under l.u.b. of bounded sequences, just as in Weyl's system. Now instead of dealing with (partial) functions of a real variable by some reduction or other to second-order functions, we treat them directly as members of

R –>> **R**. Then the classes of continuous functions, measurable functions, etc. are obtained as suitable subclasses of **R** –>> **R**.[8] Given a function space **S**, functionals on that space are simply members of **S** –>> **R**. In this way, basic concepts and examples from functional analysis are readily represented in W. In that subject I have verified (in unpublished notes) that such results as the Riesz Representation Theorem, Hahn-Banach Theorem, Uniform Boundedness Theorem and Open Mapping Theorem for separable Banach and Hilbert spaces are derivable in W – and that, finally, one can obtain the principal results of the spectral theory of bounded self-adjoint linear operators on a separable Hilbert space. Extension of this work to the case of unbounded self-adjoint operators has been carried out in a preliminary way, using an approach to their spectral theory via limits of bounded operators.

While there are clearly parts of theoretical analysis that cannot be carried out in W because they make essential use of the l.u.b. axiom applied to sets rather than sequences, or because they make essential use of transfinite ordinals or cardinals, or because they deal with non-separable spaces, *the working hypothesis that all of scientifically applicable analysis can be developed in W has been verified in its core parts.* What remains to be done is to examine results closer to the margin to see whether this hypothesis indeed holds in full generality.

6. Comparisons with the Reverse Mathematics program and constructivist and finitist foundations of analysis.

The Reverse Mathematics (R.M.) program was originated and initially developed by H. Friedman (1975) and subsequently pursued in detail mainly by Simpson and his students (cf. Simpson 1987 and 1988). The main question addressed in that program is: which set-existence principles are necessary to establish the (known) propositions of ordinary non-set-theoretical mathematics? To fix matters, though, only results which can be formulated (or reformulated) in the language of second-order arithmetic have been considered. The pattern of the work is to find for each mathematical theorem τ (which can thus be expressed) a set-existence principle σ such that $\sigma \Rightarrow \tau$ is provable in a system based on principles weaker than σ; the "reverse" part comes in showing that $\tau \Rightarrow \sigma$ is derivable (in the same system), so that σ is exactly necessary for τ. A great number of results from analysis (as well as in algebra and logic) have been examined successfully in the R.M. program.[9] Moreover, it has been found that five principles σ of increasing strength come up repeatedly in this process: RCA_0 (Recursive Comprehension Axiom), WKL_0 (Weak König's Lemma), ACA_0 (Arithmetical Comprehension Axiom), ATR_0 (Arithmetical Transfinite Recursion), and $\Pi^1_1\text{-}CA_0$ (Π^1_1-Comprehension Axiom); the subscript '0' indicates that Ind_{Set} is the only form of induction on **N** used with each set-existence principle. ATR_0 is proof-theoretically equivalent to the full progression of predicative systems referred to at the end of Section 4, while $\Pi^1_1\text{-}CA_0$ is the first patently impredicative system beyond that; we have the full l.u.b. principle for sets of reals provable in Π^1_1-CA_0. Going back down, ACA_0 is contained in our system W and is also conservative over PA. On the other hand, both RCA_0 and WKL_0 are conservative over the much weaker system PRA of Primitive Recursive Arithmetic. All of the results shown to be equivalent to RCA_0, WKL_0 or ACA_0 are thus provable in W.

There are two main differences of the R.M. program from that described in Sec. 5 with W. The first is the R.M. restriction to statements in the language of second-order arithmetic: this requires considerable coding once one moves beyond the 19th c. analysis of continuous functions (and even the representation of the latter in second-order terms is less than natural). On the other hand, the equivalences established in R.M. of results τ from ordinary mathematics with one of RCA_0,WKL_0 or ACA_0, are evidently sharper than the fact that such τ are consequence of the principles of W. In any case, the work done on RCA_0,WKL_0 or ACA_0 in analysis corroborates fully the development of all of 19th c. analysis and substantial tracts of 20th c. functional analysis within W as described in the preceding section.

Further evidence comes from the development of constructive analysis in the hands of the Bishop school (cf. Bishop and Bridges 1985). Bishop and his followers found constructive substitutes for considerable portions of classical analysis. In general, with each classical theorem τ for which this is successful is associated a constructive theorem τ^* such that LPO implies $\tau^* \Rightarrow \tau$ where LPO is the "Limited Principle of Omniscience" $\forall n(f(n)=0, \vee \exists n(f(n) \neq 0)$ for all $f:\mathbf{N} \to \mathbf{N}$. Evidently LPO is a special consequence of the Law of Excluded Middle (LEM). It was shown in (Feferman 1979) how all of Bishop's development of constructive analysis (except for his theory of Borel sets which Bishop had shown to be dispensable) could be formalized in a constructive theory of variable finite types which is proof-theoretically reducible to HA (Heyting Arithmetic). Indirectly, then, all of the classical analysis for which constructive substitutes were found in the Bishop school are accounted for by principles reducible to PA(=HA + LEM). This again acts to corroborate the work described in Sec. 5 above.[10]

A word, finally, about finitist foundations of analysis. Initial efforts in this direction were made by Goodstein (1961), by considering which results of analysis can be accounted for on the basis of PRA. That is a quantifier-free system generally acknowledged to represent part (if not all) of finitist constructions and arguments in number theory. One of the impressive results of the R.M. program is to show how much of 19th c. and 20th c. analysis can be established in WKL_0; since that is proof-theoretically reducible to PRA (cf. Sieg 1985), an exceptional amount of analysis is already accounted for on finitistically justifiable grounds.[11] In view of this it would be worthwhile setting up a subsystem of W conservative over PRA in which that same part of analysis can be formalized more directly.

7. Significance for the indispensability arguments.

The questions Q1 and Q2 raised in Section 1 take the indispensability arguments for granted. Answers given in the past to these questions have been extremely broad, on the order of: mathematical analysis is indispensable to science, the real numbers and functions and sets of reals are the basic objects of analysis, set theory provides our best account of the real number continuum and of functions and sets in general, so the entities and principles of set theory are justified by science. This sweeping passage leaves undetermined just which of those entities and principles are thereby justified, except perhaps to say that the farther reaches of set theory are evidently unnecessary for science and so may be disregarded.

The work described in the preceding two sections allows one to tell quite different and much more specific stories in response to these questions. In all of the indicated formal systems one can speak *within* the language of these systems about arbitrary real numbers, functions of real numbers, sets of real numbers, etc. Only the existence principles (closure conditions) concerning these objects are much more restricted than in the case of systems of set theory like Zermelo's. To be specific, let us concentrate on the system W. Acceptance of W and the entities with which one can deal in its language does not commit one to a Platonistic ontology of those entities, though the Platonist is free to understand W in those terms. By the fact of the proof-theoretical reduction of W to PA, the only ontology it commits one to is that which justifies acceptance of PA. But even there, the answer to Q1 and thence to Q2, is underdetermined. One view of PA is that it is about the natural numbers as independently existing abstract objects; that is again a Platonistic view, albeit an extremely moderate one. Another view is that PA is about the mental conception of the structure of natural numbers, which is of such clarity that statements concerning these numbers have a determinable truth value and their properties can be established in an indisputable intersubjective way; this is more or less the predicativistic view. Or one can make use of the fact that PA is reducible to HA to justify it on the basis of a more constructive ontology. In all these cases except the fully Platonistic point of view concerning W, it is treated in an instrumental way, its entities outside the natural numbers are regarded as "theoretical" entities, and the justification for its use lies in whatever justification we give to the use of PA; but even there we do not arrive at a unique ontology.

My conclusion from all this is that even if one accepts the indispensability arguments, practically nothing philosophically definitive can be said of the entities which are then supposed to have the same status – ontologically and epistemologically – as the entities of natural science. That being the case, what do the indispensability arguments amount to? As far as I'm concerned, they are completely vitiated. This does not mean, however, that questions Q1 and Q2 lose their interest. Rather that is retained if one regards them instead from a phenomenological point of view, and here the kind of logical work described in Sections 5 and 6 above already has much to tell us. This work needs, of course, to be continued in order to make it fully conclusive, and here it will be necessary to investigate questions at the margin, e.g. the possible essential use in physical applications of such objects as non-measurable sets or non-separable spaces, which are not accounted for in systems like W.

8. Final remarks.

As a complement to the preceding discussion one should mention Maddy's critical examination (1992) of the indispensability arguments, whose conclusion (op.cit. p. 289) is that they "do not provide a satisfactory approach to the ontology or the epistemology of mathematics", for two reasons, quite different from those given here. The first is that "fundamental mathematized science is 'idealized' (i.e. literally false)", e.g. "the analysis of water waves by assuming the water to be infinitely deep or the treatment of matter as continuous in fluid dynamics or the treatment of energy as a continuously varying quantity" (op.cit. p. 281), and hence that the mathematics involved cannot be regarded as true, other than "true in the model" (my quote marks). The second is that "[scientific] indispensability cannot account for mathematics as it is actually done."

The first of these objections puts me in mind of the oft referred to article by Wigner on "The unreasonable effectiveness of mathematics in natural science" (1960). I agree with Maddy completely about the extent to which mathematized science depends on highly idealized models. What is remarkable, then, is not the unreasonable effectiveness of mathematics so much as the *unreasonable effectiveness of (mathematized) natural science.*

In her second objection, Maddy has particularly in mind the mathematics carried on by current set-theorists, but it is not unjustly applied to the bulk of pure mathematics as it is actually practiced. Here one meets a different kind of indispensability argument, stemming from Gödel, for the need of "higher" set theory to account for that body of mathematical work. But, again, appearances are deceiving, and logical results from recent years have much to tell us about just what is needed to carry on all but the patently set-theoretical reaches of modern mathematics. It is not claimed that predicative systems account for that, but it has been established that systems far weaker than Zermelo set theory and even than second-order arithmetic suffice for the bulk of mathematical practice.

Like most scientists, philosophers of science could simply take mathematics for granted and not concern themselves with its foundations, as being irrelevant to their main concerns. But, as Hellman has emphasized in his introduction to his article in this volume, debates like those discussed here as to realism vs. (e.g.) instrumentalism, and as to the indispensability of highly theoretical concepts and principles, are equally central to the philosophy of science. Whether the kind of logical results described here will be more directly relevant to those debates remains to be seen. But as long as science takes the real number system for granted, its philosophers must eventually engage the basic foundational question of modern mathematics: "What are the real numbers, really?"

Notes

1. Invited lecture in the Symposium, "Is foundational work in mathematics relevant to the philosophy of science?" at the meeting of the Philosophy of Science Association, Chicago, Nov. 1, 1992.

2. Geoffrey Hellman's contribution to this symposium (this volume) is closely related to our discussion of these questions at a number of points.

3. The complete text for Gödel's 1933 lecture was found in his *Nachlass*; it will be reproduced in the forthcoming Vol. III of Kurt Gödel's *Collected Works*.

4. Maddy (1990) does attempt to advance a modified form of realism which would root our knowledge of sets in everyday physical experience. In my view, this is convincing only for the most elementary parts of set theory, if at all.

5. For a more extensive survey of the development of predicativity see Part I of (Feferman 1964).

6. Parsons (1983) has argued for the impredicativity of the induction principle on **N**. Nelson (1986) (in evident agreement with this view) has developed an axiom system for what he calls predicative arithmetic which drastically restricts the use of the induction principle. (The computational content of Nelson's system is related to the feasibly computable functions in the sense of computer science.) In contrast, (Feferman and Hellman forthcoming) provide predicative foundations of full arithmetic on the basis of rather weak and intuitively evident axioms for finite sets. So one sees that it is reasonable to consider notions of predicativity relative to various basic conceptions; the one dealt with in the present text has come to be called *predicativity given the natural numbers.*

7. For reasons explained in (Feferman 1988).

8. The notion of Lebesgue outer measure cannot be defined in W since it makes essential use of the g. l. b. applied to sets, not sequences. However, the measure of measurable sets can be defined as the g.l.b. of measures of a sequence of approximating open covers. Incidentally, W does not prove the existence of non-measurable sets of reals in the sense of Lebesgue, i.e. it is consistent with W to assume that all sets of reals are measurable.

9. Simpson is currently preparing a book on subsystems of second order arithmetic in which many of these results will be presented in full.

10. Naturally, one may expect that many results τ of classical analysis are not constructively provable as given; cf. e.g. (Pour-El and Richards 1989) and (Hellman 1993). That doesn't imply that τ has *no* constructive substitute in Bishop's sense.

11. Cf., e.g., (Feferman 1977 and (forthcoming)) for surveys of work in this direction, as well as (Simpson 1987).

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