Predicativity

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What is predicativity? While the term suggests that there is a *single idea* involved, what the history will show is that there are a *number of ideas* of predicativity which may lead to different logical analyses, and I shall uncover these only gradually. A central question will then be what, if anything, unifies them. Though early discussions are often muddy on the concepts and their employment, in a number of important respects they set the stage for the further developments, and so I shall give them special attention. NB. Ahistorically, modern logical and set-theoretical notation will be used throughout, as long as it does not conflict with original intentions.

Predicativity emerges: Russell and Poincaré

To begin with, the terms *predicative* and *non-predicative* (later, *impredicative*) were introduced by Russell (1906) in his struggles dating from 1901 to carry out the logicist program in the face of the set-theoretical paradoxes. Russell called a propositional function $\varphi(x)$ predicative if it defines a class, i.e., if the class $\{x : \varphi(x)\}$ exists, and non-predicative otherwise. Thus, for example, the propositional function $x \notin x$ figuring in Russell's paradox is impredicative. Since the admission of classes defined by arbitrary propositional functions in Frege's execution of his logicist program led to its demise as a result of this paradox, if the program were to be resurrected, it would somehow have to incorporate a criterion for distinguishing predicative from impredicative ones. Russell's first attempts to separate these were highly uncertain, and it was only through the engagement of Henri Poincaré in the problem starting in his article (1906) that progress began to be made. Poincaré took several paradoxes as examples to try to elicit what was common to them, namely the Burali-Forti paradox of the largest ordinal number, König's paradox of the least non-definable ordinal

[1]The subject of predicativity is one that has been of great interest to me and has periodically commanded much of my attention over the last forty years. It involves substantial developments in logic and mathematics and is of significance for the philosophy of mathematics. However, it is still unsettled how best to assess these various aspects of predicativity. On March 28, 2002, for a joint meeting of the American Philosophical Association and the Association for Symbolic Logic, Jeremy Avigad, Geoffrey Hellman and I participated in a symposium organized by Paolo Mancosu entitled "Predicativity: Problems and Prospects." In my lecture I concentrated on the idea of predicativity in its historical development and particularly its logical analysis, which has led to new problems of current interest; the present chapter is based on my text for that lecture. Complementarily, Avigad and Hellman dealt respectively with questions concerning the mathematical and philosophical significance of predicativity; their papers are to appear elsewhere. In addition, Mancosu and I are collaborating on a historically organized survey of predicativity and its relations to mathematical practice which is to include the material below, for a compendium of the history of mathematical logic under the editorship of John Dawson, Aki Kanamori and Dirk van Dalen. I greatly appreciate the help that Paolo Mancosu has given with this chapter, especially the part having to do with early developments.

number, and the Richard paradox defining, by diagonalization, a real number different from all definable real numbers; it was this last that Poincaré took as a paradigm. Note that in doing so, Poincaré shifted attention away from purported definitions of set-theoretical objects involving only purely set-theoretical notions such as those of class, membership, ordinal number and cardinal number, to purported definitions of mathematical objects more generally in which the notion of *definability* itself was an essential component. In doing so, he could be considered to be at cross-purposes with Russell. At any rate, Poincaré came up with two distinct diagnoses of the source of the paradoxes via what he regarded as "typical" examples. The first was that there is in each case a *vicious circle* in the purported definition. For example, in the case of Richard (1905), since each definition of a real number via its decimal expansion can be written out using a finite number of symbols, the set D of definable real numbers is countable. Then by Cantor's diagonal construction one can define a real number r which is distinct from each member of D; but since r is defined, it is a member of D, which is a contradiction. According to Poincaré, in this case the vicious circle lies in trying to produce the object r in D by reference to the supposed totality of objects in D; indirectly, then, r is defined in terms of itself, as one of the objects in D. Poincaré's second diagnosis is distinct in its emphasis, namely that the source of each paradox lies in the assumption of the "actual" or "completed" infinite. Again with reference to the Richard paradox, one cannot assume that there is a completed totality of *all* definable objects of a certain kind; rather, each one "comes into existence" through a definition in terms of previously defined objects. As we shall see, in his own mathematics, Poincaré did not hew to the injunction against the actual infinite. And that is related to the issue of impredicative definitions as they occur in mathematical practice to which Poincaré was to return a few years later, and that we'll also take up below.

Russell's elaborations

In his article "Les paradoxes de la logique" (1906a), Russell quickly took both the vicious circle diagnosis and, to an extent, the objection to the completed infinite as the point of departure for his further work on predicative vs. impredicative definitions of classes[2]:

> I recognize ... that the clue to the paradoxes is to be found in the vicious circle suggestion; I recognize further this element of truth in M. Poincaré's objection to totality, that whatever in any way concerns *all* or *any* or *some* of a class must not be itself one of the members of a class. (Russell 1973, p. 198)

Russell then went on to make the first of his several attempts to formulate the VCP (Vicious Circle Principle) in syntactic terms as would be appropriate for a formalism in which to redevelop logicism, namely:

[2]The quotation is from the English translation in Russell's *Essays in Analysis* (1973). Many of the original texts on the antinomies and predicativity in the period 1906-1912 are conveniently assembled in Heinzmann (1986).

> In M. Peano's language, the principle I want to advocate may be stated: "Whatever involves an apparent variable must not be among the possible values of that variable." (ibid.)

Insofar as this form of the VCP proscribed certain formulas $\varphi(x)$ from defining classes, its effect would be to exclude from φ any bound variables whose intended range includes the class $\{x : \varphi(x)\}$ as one of its values. But, in addition, the bound variable 'x' of abstraction in $\{x : \varphi(x)\}$ must not include that class in its range. These restrictions were then the lead-in to the formalism proposed in Russell's article "Mathematical logic as based on the theory of types" (1908), and on whose plan *Principia Mathematica* was erected. As is well known, Russell formulated the VCP in several different ways, and their precise significance and relation to each other has been the subject of much scrutiny and critique by a number of scholars, including Kurt Gödel (1944), Charles Chihara (1973) and Philippe de Rouilhan (1996). The formulation that is tied most closely to Russell's theory of types was given in the 1908 article as the following sharpening of the above:

> [The VCP], in our technical language, becomes: "whatever contains an apparent variable must not be a possible value of that variable." Thus [it] must be of a different type from the possible values of that variable. Thus we will say that whatever contains an apparent variable must be of a different type from the possible values of that variable; we will say that it is of a higher type. (Russell 1908, in van Heijenoort 1967, p. 163)

Before going into the actual structure of types in Russell's set up, let me draw attention to an earlier section of the article, headed *All and any* (ibid., pp. 156-159). Here, in contrast to the first quotation from Russell above, a distinction was made between the use of these two words. Roughly speaking, in logical terms, the statement that *all* objects x of a certain kind satisfy a certain condition $\varphi(x)$ is rendered by the universal quantification $(\forall x)\varphi(x)$ in which x now is a bound variable, while the statement that $\varphi(x)$ holds for any x is expressed by leaving x as a free variable. In modern terms, the logic of the latter is treated as a scheme to be coupled with a rule of substitution. The importance of this distinction for Russell has to do with the injunction against illegitimate totalities. In particular, with p a variable for propositions, he would admit "p is true or false, where p is any proposition", i.e., the scheme $p \vee \neg p$, but not the statement $(\forall p)(p \vee \neg p)$ that "all propositions are true or false" (in both cases using truth of p to be equivalent to p). Similarly for properties $P(x)$; significantly, Russell pointed out that the proposed definition of the natural numbers in the form: 'n is a finite integer' means "Whatever property φ may be, n has the property φ provided φ is possessed by 0 and by the successors of possessors" (ibid., p. 159), i.e., in symbols,

$$N(n) := (\forall\varphi)[\varphi(0) \wedge (\forall x)(\varphi(x) \rightarrow \varphi(x')) \rightarrow \varphi(n)],$$

cannot be replaced by dropping the universal quantifier over properties '$(\forall\varphi)$'.

Though the simplest paradoxes such as Russell's of the class of all non-self membered classes or the heterologicality paradox can be construed as involving arbitrary classes or properties in the form of apparent variables, they do not involve them in the form of quantified variables. Nevertheless, these considerations led Russell to ban the use of 'all' in the form of unrestricted quantification over propositions and properties, among other things. He then faced the question when it is legitimate to apply universal quantification over any given kind of object, and here he veered away from Poincaré's injunction against the "actual" infinite:

> It has often been suggested that what is required in order that it may be legitimate to speak of *all* of a collection is that the collection should be finite. Thus "all men are mortal" will be legitimate because men form a finite class. But that is not really the reason why we can speak of "all men." What is essential...is not finitude, but what may be called *logical homogeneity.* This property is to belong to any collection whose terms are all contained within the range of significance of some one function. It would always be obvious at a glance whether a collection possessed this property or not, if it were not for the concealed ambiguity in common logical terms such as *true* and *false*, which gives an appearance of being a single function to what is really a conglomeration of many functions with different ranges of significance. (ibid., p. 163)

Here a new idea became central:

> ...we can speak of *all* of a collection when and only when the collection forms part or the whole of the *range of significance* of some propositional function, the range of significance being defined as the collection of those arguments for which the function in question is significant, that is, has a value. (ibid., p. 163)

That is, in more modern terms, like all functions in ordinary mathematics, propositional functions are *partial* in the sense that they have a prescribed domain which is given in advance of the function, and outside of which they are undefined. In particular, the arguments x of a propositional function φ must somehow be "prior" to the function itself. This is achieved by the type distinction and an ordering of types, which makes the type of x lower than the type of φ. Without further restrictions, this would lead to the formalism of the *simple theory of types.* But the VCP in the form given above requires that each apparent (bound) variable in φ also has lower type than φ. Together these result in the formalism of the *ramified theory of types.* At the bottom are variables of type 0 ranging over an unspecified non-empty domain of "individuals." In the simple theory of types, types can be identified with natural numbers, classes of individuals are of type 1, classes of classes of individuals of type 2, and so on. In the ramified theory of types, as e.g., elucidated by Myhill (1974), *types* are finite descending sequences t of natural numbers of *order* or *level* the first term

m_1 of t. If $\varphi(x)$ has x of type t and m is least such that $m > m_1$ and all bound variables of φ have order less than m, then φ and its extension $\{x : \varphi(x)\}$ is of type $\langle m\rangle^\frown t$, and variables of that type are interpreted as ranging over all such classes. The atomic formulas of this language are of the form $x = y$ where x, y are of the same type and $x \in y$ where x is of a type t and y is of a type $\langle m\rangle^\frown t$. Russell says that the type of one object is lower, resp. higher, than that of another if its level is lower, resp. higher.

If the domain of individuals is finite then under this interpretation for each type t the domain of objects of type t is finite, and the ramified interpretation collapses to the simple interpretation. It also follows that if one is to define the notion of natural numbers in purely logical terms, as would be required by the logicist program, it must be assumed that the domain of individuals is infinite. The assumption of this Axiom of Infinity was the first crack in Russell's attempt to continue the logicist program within the ramified theory of types. The second came with the definition of natural number itself in the form, as above:

$$N(a) := (\forall\varphi)[\varphi(0) \wedge (\forall x)(\varphi(x) \rightarrow \varphi(x')) \rightarrow \varphi(a)]$$

where 0 is the empty class at a given type t and the successor operation takes objects of type t to objects of type t, φ is of a type $\langle m\rangle^\frown t$, and the variable '$a$' is also of type t. Here, with natural numbers interpreted as cardinal numbers in the sense of equivalence classes under the relation of one-one correspondence, t must be of a type of classes of classes. There were two problems with this. First, the notion of natural number is relative to any such type t, and second, most usual proofs of induction can't be carried out. To show, for example, for any given natural number a that $(\forall x)[N(x) \rightarrow N(a+x)]$, where $+$ is suitably defined in cardinal-theoretic terms, one usually has to carry out an induction as follows: take $\psi(x) := [N(x) \rightarrow N(a + x)]$, and establish $(\forall x)\psi(x)$ by showing $\psi(0) \wedge (\forall x)(\psi(x) \rightarrow \psi(x'))$. But ψ is of a higher type (level) than that of the quantified variable φ in the definition of N so we can't apply that definition to carry out the required induction. As Russell says, "[it] is obvious that such a state of things renders elementary mathematics impossible."(ibid. p. 167). In order to get around this quite serious obstacle, Russell introduced his "oddly devious" (in Quine's words[3]) Axiom of Reducibility. This is formulated in terms of a notion of *predicative* function $\varphi!x$ (signaled by the exclamation mark) which may be described as one in which the bound variables, if any, are all of the same or lower type than the type of x. (Note that here 'predicative' is used in a very restricted sense.) Using variables $f, g, \ldots$ to range over predicative functions, the Axiom of Reducibility states that any propositional function is coextensive with some predicative function, schematically:

$$(\exists f)(\forall x)[\varphi(x) \leftrightarrow f!x]$$

(ibid., p.171). In particular, that allows one to carry out the blocked induction above by replacing ψ by an associated predicative function g.

[3]In his introductory note to Russell (1908) in van Heijenoort (1967), p. 151.

Evidently, the Axiom of Reducibility completely vitiates the system of ramified types and makes it equivalent to simple type theory, for which however, there is no predicative justification if one assumes the Axiom of Infinity. In the introduction to the second edition of *Principia Mathematica*, (presumably) Russell wrote:

> [The Axiom of Reducibility] has a purely pragmatic justification: it leads to the required results, and to no others. But clearly it is not the sort of axiom with which we can rest content. On this subject, however, it cannot be said that a satisfactory solution is as yet obtainable. (Whitehead and Russell 1925, p. xiv)

More specifically, at the end of that introduction Russell points out various places in the development of mathematics on the basis of his formalism where the Axiom of Reducibility is required, not only to carry out usual inductions on the natural numbers, as well as their generalizations to transfinite ordinal numbers, but also in the foundations of the theory of real numbers. In conclusion he writes:

> It might be possible to sacrifice infinite well-ordered series [i.e., well-ordering relations] to logical rigour, but the theory of real numbers is an integral part of ordinary mathematics, and can hardly be the object of a reasonable doubt. We are therefore justified in supposing that some logical axiom which is true will justify it. The axiom required may be more restricted than the axiom of reducibility, but, if so, it remains to be discovered. (ibid., p. xlv)

Poincaré vs. the logicists and the Cantorians: from paradoxes to practice

In view of the lack of justification for the Axiom of Infinity and the Axiom of Reducibility one must count as a failure Russell's attempt at a purely logical predicative foundation of mathematics beginning with the definition of the natural numbers. But even if it had been successful, it would have done nothing for Poincaré. In his excellent article, "Poincaré against the logicists," Warren Goldfarb writes:

> Although the great French mathematician Henri Poincaré wrote on topics in the philosophy of mathematics from as early as 1893, he did not come to consider the subject of modern logic until 1905. The attitude he then expressed toward the new logic was one of hostility. He ... dismissed as specious both the tools devised by the early logicians and the foundational programs they urged. His attack was broad: Cantor, Peano, Russell, Zermelo, and Hilbert all figure among its objects. Indeed, his first writing on the subject is extremely polemical and is laced with ridicule and derogation. Poincaré's tone subsequently became more reasonable but his

> opposition to logic and its foundational claims remained constant. (Goldfarb 1988, p. 61)

In this article Goldfarb on the whole legitimately critiques various of Poincaré's objections to the indicated foundational programs, but the nature of those objections is relevant to a fundamental shift in the further development of the idea(s) of predicativity and its relation to mathematical practice, so we shall review them here. First, as Goldfarb explains, Poincaré's attack on the new logicians begins

> ...with the avowed aim of showing that [they] have not eliminated the need for intuition in mathematics. By showing this, he says, he is vindicating Kant... This avowal is misleading, for in Poincaré's hands the notion of intuition has little in common with the Kantian one. ... [Rather], for Poincaré, to assert that a mathematical truth is given to us by intuition amounts to nothing more than that we recognize its truth and do not need, or do not feel the need, to argue for it. Intuition, in this sense, is a psychological term; it might just as well be called "immediate conviction." (ibid., p. 63)

In particular, for Poincaré the structure of natural numbers and the associated principle of induction are given in intuition and do not require a foundation; indeed, in his view, they are presupposed in any attempt at such a foundation. In the case of the logicist program, this is seen in the very description of the axiomatic system, with its inductive generation of formulas and proofs; hence that enterprise assertedly involves a *petitio principi.* Goldfarb argues that Poincaré is mistaken in ascribing a *petitio* in this case, as he is in other cases. In a useful review of Goldfarb's article, Michael Hallett (1990) defends Poincaré on this point, at least in part. What is important for us here is not which side is correct, but the content and influence of Poincaré's views.

Poincaré's objections to the Cantorians rests on their essential use of impredicative constructions violating the VCP. Curiously, he accepted the Axiom of Choice on the grounds that it "is a synthetic judgment a priori [note the Kantian terminology]; without it the theory of cardinals would be impossible, for finite numbers as well as for infinite ones."[4] But he criticized Zermelo's 1904 proof—from the Axiom of Choice—that every set can be well-ordered, at a point that made use of the very common set-theoretical operation of the union U of a set S of sets $X, U = \{x : (\exists X)(X \in S \wedge x \in X)\}$, on the grounds that what members U has depends on which sets belong to S and what their members are. In particular, it depends on whether U belongs to S and what its members are; thus, Poincaré identified the operation of union as illegitimately impredicative. In his response to this criticism, Zermelo (1908, in van Heijenoort 1967 pp. 190-191) pointed out that "proofs that have this logical form are by no means confined to set theory; exactly the same kind can be found in analysis wherever the maximum or the minimum of a previously defined 'completed' set

[4] Poincaré (1906), p. 313.

of numbers ... is used for further inferences. This happens, for example, in the well-known Cauchy proof of the fundamental theorem of algebra, and up to now it has not occurred to anyone to regard this as something illogical." The particular proof cited by Zermelo proceeds by forming the minimum of the set of values of $|p(x)|$ where p is a polynomial over the complex numbers. Relatedly, if $f(x)$ is a continuous function on a closed interval $[a, b]$ in the real numbers, then f has a minimum value (as well, of course, a maximum value) in that interval. This seems to appeal to the general principle of greatest lower bound (g.l.b.) for the real numbers R, i.e., that any bounded subset S of R has a g.l.b. (as well as a least upper bound, l.u.b.). The existence of the g.l.b. can be recognized set-theoretically as the formation of a union in terms of the association of real numbers x with their upper Dedekind sections D_x in the rational numbers Q, where

$$D_x = \{r : r \in Q \wedge x < r\}.$$

Let $S^* = \{D_x : x \in S\}$. Then the *union* U of the sets in S^* is the Dedekind section D_m where $m = \text{g.l.b.}(S)$; in this one invokes the *Dedekind completeness* (or *continuity*) of R, according to which every Dedekind section determines a real number. Dually, the l.u.b.(S) is determined by the *intersection* of all the sets in S^*. (If one uses lower Dedekind sections, these operations are reversed.)

Poincaré responded to Zermelo's specific example by asserting that the minimum of $|p(x)|$ is equally described as the minimum of $\{|p(x)| : x \text{ is rational}\}$. More generally, for any continuous function $f(x)$ on a compact set S in the real or complex numbers its minimum is the same as the minimum of the countable set $\{f(x) : x \in S \wedge x \text{ is rational}\}$, and that does not require an impredicative step. Poincaré was correct in this response, though his argument was faulty. Simply, any bounded set S of rational numbers has a g.l.b. (and l.u.b.) by selecting a subsequence that converges to the same. In this case, one invokes *Cauchy completeness* of R for which, in contrast to Dedekind completeness, there is a predicative argument as will be explained below.

Though Poincaré rejected the actual infinite in his polemical writings (e.g., "There is no actual infinite; the Cantorians forgot that, and they fell into contradiction."–1906, p.316), his acceptance of the structure of natural numbers and of the principle of induction on it coupled with the implicit use of classical logic and its assumption of the law of excluded middle at least takes quantification over the natural numbers to be definite, and that is a form of acceptance of the actual infinite. No doubt Poincaré assumed this and much more in his mathematical practice in analysis and topology, though he might have said that he could always account for his work predicatively by more careful considerations of the sort given in his response to Zermelo. There could be some historical interest in putting Poincaré's "philosophical" principles (to the extent that they can be made precise) up against the details of his practice, but that is neither here nor there for our present survey. What counts is what influence those principles had on the development of predicativity, not whether he abided by them himself. And what is most significant about Poincaré's dispute with Zermelo in 1906-1909 is that attention was shifted away from the role of purportedly circular definitions in the production of the paradoxes to their role at the very center of mathematical practice, namely in the l.u.b. principle for the real numbers.

The other aspect of Poincaré's dispute with Cantorians lies in their extension of the actual infinite to the transfinite. Of course he was not alone in this. Other prominent critics were Leopold Kronecker in the 19th century and L. E. J. Brouwer in the 20th century; but both were more radical than Poincaré in rejecting it also at the level of the natural numbers. In Brouwer's case it lay in the identification of the law of excluded middle as the culprit in the Cantorian crimes, in its supposedly illegitimate employment when applied to infinite totalities. Since, as I have argued, Poincaré did not go that far, he can be considered to be laying out a middle ground between the constructivists and the set-theoretical platonists: namely, it is the position of being a realist with respect to the natural numbers and a definitionist in all else.

Weyl's predicative development of analysis

Hermann Weyl was the first to give substance to this middle ground, in his famous 1918 monograph *Das Kontinuum*. An exegesis of that work is contained in the article "Weyl vindicated" (Feferman 1988) and its significance has been discussed in several other articles reprinted in my collection, *In the Light of Logic* (1998). There is thus no need to go over those here at any length. Rather, for the present purposes, a relatively brief summary is sufficient, and that is most simply accomplished by quoting from myself:

> In the introduction to [*Das Kontinuum*] Weyl criticized axiomatic set theory as a "house built on sand" (though the objects of, and reasons for, his criticism are not made explicit.) He proposed to replace this with a solid foundation, but not for all that had come to be accepted from set theory; the rest he gave up willingly, not seeing any other alternative. Weyl's main aim in this work was to secure mathematical analysis through a theory of the real number system (the continuum) that would make no basic assumptions beyond that of the structure of natural numbers N. ...Weyl did not attempt to reduce...reasoning about N to something supposedly more basic. In this respect Weyl agreed with Henri Poincaré that the natural number system and the associated principle of induction constitute an irreducible minimum of theoretical mathematics, and any effort to "justify" that would implicitly involve its assumption elsewhere... . ...unlike Brouwer,Weyl accepted uncritically the use of classical logic at this stage (though at a later date he was to champion Brouwer's views). (op. cit., pp. 51ff)

In Weyl's redevelopment of analysis, the rational numbers are reduced to the natural numbers in a standard way going back to Kronecker. But to do analysis, one needs representations of real numbers either as sets or sequences of rational numbers, and that reduces to the question of what sets or sequences of natural numbers may be asserted to exist. Having accepted the natural number system with its basic inductively defined operations such as addition and multiplication, Weyl accepted that each subset of N of the form $\{n \in N : A(n)\}$ exists, where A is an arithmetical formula, i.e., one that contains no quantifiers ranging over sets, only over natural numbers. Beyond those, what other kinds of sets may be asserted to exist on predicative grounds? Adapting ramified type theory to this specific second-order context, one could consider sets of level 1, 2, 3 and so on, where the sets of level 1 are the arithmetically definable ones, those of level 2 are defined by formulas in which all second order quantifiers range over sets of level 1, those of level 3 allow second order quantifiers only over sets of level 1 or sets of level 2, and so on. Carrying this over to analysis, one would correspondingly then have real numbers of different levels 1, 2, 3, ... But if S is a bounded set of real numbers of some level k, its g.l.b. and l.u.b. are defined by applying the operations of union, resp. intersection to corresponding sets S^* of upper Dedekind sections of the rational numbers, and that requires quantification over sets of level k. Thus the g.l.b., resp., l.u.b. of S would be of level $k + 1$. Weyl concluded that a development of analysis in such ramified terms would be unworkable; at the same time, he rejected the Axiom of Reducibility as untenable on predicative grounds. His solution was to confine himself to arithmetical sets of natural numbers and the associated sets and sequences of rational numbers. With real numbers identified not as Dedekind sections but rather as Cauchy (or fundamental) sequences $\langle r_n \rangle_{n \in N}$ of rational numbers, sequences of real numbers can be treated as double- sequences $r_{k,n}$ of rational numbers. Then one can show that if $s = \langle x_k \rangle_{k \in N}$ is a Cauchy sequence of real numbers given as $x_k = \langle r_{k,n} \rangle_{n \in N}$, then its limit $t = \langle q_n \rangle_{n \in N}$ exists arithmetically definable from the double-sequence $\langle r_{k,n} \rangle_{k,n \in N}$. Thus, in this form, the Cauchy completeness of the real number system is justified by the arithmetical comprehension axiom.

With the real number system secured in this way, Weyl could move on to see which parts of analysis could be redeveloped in this restricted version of predicativity, given the natural numbers, which one might call *arithmetical analysis*.[5] His main achievement was to show that substantially all of 19th century analysis of piece-wise continuous functions could be accounted for in this way, since any continuous function f of real numbers is completely determined by its behavior at rational arguments, and hence can be represented at the second-order level. Subsequent research beginning in the 1970s has been able to extend Weyl's program much farther into 20th century analysis; some more details about the reach of that will be indicated below. What is important for this part of our story is not that Weyl brought forth a new idea about predicativity, but rather that he showed the exceptional viability mathematically of the restricted part of predicativity, given the natural numbers, based without ramification on the second-order arithmetical comprehension axiom.

[5]The first formulation of arithmetical analysis in modern logical terms was given by Grzegorczyck (1955).

Predicativity sidelined: 1920-1950

If predicativity was to progress following these early developments it would need a leader and practitioners. As it turned out, neither were there, for a number of reasons. Its first champion, Poincaré, died in 1912, and in any case he had not engaged in the essential clarification or implementation of his ideas. As for Russell, predicative logicism in the form of ramified type theory was compromised by the Axiom of Reducibility. Then, following the exhausting task of producing the first edition of *Principia Mathematica* (1910-1913), Russell turned to general problems in philosophy from an analytic perspective, and during World War I was drawn aside into pacifist politics. Finally, Wittgenstein's critiques of Russell's logic and philosophy struck serious blows to his confidence in his ideas.

The case of Weyl is different: despite the relative success of his arithmetically predicativist program in *Das Kontinuum*, he fell under the spell of Brouwer as he became more familiar with the work of that crusading intuitionist. In 1921 he wrote, "I now abandon my own attempt and join Brouwer."[6] He then engaged in contributing to intuitionistic ideas and their relation to practice, and championed these in the following years, but eventually he became rather pessimistic about its prospects. Years later he wrote:

> Mathematics with Brouwer gains its highest intuitive clarity. He succeeds in developing the beginnings of analysis in a natural manner, all the time preserving the contact with intuition much more closely than had been done before. It cannot be denied, however, that in advancing to higher and more general theories, the inapplicability of the simple laws of classical logic eventually results in almost unbearable awkwardness. (Weyl 1949, p. 54)

The foundational program most prominently in competition with that of Brouwer was Hilbert's proof-theoretical consistency program. By restricting to finitist methods, that was more radical than Brouwer's, but it was much more ambitious in its aim to "secure" the practice of non-constructive mathematics via consistency proofs of appropriate formal systems. Though that was dashed by Gödel's theorem on unprovability of consistency in 1931, the program took on new life with Gentzen's consistency proof for arithmetic and was transformed by the employment of limited transfinite methods.

What really pushed predicativity to the sidelines, however, was the success of axiomatic set theory—as developed by Zermelo, Skolem and Fraenkel—in allaying fears about the paradoxes. Though not demonstrably consistent, intensive development of the subject without running into any difficulties gave comfort and confidence to its practitioners and gradually won the support of mathematicians at large. Nor did the impredicativity that Poincaré and Weyl had located in the l.u.b. principle in the real numbers generate any concern.

[6] Weyl's intuitionistic excursion is fully described and exemplified with relevant articles in Part II of Mancosu (1998); cf. p. 98 op. cit. for the 1921 quote.

For, in mathematical practice the real numbers are regarded as a definite completed totality independent of human constructions and definitions, not some vague collection of numbers "growing" under successive definitions. Then the l.u.b. of a bounded subset S of R merely serves to single out a specific element of R, not to "bring it into existence." In this respect, it is like the least number operator in the set of natural numbers, which serves to single out a specific number in N without our always being able to say *which* one it is in ordinary systems of representation. (For example, in the solution of Waring's problem, given k, there is a least n such that every natural number is a sum of at most n k^{th} powers; for all but the first few k, the specific value of n is unknown.) Mathematicians were thus insensitive to impredicativity in practice.

A story here, recounted in my book, is apropos:

> ...a famous wager was made in Zürich in 1918 between Weyl and George Pólya, concerning the future status of the following two propositions: (1) Each bounded set of real numbers has a precise upper bound. (2) Each infinite subset of real numbers has a countable subset. [The latter requires the Axiom of Choice.] Weyl predicted that within twenty years either Pólya himself or a majority of leading mathematicians would admit that the concepts of number, set and countability involved in (1) and (2) are completely vague, and that it is no use asking whether these propositions are true or false, though any reasonably clear interpretation would make them false... . the loser was to publish the conditions of the bet and the fact that he lost in the *Jahresberichten der Deutschen Mathematiker Vereinigung...* (Feferman 1998, p. 57)

The wager was never settled as such, for obvious political reasons. According to Pólya (1972), "The outcome of the bet became a subject of discussion between Weyl and me a few years after the final date, around the end of 1940. Weyl thought he was 49% right and I, 51%; but he also asked me to waive the consequences specified in the bet, and I gladly agreed." Pólya showed the wager to many friends and colleagues, and, with one exception, all thought he had won.

In axiomatic Zermelo-Fraenkel (**ZF**) set theory, the fundamental source of impredicativity is in the Separation Axiom scheme, which asserts for each well-formed formula $\varphi(x)$ (possibly with parameters) of the language of **ZF** the existence of the set $\{x : x \in a \wedge \varphi(x)\}$ for any set a. Since the formula φ may contain quantifiers ranging over the supposed "totality" of all sets, this is impredicative according to the VCP. Mathematically, this prima facie impredicativity is given teeth by the assumption of the Axiom of Infinity, guaranteeing the existence of the set ω of finite ordinals (or the natural numbers) and by the assumption of the Power Set Axiom, guaranteeing for any set a the existence of $P(a) = \{x : x \subseteq a\}$. Without the Axiom of Infinity, the impredicativity of Separation becomes innocuous, since the system has a model in the hereditarily finite sets. With the Axiom of Infinity, the axioms of Power Set and Separation lead, among other things to the existence of the real number system, represented e.g. as the set of all upper Dedekind sections in the rational numbers, and thence to the l.u.b. principle for arbitrary bounded sets of real numbers.

Though predicativity went into hibernation until the 1950s, one important technical development in axiomatic set theory would prove to be significant for it. This was Gödel's model of the constructible sets (1939). The "standard interpretation" of **ZF** set theory, due to Zermelo in 1930, is in the *cumulative hierarchy* $\langle V_a \rangle_{a \in On}$, defined by transfinite recursion on the class On of (finite and transfinite) ordinal numbers α by transfinite iteration of the power set operation P starting with the empty set $\emptyset$ as follows:

$$V_0 = \emptyset, V_{\alpha+1} = P(V_\alpha), \text{ and for limit } \alpha, V_\alpha = \bigcup_{\beta<\alpha} V_\beta.$$

This is a *cumulative theory of types*, unlike simple type theory, since as may be shown by induction on α, if $\beta < \alpha$ then $V_\beta \subseteq V_\alpha$. Gödel defined the *constructible hierarchy* $\langle L_\alpha \rangle_{\alpha \in On}$ by modifying the successor step, replacing the power set operation by an operation $P_{\text{Def}}(a)$ which, for any set a, consists of all sets of the form $\{x : x \in a \wedge \varphi^a(x)\}$ where the superscript 'a' indicates that all quantifiers in φ are restricted to range over a, and where all parameters in φ are elements of a. Then L_α is defined recursively by:

$$L_0 = \emptyset, L_{\alpha+1} = P_{\text{Def}}(L_\alpha), \text{ and for limit } \alpha, L_\alpha = \bigcup_{\beta<\alpha} L_\beta.$$

The constructible hierarchy thus stands to the cumulative hierarchy as Russell's ramified theory of types stands to the simple theory of types. In fact, the constructible hierarchy may be considered to be entirely predicative except perhaps in its free use of arbitrary ordinals. Since ordinals are the order types of well-ordered sets and those are defined impredicatively by the condition that any non-empty subset has a least element, the constructible hierarchy is not on the face of it predicative. But it may be considered to be predicative in a modified sense, relative to the notion of arbitrary well-ordering or ordinal number.

An interesting restriction of the notion of constructibility is the L hierarchy taken up to the least uncountable ordinal ω_1, that comes into the next part of our story. There and in the four succeeding sections[7], somewhat technical notions are involved, and for the sake of brevity, it must be assumed that the reader is familiar with the basic concepts involved. Otherwise, the sections can be skimmed.

Predicativity in transition, as a chapter of definability theory

The logical analysis of predicativity reemerged in the 1950s as a chapter in the extension of recursion theory to various "higher" definability notions,

[7]Through "The outer mathematical bounds of predicativity."

especially for sets of natural numbers. A systematic study of hierarchies of definitions of sets of natural numbers was undertaken in the early 1950s by Kleene and, independently, Davis and Mostowski. In Kleene's hands, this took the following form, using the "Turing jump" operator which takes one from any set X to a set X' which is universal for sets that are Σ^0_1 in X. The sets which are obtained from N by finite iteration of the jump operation are, up to relative recursiveness, all the arithmetically definable sets. These thus take for granted the definiteness of quantification over N, and in that sense N as a completed totality. Kleene (1955) defined a transfinite extension of this hierarchy, using the set O of Church-Kleene notations for constructive ordinals. For $a \in O$, we write $|a|$ for the ordinal α denoted by a. As an analogue of the least uncountable ordinal ω_1, the least ordinal not constructive in the sense of Church-Kleene (i.e., the least α such that there is no $a \in O$ with $|a| = \alpha$), is denoted ω_1^{CK}. It was shown by Spector (1955) that the order types of recursive well-orderings of N are exactly the ordinals less than ω_1^{CK}. Kleene's extension of the finite jump hierarchy, and thence of the arithmetical sets, is defined inductively for each $a \in O$ yielding a subset H_a of N, as follows. $H_0 = \emptyset$ and the step from H_a to its successor set is by the jump operation; finally, at a limit notation, one takes the direct sum of the sets associated with the recursive sequence approaching it. Then a set is called *hyperarithmetic* if it is recursive in H_a for some $a \in O$. The collection of all such sets is denoted HYP, which properly extends the class of arithmetical sets. One of the main results of Kleene (1955) is that

$$HYP = \Delta^1_1,$$

i.e., the collection of sets definable both in Π^1_1 form $\{n : (\forall X)A(n, X)\}$ and in Σ^1_1 form $\{n : (\exists X)B(n, X)\}$ for A, B arithmetical.

The connection with predicative definitions in the sense of the *ramified analytic hierarchy* was established as follows. The basic step in that hierarchy consists in passing from a collection D of subsets of N to a new collection D^* by putting a set S in D^* just in case there is a formula $\varphi(x)$ of second-order arithmetic such that for all n,

$$n \in S \leftrightarrow (\varphi(n))^D,$$

where the superscript 'D' indicates that all second-order quantifiers in φ are relativized to range over D. Then we can define the collections R_α for arbitrary ordinals α by

$$R_0 = \emptyset, R_{\alpha+1} = (R_\alpha)^*, \text{ and for limit } \alpha, R_\alpha = \bigcup_{\beta<\alpha} R_\beta.$$

Thus the definition of the R_α proceeds exactly like the L_α in Gödel's constructible hierarchy, except that at each stage one is confined to collections of subsets of N. Though R_α is defined for all α, it is not hard to show that we get nothing new for uncountable α, in fact there is a countable α such that $R_\alpha = R_{\alpha+1}$. Kleene (1959) proved that

$$HYP = R_{\omega_1^{CK}}.$$

However, HYP does not exhaust the ramified hierarchy, as follows from results of Gandy (1960) and Spector (1960).

It was tentatively proposed by Kreisel (1960) to identify the predicatively definable sets, given the natural numbers, with the members of HYP, essentially for the following reasons. Call an ordinal α predicative(ly definable) if it is the order type of a predicatively definable well-ordering $\prec$ of the natural numbers; then a set should be considered to be predicatively definable if it belongs to R_α for some predicative α. Clearly, then, on those grounds all recursive ordinals are predicative, and so by Kleene's equation above, each member of HYP ought to be accepted as predicative. For the converse, the predicative ordinals should only be taken to be those generated by the following "boot-strap" condition: if α is a predicative ordinal and $\prec$ is a well-ordering relation in R_α and if β is the order-type of $\prec$, then β is predicative. But then the predicative ordinals do not go beyond the recursive ordinals, using the result of Spector (1955) that every hyperarithmetic well-ordering has order type less than ω_1^{CK}: for if α is recursive and the well-ordering $\prec$ is in R_α then it is hyperarithmetic, and hence its order-type β is recursive. This part of the argument shows that the predicative sets do not go beyond HYP.

Actually, analogous considerations leading to this identification had been proposed somewhat earlier by Wang (1954), who introduced ramified formal systems Σ_α which have as a natural model the constructible sets to level $\omega + \alpha$ (since Wang starts with the hereditarily finite sets). The ordinals regarded to be predicative are again generated by a "boot-strap" condition: if α is predicative and β is defined by a well-ordering relation expressed in Σ_α then β is predicative. The system Σ is taken to be the union of the Σ_α for α predicative. Spector (1957) established that the predicative ordinals in this sense are exactly those less than ω_1^{CK} and the sets of natural numbers definable in Σ are exactly the HYP sets. In a suitable sense, the sets definable in Σ are those which have HYP structure on their transitive closure.[8]

Though the considerations leading to the identification of the predicative ordinals, resp. sets of natural numbers, with the recursive ordinals, resp. hyperarithmetic sets, have a certain plausibility, they ignored one crucial point if predicativity is only to take the natural numbers for granted as a completed totality, namely that they involve in an essential way (both from above and below) the impredicative notion of being a well-ordering relation. A step away from that would be to talk of *predicatively provable well-orderings* in a way to be explained next.

Predicative provability in the 1960s

The idea, to begin with, is that instead of dealing with the collections R_α, one deals (as in the systems Σ_α) with a transfinite progression of formal systems

[8]Independently of Wang, Lorenzen (1955) dealt with systems $\mathbf{S}_\alpha$ of a character similar to the Σ_α but did not impose precise conditions on the ordinals to be admitted. Lorenzen's main concern there is with what parts of mathematics can be developed predicatively.

of ramified analysis $\mathbf{RA}_\alpha$, using variables $X^\beta, Y^\beta, Z^\beta, \ldots$ for each $\beta \leq \alpha$. Here ordinals are not to be considered set-theoretically but rather as notations chosen from O by some natural procedure. The formal systems $\mathbf{RA}_\alpha$ incorporate as axioms comprehension principles expressing closure under the appropriate ramified definitions. These can only provide closure conditions on the sets at each level; the minimal model of $\mathbf{RA}_\alpha$ is given by $\langle R_\beta \rangle_{\beta \leq \alpha}$, but larger collections can also satisfy the axioms of $\mathbf{RA}_\alpha$. In particular, we can interpret the variables $X^0, Y^0, Z^0, \ldots$ as satisfying the closure conditions of *any* larger R_β.

The crucial new point (compared to Wang's systems) is that the predicative ordinals are those that cannot only be *defined* by (what happen to be) well-ordering relations in the given systems but must also be *proved* previously to be such relations. The problem is how to meet this requirement without unrestricted second-order quantification; the answer comes from the provability condition as follows. Given $\prec$ a binary recursive relation in the natural numbers, let $WO^\beta(\prec)$ express that $\prec$ is a linear ordering such that every non-empty subset X^β of its field has a $\prec$-least element. Then if one proves $WO^0(\prec)$, it follows that one can "lift" the proof to establish $WO^\beta(\prec)$ for each β that comes to be accepted. In this sense, the proof of the predicatively meaningful statement $WO^0(\prec)$ can insure all predicatively meaningful consequences $WO^\beta(\prec)$ of the impredicative statement of well-ordering $WO(\prec)$; indeed, from the outside it insures that $\prec$ is a well-ordering. Now Kreisel's proposal (1958) can be formulated as follows. The predicative(ly provable) ordinals are generated from 0 by the boot-strap or *autonomy* condition: if α is predicative and $\mathbf{RA}_\alpha$ proves $WO^0(\prec)$ for a given recursive $\prec$, and β is the order type of $\prec$ then β is predicative. The least non-predicatively provable ordinal is then proposed to be the least ordinal which cannot be obtained in that way.

Kreisel called for an independent characterization of the limit of the predicatively provable ordinals in the sense just described. That problem was solved independently by Schütte (1965, 1965a) and me ("Systems of predicative analysis," 1964), in terms of the Veblen hierarchy $\langle \chi_\alpha \rangle_\alpha$ of critical functions of ordinals defined for each ordinal α by: $\chi_0(\xi) = \omega^\xi$ and when $\alpha \neq 0, \chi_\alpha$ enumerates in order of size the set of common fixed points ξ of all χ_β for each $\beta < \alpha$, i.e., $\{\xi : \text{for all } \beta < \alpha, \chi_\beta(\xi) = \xi\}$. Then the function of ξ given by $\chi_\xi(0)$ is normal; let Γ_α be the α^{th} fixed point ξ of the equation $\chi_\xi(0) = \xi$. Then what Schütte and I showed is that the least non-predicatively provable ordinal is Γ_0. Outlines of proofs are to be found in the cited references. A full exposition can be found in Schütte's book *Proof Theory* (1977), Ch. VIII.

Predicatively justifiable systems

Taking the intrinsic interest of predicativity given the natural numbers for granted, after determining Γ_0 to be the least non-predicatively provable ordinal according to the preceding proposal, one had to return to the question of how much mathematics could be developed predicatively. This was pursued both theoretically, via alternative formal systems, and by means of case studies, to be discussed below. On the theoretical side, since (as Weyl stressed) ramified theories are unsuitable as a framework for the development of analysis, the first question was to see which unramified systems could be justified on predicative grounds. A formal system **T** is said to be *predicatively justifiable* if it is proof-theoretically reducible to one of the systems $\mathbf{RA}_\alpha$ for $\alpha < \Gamma_0$ and *locally predicatively justifiable* if it is proof-theoretically reducible to the union of all these systems.[9] If **T** has the same proof-theoretic strength as that progression, then its proof-theoretic ordinal is Γ_0. In that case, though the system **T** as a whole may not be justifiable predicatively, each theorem φ of **T** rests on predicative grounds, at least indirectly. In practice, more can be said: **T** is conservative over the autonomous ramified progression for arithmetic sentences, i.e., if φ is arithmetical and provable in **T** then it is provable in that progression. For second-order **T** this can often be strengthened to conservativity for Π^1_1 sentences. In particular, in that case, any provable well-ordering of **T** is also predicatively provable.

The first two examples of unramified second-order locally predicatively justifiable systems were given in the paper Feferman (1964). The first of these was obtained by replacing the progression of ramified theories $\mathbf{RA}_\alpha$ by a progression $\mathbf{HC}_\alpha$ of unramified second order theories, based on the Hyperarithmetic (or Δ^1_1) Comprehension Rule:

$$(\Delta^1_1 - \mathrm{CR}) \quad \text{From } (\forall x)[P(x) \leftrightarrow Q(x)] \text{ infer } (\exists X)(\forall x)[x \in X \leftrightarrow P(x)],$$

where $P(x)$ is any Π^1_1 formula and $Q(x)$ is any Σ^1_1 formula (parameters allowed).

The motivation for $\Delta^1_1 - \mathrm{CR}$ is the recognizable absoluteness (or invariance) of provably Δ^1_1 definitions, in the following sense. At each stage one has recognized certain closure conditions on the "open" universe of sets, and the definitions $D(x)$ of sets introduced at the next stage should be independent of what further closure conditions may be accepted. In the words of Poincaré, the definitions used of objects in an incomplete totality should not be "disturbed by the introduction of new elements." Thus if U represents a universe of sets (subsets of N) satisfying given closure conditions and is extended to U' (satisfying the same closure conditions and possibly further ones) one wants D to be *provably invariant or absolute* in the sense that $(\forall x)[D^U(x) \leftrightarrow D^{U'}(x)]$. This requirement is easily seen to hold for provably Δ^1_1 formulas D, i.e., those for which $(\forall x)[D(x) \leftrightarrow P(x)]$ has been proved where P and Q satisfy the hypothesis of the rule $\Delta^1_1 - \mathrm{CR}$.[10] The progression of theories $\mathbf{HC}_\alpha$ is then obtained by suitable transfinite iteration of the rule $\Delta^1_1 - \mathrm{CR}$. Again, one can apply the notion of autonomy to such a progression; it was shown that the least non-autonomous ordinal of this progression is Γ_0, the union of the $\mathbf{HC}_\alpha$ for $\alpha < \Gamma_0$ is of the same proof-theoretical strength as the union of the autonomous ramified progression, and one has conservativity for Π^1_1 formulas.

[9] See Feferman (1998) p. 193 for the notion of proof-theoretic reduction. Previously, I have used 'predicatively reducible' for what is written here as 'predicatively justifiable.'

[10] It has been shown in Feferman (1968) that every provably invariant formula is equivalent to a provably Δ^1_1 formula, so the rule $\Delta^1_1 - \mathrm{CR}$ is fully general for this requirement.

The second locally predicatively justifiable system introduced in Feferman (1964) was obtained by replacing the autonomous progression of the $\mathbf{HC}_\alpha$ by a single second-order system, denoted **IR**. This is axiomatized by the rule $\Delta^1_1 -$ CR together with what is called the Bar Rule in the proof-theoretic literature, which allows one to infer the full scheme of transfinite induction on a recursive ordering $\prec$ when $WO(\prec)$ has been established; correspondingly, one has a rule for inferring a scheme of transfinite recursion on $\prec$ under the same hypothesis. (The 'I' in '**IR**' is for Induction, and the 'R' is for Recursion.) The main result concerning **IR** stated in Feferman (1964) is that it proves the same theorems as the union of the $\mathbf{HC}_\alpha$ for $\alpha < \Gamma_0$, and thus it is predicatively reducible with conservativity for Π^1_1 formulas.

The system of what is often referred to as second order arithmetic has the *Full* (Π^∞_1) *Comprehension Axiom Scheme*

$$(\Pi^\infty_1 - \text{CA}) \qquad (\exists X)(\forall x)[x \in X \leftrightarrow \varphi]$$

where φ is an arbitrary second-order (Π^∞_1) formula which does not contain 'X' free. In 1976, Harvey Friedman introduced several subsystems of second-order arithmetic whose common feature is that induction is restricted to its second-order form, the *Induction Axiom*

$$(\text{IA}) \qquad (\forall X)[(0 \in X) \wedge (\forall x)(x \in X \rightarrow x' \in X) \rightarrow (\forall x)(x \in X)].$$

In the presence of the full comprehension axiom one can infer from the induction axiom IA the *Induction Scheme* considered as the collection of all formulas of the form

$$(\text{IS}) \qquad \varphi(0) \wedge (\forall x)[\varphi(x) \rightarrow \varphi(x')] \rightarrow (\forall x)\varphi(x)$$

for arbitrary second-order formulas φ. But in the restricted systems considered by Friedman in which the comprehension axiom is considerably weakened, that step is not possible. Following Friedman, these systems are indicated by a subscript '0.' An obvious choice for such to consider with respect to the present subject is the system $\mathbf{ACA_0}$ obtained by replacing the full comprehension axiom scheme by the sub-scheme in which only arithmetical formulas φ (no bound second-order variables) are admitted. It is a classical result of proof theory that the system $\mathbf{ACA_0}$ is a conservative extension of the first-order system of Peano Arithmetic **PA**. On the other hand, the system **ACA**, which is obtained by use of the full induction scheme (IS) in place of (IA), is stronger than **PA**; it is still predicatively justifiable, but its proof theoretical ordinal is far below Γ_0.

Another locally predicatively justifiable system introduced in Friedman (1976) is denoted $\mathbf{ATR_0}$. It has a certain similarity to **IR**, using axioms in place of rules, as follows. In place of the $\Delta^1_1 -$ CR one has the $\Delta^1_1 -$ CA, i.e., the axiom scheme

$$(\forall x)[P(x) \leftrightarrow Q(x)] \rightarrow (\exists X)(\forall x)[x \in X \leftrightarrow P(x)]$$

where P is Π^1_1 and Q is Σ^1_1. In place of the transfinite recursion rule of **IR** one has an axiom expressing that for all well-ordering relations Z one can construct the Turing jump hierarchy along Z starting with any set X, i.e., the relative hyperarithmetical hierarchy along Z. The transfinite induction rule in **IR** is not replaced by the corresponding axiom, as that would be too strong; in place of it, the system $\mathbf{ATR_0}$ uses only the induction axiom (IA) for natural numbers as above. Friedman's main result announced in 1976 was that $\mathbf{ATR_0}$ is locally predicatively justifiable with proof-theoretical ordinal Γ_0 and is conservative over **IR** for Π^1_1 sentences; a full proof of this with further interesting results about $\mathbf{ATR_0}$ is given in the paper Friedman, McAloon and Simpson (1982).

Various wholly or locally predicatively justifiable systems of higher type are surveyed in Avigad and Feferman (1998), sections 8.2 and 8.3. Systems of set theory related to **IR** were treated in Feferman (1974), while ones related to $\mathbf{ATR_0}$ have been dealt with in Simpson (1982). See also Simpson's article (2002),"Predicativity: the outer limits."

The mathematical reach of predicativity: positive developments

Having established theoretical bounds to predicativity and workable unramified predicatively justifiable systems, the next questions of interest are to see what parts of "everyday" mathematics can be carried out within those bounds, and what parts are essentially impredicative; the latter question is treated in the next section. It turns out in practice that if a known mathematical result can be established predicatively, it can already be done in a system conservative over Peano Arithmetic (**PA**). In other words, the predicative part of everyday mathematics is *robustly predicative*. There is no theorem which can establish this; one can only depend on various case studies for confirming evidence. One avenue is that pursued in the so-called Reverse Mathematics (R.M.) program due to Friedman and carried on most extensively by Simpson in his book, *Subsystems of Second Order Arithmetic* (1999). The principal relevant system for the positive work on predicative mathematics in that framework is $\mathbf{ACA_0}$, described above, and conservative over **PA**. Another relevant system is much weaker, being based only on Weak König's Lemma, i.e., the familiar tree lemma for infinite branching binary trees; it is shown op. cit. that the associated system $\mathbf{WKL_0}$ is conservative over the system **PRA** of Primitive Recursive Arithmetic. A second avenue in which the positive reach of predicative mathematics has been studied is via a system **W** (for 'Weyl') of flexible finite types introduced in the final part of the article "Weyl vindicated" (Feferman 1988). The main proof-theoretical result concerning **W** is that it is a conservative extension of **PA** (Feferman and Jäger 1993, 1996). Only some points of general character concerning both of these approaches will be indicated here.

In **W**, types are variable; for any two types X and Y, one has existence of their cartesian product $X \times Y$, and the type $Y^{(X)}$ of all partial functions from X to Y; finally, for any type X and bounded formula φ one has the subtype $\{x \in X : \varphi(x)\}$ of X determined by φ. Starting from the type N of natural numbers, one can introduce as usual the type Q of rational numbers, and then the type R of real numbers considered as Cauchy sequences of rational numbers, identified under the usual equivalence relation. Then (partial) functions of real numbers can be treated as members of $R^{(R)}$, and various spaces of such functions (e.g., continuous, measurable, etc.) form examples for Hilbert spaces and Banach spaces. Given any such space S, the functionals from S to R are then certain members of $R^{(S)}$. In practice, only separable spaces can be treated predicatively, via a given countable dense subset.

As for the program initiated by Weyl in 1918, the full classical analysis of continuous functions can be carried out directly in **W**. Turning to more general classes of functions from 20th century analysis such as come out of the Lebesgue theory of measure, one first notes that the general notion of outer measure can't be defined in **W**, since it makes essential use of the g.l.b. as applied to sets of reals, not sequences of reals. But the notion of measurable set can be treated via sequences of open covers whose measure is directly defined, and the notion of measurable function can be defined in terms of that or in terms of sequences of approximating step functions. One cannot prove the existence of Lebesgue non-measurable sets of reals; put in other terms, it is consistent with **W** that all sets of reals are Lebesgue measurable. Various parts of standard functional analysis have been verified in unpublished notes: the Riesz representation theorem, Hahn-Banach theorem, uniform boundedness theorem and open mapping theorem for separable Hilbert and Banach spaces. Finally, one can obtain the principal results of the spectral theory of bounded as well as unbounded self-adjoint linear operators on a separable Hilbert space. Subsequent to this work, it has been shown in the dissertation of Feng Ye (1999) that much of this work can already be carried out in a constructive subsystem of **W**, conservative over **PRA**.

By comparison, in the R.M. program sharper results have been obtained, of the form that over a weak base system $\mathbf{RCA_0}$, various of these results in analysis are actually equivalent to $\mathbf{ACA_0}$ (if not already to $\mathbf{WKL_0}$); these are detailed in Simpson (1999). However, since the systems considered in the R.M. program are all subsystems of second-order arithmetic, there is a cost involved, namely that higher type notions such as those of functions of real numbers, function spaces and functionals, cannot be dealt with directly but must somehow be coded in second-order terms. If one is not concerned with obtaining exact equivalences between mathematical results and the second-order set and function existence principles on which they ultimately rest, the positive predicative development of analysis is carried out much more naturally in systems like **W**.

The outer mathematical bounds of predicativity

Turning now to mathematical results which cannot be carried out predicatively, the easiest proofs of independence are those which can be established by models in the hyperarithmetic sets. For example, independence of the existence of non-measurable sets of reals stems from the fact that every Δ^1_1 set of reals has HYP Lebesgue measure. Another example using HYP that was found by Kreisel (1959) is the impredicativity of the Cantor-Bendixson theorem, which asserts that every closed set of reals is the union of a perfect set and a countable (scattered) set. The HYP model can also be used to obtain examples of impredicative theorems in Abelian group theory (Feferman 1975).

For independence results closer to the bounds of predicativity, one must fall back on proof-theoretic results. For example, any theorem equivalent to $\mathbf{ATR_0}$ is at the exact limit Γ_0 of predicativity, and is thus impredicative. Simpson (2002) gives a number of examples of theorems from descriptive set theory that are equivalent to $\mathbf{ATR_0}$ over $\mathbf{RCA_0}$, such as that every uncountable closed (or analytic) set contains a perfect subset. Also, $\mathbf{ATR_0}$ is equivalent to comparability of countable well-orderings. Other independence results have come from combinatorics rather than analysis, set theory, or algebra. They are of interest because the statements are arithmetical, in fact in Π^0_2 form. One group of these has to do with variants of the Ramsey coloring theorem, such as the Paris-Harrington (1977) version independent of **PA**. Friedman, McAloon and Simpson (1982) have given a mathematically natural finite combinatorial theorem which is equivalent to $\mathbf{ATR_0}$ over $\mathbf{RCA_0}$. Friedman showed that certain simple Π^0_2 consequences of Kruskal's theorem about embeddings of finite trees are not even provable in $\mathbf{ATR_0}$ (cf. Simpson 2002 for references and further developments).

Predicativity and the indispensability arguments[11]

The idea that natural science justifies a part of mathematics because of its indispensability to science—and that that is the *only* part of mathematics that is justified—is due to Willard Quine and Hilary Putnam among others. Famously, Quine has written:

> So much of mathematics as is wanted for use in empirical science is for me on a par with the rest of science. Transfinite ramifications are on the same footing insofar as they come of a simplificatory rounding out, but anything further is on a par with uninterpreted systems. (Quine 1984, p. 788).

Quine argued that we need the power set operation in set theory to establish the existence of the real numbers R and then its application once more to obtain such sets as that of all functions from R to R. That led him by a "simplificatory rounding out" to acceptance of finite iterations of the power set, but not its ω^{th} iteration over the natural numbers. In other words, Quine was led thus to accept Zermelo set theory but nothing stronger, which he looked upon as "mathematical recreation and without ontological rights" (Quine 1986, p. 400). Penelope Maddy has characterized the indispensability arguments (for critical purposes), as follows:

[11] Re the indispensability arguments, see also the relevant sections of the chapters by Resnik and Maddy in this volume.

> We have good reason to believe our best scientific theories, and mathematical entities are indispensable to those theories, so we have good reason to believe in mathematical entities. Mathematics is thus on an ontological par with natural science. Furthermore, the evidence that confirms scientific theories also confirms the required mathematics, so [that part of] mathematics and science are on an epistemological par as well. (Maddy 1992, p. 78)

In (Feferman 1993, reprinted in 1998) I considered the significance for the indispensability arguments of the positive developments of parts of mathematics by predicative means in the following terms (1998, p. 284): if one accepts the indispensability arguments [of course one might argue against them on philosophical grounds] there remain two critical questions:

Q1. Just which mathematical entities are indispensable to current scientific theories?

Q2. Just what principles concerning those entities are needed for the required mathematics?

The positive developments described above are brought to bear on these questions as follows. On their basis, I had formulated *the working hypothesis that all of scientifically applicable analysis can be developed in the system* **W**, and argued that this has been verified in its core parts (cf. 1998, pp. 280-283 and 293-294). Of course, there are results of theoretical analysis which cannot be carried out predicatively, either because they are essentially impredicative in their very formulation, or because they are independent of predicative systems such as the examples given above. However, none of those affects the working hypothesis because they do not figure in the applicable mathematics. What is more to the point are examples closer to the margin scientifically, e.g., the proposed use of certain non-separable spaces for a quantum-mechanical model involving infinitely many degrees of freedom in Emch (1972, p. 103); *contra* that one can appeal to the arguments on physical grounds by Streater and Wightman (1978, p. 87).[12] Of course the working hypothesis may yet prove to be wrong by other examples, but as of now, all evidence is in its favor. If one accepts it, one can return to the questions Q1 and Q2 as follows:

> By the fact of the proof-theoretical reduction of **W** to **PA**, the only ontology it commits one to is that which justifies acceptance of **PA**. But even there, the answer to Q1 and thence to Q2 is underdetermined. One view of **PA** is that it is about the natural numbers as independently existing abstract objects; that is ... a platonistic view, albeit an extremely moderate one. ... Or one can make use of the fact that **PA** is reducible to **HA** [Heyting Arithmetic] to justify

[12]See Feferman (1998) p. 282 for a fuller discussion.

> it on the basis of a more constructive ontology. (Feferman 1998, p. 296).

From this and related arguments, I drew the conclusion that

> ...if one accepts the indispensability arguments, practically nothing philosophically definitive can be said of the entities which are then supposed to have the same status—ontologically and epistemologically—as the entities of natural science. That being the case, what do the indispensability arguments amount to? As far as I'm concerned, they are completely vitiated. (ibid., p. 297).

Rethinking predicativity II: 1970-1996.

In Kreisel's article "Principles of proof and ordinals implicit in given concepts" (1970), he criticized existing proof theory for "*the lack of a clear and convincing analysis of the choice of methods of proof*," and took as his ultimate aim "*the discovery of objective criteria for such a choice*" [italics in the original]. "What one is after is a (phenomenological) description of certain kinds **P** of mathematical reasoning; the *objective question* is then simply this: whether the proofs represented ... by derivations of a given formal system **F** are in **P** (soundness of **F**); [and] whether all proofs in **P** are represented in **F** (completeness with respect to ... provability in **P**)." The particular kinds of reasoning **P** considered with respect to this aim were described as an answer to the following:

> What principles of proof do we recognize as valid once we have understood (or, as one sometimes says, 'accepted') certain given concepts? (Kreisel 1970, p. 489)

As a further elaboration, "[t]he process of recognizing the validity of such principles (including the principles for *defining* new concepts, that is, formally, of extending a given language) is here conceived as a *process of reflection...* . ... Granted that we have to do with an area **P** which lends itself to the kind of analysis indicated, it is evident that *ordinals* play a basic role. They index the stages in the reflection process."

The two principal basic concepts considered by Kreisel (op. cit., p. 490) are, in his terminology:

1. The concepts of ω-sequence and ω-iteration.

2. The concepts of set of natural numbers and numerical quantification.

Kreisel pointed out that this is related to earlier work on autonomous progressions for *finitist mathematics* (in Kreisel 1958 and 1965) and for *predicative mathematics* (in Feferman 1964).

This rethinking of predicativity, and relatedly, of finitism, was persuasive to me except for the idea that one would expect the stages of reflection to be indexed by (possibly) transfinite ordinals, though such ordinals might well be used metamathematically in an evaluation of the proof-theoretical strength of the system **F** proposed to represent **P**. In my view, the **F** considered for a given **P** should not be taken to involve the notions of ordinal or well-ordering in any way that is not already contained in the basic concepts of **P**. The formulation of this went through several stages, marked among others by the publications (Feferman 1979 and 1991), arriving most recently at the general notion of unfolding, as first explained in "Gödel's program for new axioms: why, where, how, and what?" (Feferman 1996). That also pointed to the possible applicability of the notion to systems of set theory at the other extreme to finitism, in which certain axioms for "large cardinals" would be derived that would fulfill Gödel's view that the familiar systems such as **ZFC** "can be supplemented without arbitrariness by new axioms which are only the natural continuation of those set up so far" (Gödel 1947, p. 520). The idea of unfolding is outlined next.

Predicativity as unfolding[13]

It is of the essence of the notion of unfolding that we are dealing with schematically presented formal systems. In the usual conception, *formal schemata* for axioms and rules of inference employ *free predicate variables* $P, Q, \ldots$ of various numbers of arguments $n \geq 0$. An appropriate substitution for $P(x_1, \ldots, x_n)$ in such a scheme is a formula $\varphi(x_1, \ldots x_n)$, possibly with additional parameters. Familiar examples of *axiom schemata* in the propositional and predicate calculus are

$$\neg P \rightarrow (P \rightarrow Q) \qquad \text{and} \qquad (\forall x)P(x) \rightarrow P(t).$$

The induction axiom scheme in non-finitist arithmetic is given by

$$P(0) \wedge (\forall x)[P(x) \rightarrow P(x')] \rightarrow (\forall x)P(x).$$

In set theory, familiar axiom schemes which can be represented similarly are those for Separation and Replacement.

Also rules of inference may be represented schematically, such as Modus Ponens in the propositional calculus and Universal Generalization in the predicate calculus, given respectively by

$$P, P \rightarrow Q/Q \qquad \text{and} \qquad [P \rightarrow Q(x)]/[P \rightarrow (\forall x)Q(x)].$$

In finitist arithmetic, in which quantification over the natural numbers is not accepted, the basic principle of induction is given by the rule:

$$P(0), P(x) \rightarrow P(x')/P(x).$$

[13]This section is adapted from Feferman (1996).

The informal philosophy behind the use of schemata in the concept of unfolding is their *open-endedness*. That is, they are not conceived of as applying to a specific language whose stock of basic symbols is fixed in advance, but rather as applicable to *any* language which one comes to recognize as embodying meaningful basic notions. Put in other terms, *implicit in the acceptance of given schemata is the acceptance of any meaningful substitution instances.* But it need not be determined in advance *which* substitution instances are to be accepted. Thus, for example, if one accepts the axioms and rules of the classical propositional calculus given in schematic form, one will accept all substitution instances of these schemata in any language which one comes to employ.

The question which the notion of unfolding is supposed to address is: *given a schematic system S, which operations and predicates—and which principles concerning them—ought to be accepted if one has accepted S*? The answer for operations is straightforward: *any operation from and to individuals is accepted which is determined explicitly or implicitly from the basic operations of S.* Moreover, the *principles* which are added concerning these operations are just those which are derived from the way that they are introduced. Ordinarily, we would confine ourselves to the *total operations* obtained in this way, i.e., those which have been proved to be defined for all values of their arguments, but it should not be excluded that the introduction might depend on prior *partial operations*, e.g., those introduced by recursive definitions of a general form. The question concerning predicates in the unfolding of S is treated in operational terms as well, i.e.: *which operations on and to predicates–and which principles concerning them—ought to be accepted if one has accepted S*? For this, it is necessary to tell at the outset *which logical operations on predicates are taken for granted in S.* For example, in the case of non-finitist classical arithmetic, these would be (say) the operations $\neg$, $\wedge$ and $\forall$, while in the case of finitist arithmetic we would be limited to positive propositional connectives and (in one formulation) the $\exists$ operator.

As a general background theory to unfolding for arbitrary S, one assumes a kind of proto-mathematical theory of operations and predicates, which makes only those assumptions that appear in every mathematical theory. The theory of operations can be typed or untyped; the latter, which is formally simpler, is taken to be a form of partial combinatory algebra with pairing and projection operations. These provide for closure of both operations and predicates under explicit definition. In addition, the combinatory setup allows one to construct a *generalized recursion* or *fixed point operator* which assures the uniform implicit definition from any given f of a (possibly partial) operation g satisfying

$$g = fg.$$

One general (logic independent) operation on predicates is given a distinguished role, since we consider the case when it is not used. Given a total operation f on a domain M to n-ary predicates over M, say $fx = P_x$ for each $x \in M$, the *join operation* J is used to form the predicate $J(f) = P$ satisfying for all $x, x_1, \ldots, x_n$:

$$P(x, x_1, ..., x_n) \leftrightarrow P_x(x_1, \ldots, x_n).$$

The *operational unfolding* of a schematic system $\mathbf{S}$, in symbols $\mathbf{U_0}(\mathbf{S})$, makes use only of the background theory of operations over the given operations of $\mathbf{S}$, i.e., it does not make use of any operations on or to predicates. The *full (operational and predicate) unfolding of* $\mathbf{S}$, in symbols $\mathbf{U}(\mathbf{S})$ also admits the background theory of operations on and to predicates over the given logical operations of $\mathbf{S}$, including the join operator J. The *intermediate (operational and predicate) unfolding of* $\mathbf{S}$, in symbols $\mathbf{U_1}(\mathbf{S})$, is the same without the join operator. Systems are unfolded by establishing the definedness and uniqueness of more and more operations of the various kinds. These serve to expand the language and thence the formulas φ which can be admitted to the

(Substitution Rule) $\quad A(P)/A(\hat{x}.\varphi(x))$.

As an example, the schematic system **NFA** of Non-Finitist Arithmetic is given with basic operations on individuals of successor Sc, predecessor Pd and the 0-ary operation (constant) 0, and a schematic predicate symbol $P(x)$. In the intermediate and full unfoldings, the basic logical operations assumed are $\neg, \wedge$ and $\forall$. The basic axioms of **NFA** are simply the following three, where we write x' for $Sc(x)$:

Ax 1. $x' \neq 0$.
Ax 2. $Pd(x') = x$.
Ax 3. $P(0) \wedge (\forall x)[P(x) \rightarrow P(x')] \rightarrow (\forall x)P(x)$.

In my paper with Thomas Strahm, "The unfolding of non-finitist arithmetic" (2000) the following results are proved, where $\equiv$ is the relation of proof-theoretical equivalence, **PA** is the usual system of Peano Arithmetic, and $\mathbf{RA}_{<\alpha}$ is the union of the systems $\mathbf{RA}_\beta$ of ramified analysis to level β, for $\beta < \alpha$.

(i) $\mathbf{U_0}(\mathbf{NFA}) \equiv \mathbf{PA}$

(ii) $\mathbf{U_1}(\mathbf{NFA}) \equiv \mathbf{RA}_{<\omega}$.

(iii) $\mathbf{U}(\mathbf{NFA}) \equiv \mathbf{RA}_{<\Gamma_0}$.

In other words, the full operational and predicate unfolding of **NFA** is proof-theoretically equivalent to predicative analysis as characterized via the autonomous progression of ramified theories. The first step in getting **PA** contained in $\mathbf{U_0}(\mathbf{NFA})$ is to establish successively the definedness of all primitive recursive functions.[14]

In an abstract with Strahm, "Unfolding finitist arithmetic" (2001) we have announced the following result for a system **FA** of Finitist Arithmetic with

[14]Actually, in Feferman and Strahm (2000) we made use of a background theory of typed operations with general Least Fixed Point operator, but we have also verified the results as stated here for the untyped background theory.

the restricted operations on predicates indicated above, and with the induction axiom replaced by the quantifier-free induction rule.[15] Here we obtained, in terms of the system **PRA** of Primitive Recursive Arithmetic:

$$\mathbf{U_0}(\mathbf{FA}) \equiv \mathbf{U_1}(\mathbf{FA}) \equiv \mathbf{U}(\mathbf{FA}) \equiv \mathbf{PRA}.$$

This is in accord with Tait's argument (1981) that finitism is formally represented by **PRA**.

Kreisel's work on finitism in terms of certain autonomous progressions (1958 and 1965) led to a system whose proof-theoretical strength is Peano Arithmetic **PA**. It should be possible to expand the system **FA** to a system $\mathbf{FA}^*$ whose unfolding is exactly **PA** in strength. The guess is that this would be obtained by adding a suitable form of the Bar Rule in the language of **FA** informally expressed as follows: if $\prec$ is a decidable ordering and it has been proved (with free variable f) that f is not descending in $\prec$, then one can apply transfinite induction to $\prec$.

Conjecture. $\mathbf{U}(\mathbf{FA}^*) \equiv \mathbf{PA}$.

If this is right, what gives Kreisel's characterization of finitism its unexpected strength is his implicit use of a notion of finitist well-ordering.

What is predicativity? (Summary)

The idea of predicativity started in a negative frame of mind with the identification of a vicious circle in the use of definitions purported to single out an object D from a supposed totality V by essential reference to the entirety of V. Such a definition was considered to be problematic when V is in some sense essentially ill-defined, not "actual" or "complete." To begin with, the focus was on the supposed totalities of all sets or all ordinal or cardinal numbers in the framework of Cantorian set theory. The radical diagnosis of Kronecker, Poincaré and Brouwer saw a threat to mathematics in *all* assumptions of the actual infinite, even down to the natural numbers. Hilbert too, in his consistency program to secure infinitistic mathematics was infected in this way and, in the words of Paul Bernays, "it became his goal to do battle with Kronecker with his own weapon of finiteness" by way of inoculation.

Less radical at first, Weyl accepted the set N of natural numbers as a completed totality, but not the real numbers, or what comes to the same thing foundationally, the totality of all subsets of N. Thus was born the concept of predicativity given the natural numbers, and the investigation of its reach in practice. That there is a fundamental difference between our understanding of the concept of natural numbers and our understanding of the set concept, even for sets of natural numbers, is undeniable. The study of predicativity given the

[15] Preparation of a full presentation of the system **FA** and the proof of the theorem about it is in progress.

natural numbers is thus of special foundational significance, and is the one that has received the most detailed investigation, as described in the sections above. This is *not* to say that *only* what is predicative in that sense is justified. What we are dealing with here are *questions of relative conceptual clarity and foundational status*. Parallel to the development of predicative mathematics was the pursuit of finitist mathematics in the hands, to begin with, of Skolem (1923) and, much later, Goodstein (1957). Both were overshadowed in the foundational wars by the intuitionists, the most radical of the radicals.

With the rise in the 1950s of metamathematics as a substantial discipline and broadly applicable tool, these directions became the subject of logical analysis at the hands of philosophically motivated but not necessarily ideologically committed logicians such as Kleene and Kreisel. Kleene was concerned, in effect, with what *definition processes* (recursive, hyperarithmetic, etc.) were implicit in accepting given notions. Kreisel shifted the attention to what *proof processes* were implicit in these. The notion of unfolding of schematic formal systems evolved from his 1970 paper "Principles of proofs and ordinals implicit in given concepts." The aim was to put that in more general form by not assuming the notion of ordinal as part of the basic description. The first results described above are gratifying in this respect: the full unfolding of the system **NFA** of Non-Finitist Arithmetic is equivalent to the characterization in terms of the autonomous progression of ramified systems, of predicativity given the natural numbers, while that of the system **FA** of Finitist Arithmetic is equivalent to primitive recursive arithmetic.

Are those the only two senses of predicativity to be considered? In one direction, stronger notions of predicativity than that, given the natural numbers, were suggested by work in the late 1950s of Paul Lorenzen, John Myhill and Hao Wang. In particular, the paper of Lorenzen and Myhill (1959) involves the acceptance in a certain constructive form of the countable ordinals and of inductive proof and definition on them. In metamathematical terms, their principles would legitimate a system of strength at least that of $\mathbf{ID}_1$, the theory of one generalized inductive definition. In quite the opposite direction, Edward Nelson published a monograph, *Predicative Arithmetic* (1986), which perhaps more properly should have been entitled *Strictly Finitist Arithmetic*, whose admissible principles place it within the systems of "feasible" or "polytime" arithmetic that have been developed by Sam Buss and others, systems that form a very weak fragment of primitive recursive arithmetic (cf. Buss 1986). And somewhere between these are two papers of Geoffrey Hellman and myself, "Predicative foundations of arithmetic" (1995) and "Challenges to predicative foundations of arithmetic" (2000), where we argue in particular for a concept of predicativity given the notion of finite set, and in general for predicativity as a *relative* rather than an *absolute* concept. In addition, concepts of predicativity given the notion of finitist ordinal and even of predicativity given the notion of the cumulative hierarchy of sets have been indicated above. These open up a series of more or less specific problems to be tackled in terms of the concept of unfolding. To begin with, these problems call for the formulation of basic schematic systems as simple and natural as **NFA** and **FA** for the weaker or stronger notions indicated in the various mentioned developments, and then the determination of the reach of the corresponding systems of unfolding.

So if one accepts this more general standpoint, one answer to the question "What is predicativity?" is that it is a concept applicable to different foundational stances given by the rejection of the actual infinite for various domains, coupled with its possible limited acceptance for others. The logical problem in each case is to characterize exactly the limits of that particular stance. The potential value for philosophy then is to be able to say in sharper terms what arguments may be mounted for or against taking such a stance.

I wish to conclude with a brief indication—to the extent that it is relevant—of my own point of view philosophically, which has both negative and positive aspects.[16] On the negative side, I am a confirmed anti-platonist. On the positive side, I am a realist insofar as the natural numbers are concerned, i.e., I believe that statements about the natural number structure have a determinate truth value independent of human proofs and constructions. What more? It might be thought that since I have spent so many years working on predicativity that I consider it the be-all and end-all in non-platonistic foundations. Rather, as I wrote in the Preface to *In the Light of Logic*, p. ix, I believe that

> [predicativity given the natural numbers] should be looked upon as the philosophy of how we get off the ground and sustain flight mathematically without assuming more than the basic structure of natural numbers to begin with. There are less clear-cut conceptions which can lead us higher into the mathematical stratosphere, for example that of various kinds of sets generated by infinitary closure conditions. That such conceptions are less clear-cut than the natural number system is no reason not to use them, but one should look to see where it is necessary to use them and what we can say about what it is we know when we do use them.

[16] See also Feferman (1998), pp. 123-124.

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