On the strength of some semi-constructive theories

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ABSTRACT. Most axiomatizations of set theory that have been treated metamathematically have been based either entirely on classical logic or entirely on intuitionistic logic. But a natural conception of the set-theoretic universe is as an indefinite (or "potential") totality, to which intuitionistic logic is more appropriately applied, while each set is taken to be a definite (or "completed") totality, for which classical logic is appropriate; so on that view, set theory should be axiomatized on some correspondingly mixed basis. Similarly, in the case of predicative analysis, the natural numbers are considered to form a definite totality, while the universe of sets (or functions) of natural numbers are viewed as an indefinite totality, so that, again, a mixed semi-constructive logic should be the appropriate one to treat the two together. Various such semi-constructive systems of analysis and set theory are formulated here and their proof-theoretic strength is characterized. Interestingly, though the logic is weakened, one can in compensation strengthen certain principles in a way that could be advantageous for mathematical applications.[1]

1 Introduction

There are various foundational frameworks in which the full universe or domain of its objects is considered to be indefinite but for which certain predicates and logical operations on restricted parts of the universe are considered to be definite. For example, on one view in the case of set theory, each set is considered to be a definite (or "completed") totality, so that the membership relation and bounded quantification are definite, while the universe of sets at large is an indefinite (or "potential") totality. The idea carries over to frameworks with more than one universe, some of which may be regarded as definite while others are indefinite. For example, in the case of predicativity, the natural numbers are considered to form a definite totality, while the universe of sets (or functions) of natural numbers forms an indefinite totality. Thus quantification over the natural numbers is taken to be definite, but not quantification applied to variables for sets or functions of natural numbers. Most axiomatizations of set theory that have been treated metamathematically have been based either entirely on classical logic or entirely on intuitionistic logic, while almost all axiomatizations of predicative systems have been in classical logic. But it has been suggested on philosophical grounds that it is more appropriate to restrict

[1]For my friend and colleague Grisha Mints, on the occasion of his 70th birthday, June 7, 2009, with special appreciation for his steadfast support of logic at Stanford.

the application of classical logic to definite predicates and quantifiers and to take the basic logic otherwise to be intuitionistic. We shall show here, for various examples — including the ones that have been mentioned — that while this may provide a more philosophically satisfying formal model of the given foundational framework — there is no difference in terms of proof-theoretical strength from the associated system based on full classical logic; in that respect they are equally justified. On the other hand, the semi-constructive systems in general have a further advantage that they admit more formally powerful principles — such as the unrestricted axiom of choice — without increase of strength, and this can be advantageous when considering what mathematics can be accounted for in the given systems.

The initial stimulus for my work here was the paper of Coquand and Palmgren (2000) in which they give a constructive sheaf model for a theory of finite types over the natural numbers together with a domain of countable tree ordinals, formulated in intuitionistic logic plus the so-called *numerical omniscience scheme* for ϕ an arbitrary formula:

$$\text{(NOS)} \quad \forall n[\phi(n) \vee \neg\phi(n)] \rightarrow \forall n\phi(n) \vee \exists n\neg\phi(n)$$

A special case of this non-constructive principle is what Errett Bishop (1967) called the *limited principle of omniscience*:

$$\text{(LPO)} \quad \forall n f(n) = 0 \vee \exists n f(n) \neq 0$$

Bishop pointed out that all the results in classical analysis for which he found constructive substitutes follow from those substitutes plus LPO. In Feferman (2001) I reported determination of a bound on the proof-theoretical strength of the Coquand-Palmgren system by means of an extension of Gödel's *Dialectica* (functional) interpretation interpretation using non-constructive operators. That method goes back to Feferman (1971), with further applications in Feferman (1977), (1979); semi-constructive systems there played an essential intermediate role in the use of the method to determine the proof-theoretical strength of various classical systems. Here, by contrast, they are at the center of our attention.

What's needed in the following about Gödel's (D-)interpretation is reviewed in section 2; the reader is referred to the exposition of the basic interpretation and various of its extensions in Avigad and Feferman (1998) for more details. In section 3, we take up the particular extension by means of the non-constructive minimum operator, where it is also shown by a simple argument that (NOS) follows from (LPO) under the assumption of the Axiom of Choice (AC), as expressed in our finite type systems and verified by the D-intepretation. This is followed in section 4 by determination of the strength of some semi-constructive theories of finite type over the natural numbers, and then in section 5 the same augmented by the type of countable tree ordinals. Both sections 4 and 5 make direct use of previously established results. The main new results are in section 6, which deals with the strength of some semi-constructive systems of admissible set theory with strong choice principles plus a generalization of NOS called the Bounded Omniscience Scheme (BOS) and where the method of non-constructive D-interpretation is adapted in a new way. I conclude in section

7 with comparison with work on a various related systems and some open questions, in particular with strong systems of semi-intuitionistic set theory due to Poszgay (1971, 1972), Tharp (1971), Friedman (1973) and Wolf (1974) and, by contrast, a system conservative over PA, due to Friedman (1980).

The reader familiar with the material of Avigad and Feferman (1998) through its section 8 is encouraged to skip directly to the new results here in section 6, after taking note of Theorem 2 in section 3.2 below.

2 Review of Gödel's *Dialectica* interpretation

The main basic notions, notation and results concerning this interpretation are recalled here from Avigad and Feferman (1998). For each formula ϕ, the *double-negation* (or *negative*) *translation* of ϕ is denoted by ϕ^N, and the *Dialectica* (or *D-*) *interpretation* of ϕ is denoted by ϕ^D. The N-translation in general takes one from classical theories to formally intuitionistic theories, and the D-interpretation is then applied to take one to a quantifier free theory of functionals of finite type. The basic example is given by the N-translation of Peano Arithmetic PA into Heyting Arithmetic HA (Gödel, Gentzen) followed by the D-interpretation of HA into a quantifier-free theory T of primitive recursive functionals of finite type (Gödel). The latter extends directly to a D-interpretation of a finite type extension HA^ω of HA into T, and then by making a similar extension PA^ω of PA, we obtain a composite ND-interpretation of that system into T. Finally, T has a model in HEO, the Hereditarily Extensional (Recursive) Operations of finite type, and that can be formalized in PA. Thus all these systems are of the same proof-theoretical strength.

For the finite type theories involved, the finite type symbols (t.s.) $\sigma, \tau, \ldots$ are generated as follows: (1) 0 is a t.s. and (ii) if σ, τ are t.s., then so also is $\sigma \to \tau$. These theories have infinitely many variables $x^\tau, y^\tau, z^\tau, \ldots$ of each type τ; type superscripts are suppressed when there is no ambiguity. We occasionally use other kinds of letters like $f, g, \ldots n, m, \ldots$ as well as capital letters $X, Y, \ldots$ for variables of appropriate types. Terms $s, t, \ldots$ are generated from the variables and constants (to be described) by closure under application ts (or $t(s)$) when t is of a type $\sigma \to \tau$ and s is of type σ, the result being of type τ.

Application in terms such as rst is read by association to the left, while the t.s. $\rho \to \sigma \to \tau$ is read by association to the right. Each higher type symbol σ can be written in the form $\sigma = (\sigma_1 \to \ldots \to \sigma_k \to 0)$; then equality $s = t$ at type σ is informally regarded as an abbreviation for $sx_1 \ldots x_k = tx_1 \ldots x_k$ where the x_i are fresh variables of type σ_i for $i = 1, \ldots, k$. The formal axioms and rules that we take to govern equality at higher types are given by the so-called weakly extensional approach due to Spector and described in Avigad and Feferman (1998), p. 350.

The type $0 \to 0$ is also denoted 1, and the type $1 \to 0$ is denoted 2. The constant symbols include 0 of type 0 and Sc of type 1. In addition we have symbols K, S for the usual combinators in all appropriate types, satisfying equations of the form $Kst = s$ and $Srst = rt(st)$; the typed λ-calculus is then introduced by definition as usual. Finally we have constant symbols R for the recursors in all appropriate types, satisfying equations of the form

$$Rxy0 = x \text{ and } Rxyn' = yn(Rxyn)$$

where n is a variable of type 0 and $n' = \mathrm{Sc}(n)$. The formulas $\phi, \psi, \ldots$ of T are generated from the equations $s = t$ and the falsity $\bot$ by closure under $\phi \wedge \psi$, $\phi \vee \psi$, and $\phi \rightarrow \psi$; then $\neg\phi$ is defined as $\phi \rightarrow \bot$. The formulas of PA^ω and HA^ω are in addition closed under universal and existential quantification, $\forall x\phi$ and $\exists x\phi$, w.r.t. variables x of all finite types. The underlying logic of PA^ω is classical while that of HA^ω and T is intuitionistic. The axioms in all three systems are as usual for 0 and Sc, and as indicated above for the symbols K, S and R. The induction axiom scheme is given as usual in these systems, while it is formulated as a rule in T. One may show that quantifier-free (QF) formulas (with all equations restricted to terms of type 0) are decided, i.e. satisfy the law of excluded middle (LEM) in all of these systems.

For a formula ϕ of the quantified finite type language, the D-interpretation of ϕ is of the form

$$\phi^D = \exists x \forall y \phi_D(x, y)$$

where x, y are sequences (possibly empty) of finite type variables and ϕ_D is a QF formula whose free variables are those of ϕ in addition to those of x and y. The inductive definition of the D-interpretation for formulas ϕ, ψ with ϕ^D as above, and $\psi^D = \exists u \forall v \psi_D(u, v)$ is as follows:

(i) For ϕ an atomic formula, x and y are both empty and $\phi^D = \phi_D = \phi$.

(ii) $(\phi \wedge \psi)^D = \exists x, u \forall y, v[\phi_D(x, y) \wedge \psi_D(u, v)]$

(iii) $(\phi \vee \psi)^D = \exists z, x, u \forall y, v[(z = 0 \wedge \phi_D(x, y)) \vee (z = 1 \wedge \psi_D(u, v))]$

(iv) $(\forall z \phi(z))^D = \exists X \forall y, z \phi_D(Xz, y, z)$

(v) $(\exists z \phi(z))^D = \exists x, z \forall y \phi_D(x, y, z)$

(vi) $(\phi \rightarrow \psi)^D = \exists U, Y \forall x, v[\phi_D(x, Yxv), \rightarrow \psi_D(Ux, v)]$

With $\neg\phi$ defined as $\phi \rightarrow \bot$, we have

(vii) $(\neg\phi)^D = \exists Y \forall x \neg\phi_D(x, Yx)$

The reasoning behind (iv) lies in the constructive acceptance of the Axiom of Choice, here taken as the following scheme in all finite types:

$$\text{(AC)} \qquad \forall x \exists y \phi(x, y) \rightarrow \exists Y \forall x \phi(x, Yx)$$

The reasoning behind (vi) lies in a chain of steps[2], an intermediate one of which lies in transforming $[\exists x \forall y \phi_D(x, y) \rightarrow \exists u \forall v \psi_D(u, v)]$ into

(vi)* $\forall x \exists u \forall v \exists y[\phi_D(x, y) \rightarrow \psi_D(u, v)]$

Implicitly, that makes use of a principle called Independence of Premises (IP) that is not intuitionistically justified. Moreover, another one of the steps to (vi) implicitly makes use of the finite type forms of Markov's Principle,

$$\text{(MP)} \qquad \forall x(\neg\neg\exists y \phi \rightarrow \exists y \phi) \text{ for QF } \phi$$

[2]See Avigad and Feferman (1998) pp. 346-347 for all the steps involved.

which is also not justified intuitionistically. Nevertheless, Gödel's interpretation gives a constructive reduction of both AC and MP.[3] In the case of AC this is immediate, and in the case of MP, it is quite easy given that all QF formulas in the systems we're dealing with are decided. The additional power of the D-interpretation below comes from the fact that $(\mathrm{AC})^{\mathrm{D}}$ and $(\mathrm{MP})^{\mathrm{D}}$ are verified quite generally.

The following is the direct extension from HA to its finite type version of Gödel's main result (1958):

Theorem 1. If HA^{ω} + AC + MP proves ϕ and $\phi^D = \exists x \forall y \phi_D(x, y)$ then for some sequence of terms t of the same type as the sequence x of variables, T proves $\phi_D(t, y)$.

The main use of the recursor constants R is in verifying the D-interpretation of the induction axiom scheme in HA^{ω}.

By QF-AC we mean the scheme AC restricted to QF formulas ϕ. Since for such ϕ we have ϕ^N equivalent to ϕ, we see that $(\text{QF-AC})^{\mathrm{ND}}$ is also verified in T.

Corollary 1. PA^{ω} + QF-AC is N interpreted in HA^{ω} and so it is ND interpreted in T.

By an analysis of the reduction of the terms of T to normal form, one sees that its closed terms denote recursive functions defined by ordinal recursions on proper initial segments of the natural well-ordering of order type ϵ_0. Hence all the systems PA, PA^{ω}, HA, HA^{ω} and T have this same class of provably recursive functions.

3 The non-constructive minimum operator and interpretation of NOS

3.1 The non-constructive minimum operator

One way to arrange for arithmetical formulas to satisfy the Law of Excluded Middle, LEM, in a system based as a whole on intuitionistic logic is to make them equivalent to QF formulas by adjunction of a numerical quantification operator E_0 of type 2 satisfying the axiom

$$(E_0) \qquad E_0 f = 0 \leftrightarrow \exists x (fx = 0)$$

for f, x variables of type 1 and 0, resp. In order for this to satisfy the ND-interpretation we need to verify the following two implications:

$$fx = 0 \to E_0 f = 0 \text{ and } E_0 f = 0 \to \exists x (fx = 0)$$

The first of these is automatically taken care of, and the N-interpretation of the second is taken care of by the verification of MP, but in order to get its further D-interpretation we need to have a functional X which satisfies $E_0 f = 0 \to f(Xf) = 0$, and hence $fx = 0 \to f(Xf) = 0$. To take care of this we adjoin a new constant symbol μ with axiom:

$$(\mu) \qquad fx = 0 \to f(\mu f) = 0$$

[3] It also verifies the interpretation of IP, but we shall not make use of that fact.

We call μ the *non-constructive minimum operator*, though properly speaking that would need an additional axiom specifying that μf is the least x such that $fx = 0$ if $\exists x(fx = 0)$ and (say) is 0 otherwise; in fact, that is definable from μ using the primitive recursive bounded minimum operator.

3.2 The LPO axiom and the Numerical Omniscience Scheme

As stated, under the axiom (μ) every arithmetical formula is equivalent to a QF formula in intuitionistic logic; various consequences of this for semi-constructive systems incorporating that axiom will be dealt with in the next section. In particular, we can derive the NOS scheme for arithmetical formulas from that assumption. But in the presence of AC we can do even more. We here understand by the NOS, the scheme described in sec. 1 where we allow ϕ to be any formula of the language of HA^ω, and by LPO the statement given in sec. 1, where 'f' is a variable of type 1.

Theorem 2.

(i) HA^ω + (μ) proves LPO.

(ii) HA^ω + (LPO) + AC proves NOS.

Proof. (i) is immediate, and for (ii) we note that if $\forall n[\phi(n) \vee \neg\phi(n)]$ holds, then so also does $\forall n \exists k[k = 0 \wedge \phi(n) \vee k = 1 \wedge \neg\phi(n)]$. Hence by AC there exists f of type 1 such that $\forall n[(f(n) = 0 \leftrightarrow \phi(n)) \wedge (f(n) = 1 \leftrightarrow \neg\phi(n))]$. □

4 Semi-constructive systems of finite type over the natural numbers

4.1 Primitive recursion in a type 2 functional and Kleene's variant

The system HA^ω + (μ) + (AC) offers itself immediately for consideration as a semi-constructive system of interest; this is a predicative system that is somewhat stronger than PA. But we shall also consider systems using operators F of type 2 stronger than μ. Given any such F, Shoenfield defined a hierarchy H^F_α of functions for α less than the first ordinal not recursive in F, such that the 1-section of F (i.e., the totality of type 1 functions recursive in F) consists of all those functions that are primitive recursive in the usual sense in some such H^F_α. Now the normalization of terms of the system T augmented by such an F shows that its 1-section consists of all those functions primitive recursive in some H^F_α for $\alpha < \epsilon_0$. In particular, the 1-section of the functionals defined by closed terms of T augmented by μ consists of the functions in the HYP hierarchy up to (but not including) ϵ_0.

We shall also consider an interesting subsystem of $\hat{\mathrm{T}}$ augmented by such type 2 functionals F, obtained by restricting the induction and recursion principles. The motivation for that restriction lies in the fact that the recursors R with values of higher type have a kind of impredicative character. For example, for values of $Rfgn$ of type 2, thought of as $\lambda h.Rfgnh$, we have $Rfgn' = gn(\lambda h.Rfgnh))$ and *that* evaluated at a given function h_1 makes prima-facie reference to *the* values of $Rfgn$ at all functions h. It is easily shown that non-primitive recursive functions such as the Ackermann function may be generated in this way. Kleene (1959) introduced restricted recursors $\hat{R}$ satisfying the recursion equations

$$\hat{R}xy0z \text{ and } \hat{R}xyn'z = y(\hat{R}xynz)$$

where z is a sequence of variables such that xz is of type 0. He showed that the 1-section of the functionals generated from 0, Sc, the K, S combinators and the $\hat{R}$ recursors by closure under application are exactly the primitive recursive functions. Thus taking $\hat{\mathrm{T}}$ to be the subsystem of T with constants $\hat{R}$ in place of the constants R, and corresponding change of axioms, we have that the 1-section of $\hat{\mathrm{T}}$ consists exactly of the primitive recursive functions in the usual sense. Now all this may be relativized to a type 2 functional F to show that the 1-section of $\hat{\mathrm{T}}$ augmented by F consists exactly of the functions primitive recursive in H_n^F for some $n < \omega$. In particular, the 1-section of $\hat{\mathrm{T}} + (\mu)$ consists of all the arithmetically definable functions.

By Res-PA$^\omega$ and Res-HA$^\omega$ we mean the systems using the $\hat{R}$ recursors in place of the R recursors and with the axiom of induction restricted to QF-formulas.

4.2 The strength of some semi-constructive systems based on the non-constructive minimum operator

We begin with semi-constructive variants of predicative systems, i.e. systems whose strength is at most that of ramified analysis up to the Feferman-Schütte ordinal Γ_0, or equivalently, the union of the (Π^0_1-CA$_\alpha$) systems for $\alpha < \Gamma_0$.

Theorem 3.

(i) The systems Res-HA$^\omega$ + (AC) + (MP) + (μ) and Res-PA$^\omega$ + (QF-AC) + (μ) are proof-theoretically equivalent to and conservative extensions of PA; furthermore they are conservative extensions of the 2nd order system ACA$_0$ for Π^1_2-sentences.

(ii) The systems HA$^\omega$ + (AC) + (MP) + (μ) and PA$^\omega$ + (QF-AC) + (μ) are proof-theoretically equivalent to — and conservative extensions for Π^1_2-sentences of — the 2nd-order systems (in decreasing order) Σ^1_1-DC, Σ^1_1-AC, and the union of the (Π^0_1-CA$_\alpha$) systems for $\alpha < \epsilon_0$.

(iii) The systems HA$^\omega$ + (AC) + (MP) + (μ) + (Bar-Rule) and PA$^\omega$ + (QF-AC) + (μ) + (Bar-Rule) are proof-theoretically equivalent to — and conservative extensions for Π^1_2 sentences of — the 2nd order systems (in decreasing order) Σ^1_1-DC + (Bar-Rule), Σ^1_1-AC + (Bar-Rule), and the union of the (Π^0_1-CA$_\alpha$) systems for $\alpha < \Gamma_0$.

(iv) There is no increase in strength when the NOS scheme is added to the semi-constructive systems in (i)-(iii).

Proofs. The result (i) is from Feferman (1977), (ii) is from Feferman (1971) and (iii) is from Feferman (1979). The ideas for their proofs are exposited in Avigad and Feferman (1998), sec. 8. Briefly, the proof of (i) uses the fact that the D-interpretation of the semi-constructive system Res-HA$^\omega$ + (AC) + (μ)

and the ND-interpretation of the classical system Res-PA$^\omega$ + (QF-AC) + (μ) both take us into $\hat{\mathrm{T}}$ + μ, which is interpreted in PA preserving arithmetical sentences (as translated using μ). For the conservation statement, one notes that under the μ axiom, every Π^1_2 sentence is equivalent to one of the form $\forall f \exists g \phi(f, g)$, where ϕ is quantifier-free, hence if provable, it is preserved under the N-translation using (MP) and then under the D-interpretation one obtains a type 2 term t such that $\hat{\mathrm{T}}$ + (μ) proves $\phi(f, tf)$. That term defines $g = tf$ arithmetically from f. The main steps of the proof of (ii) follow the same lines, concluding with the interpretation in T + (μ), whose 1-section consists of the functions in the HYP hierarchy up to (but not including) ϵ_0, as described in 4.1 above. For (iii) the main new work goes first into the D-interpretation of HA$^\omega$ + (AC) + (MP) + (μ) + (Bar-Rule) in the extension of T + (μ) by two new rules, (BR) and (TR). These rules involve expressing in QF form, well-foundedness of any specific segment $\preceq_a$ of a given arithmetical well-ordering as the open formula $\forall x[\forall y(y \prec_a x \to y \in X) \to x \in X]$, denoted $I(\preceq_a, X)$, where X is a set-variable (i.e., a characteristic function at type 1). Then, for the natural well-ordering $\prec$ of order type Γ_0, the version BR of the Bar-Rule used in this context allows one to pass from $I(\preceq_a, X)$ for any specific a to the result $I(\preceq_a, \phi)$ of substituting in it any formula $\phi(x)$ of the system for the formula $x \in X$, while the rule (TR) allows one to introduce a transfinite recursor on the given segment under the same hypothesis. One gets up to each ordinal less than Γ_0 by a boot-strapping argument, and the proof that one doesn't go beyond is via a normalization argument. See Feferman (1979) pp. 87-89 for more details. Finally, (iv) is immediate by Theorem 2.

4.3 The strength of some semi-constructive systems based on μ plus the Suslin-Kleene operator

For a given f, let Tree(f) be the tree consisting of all finite sequence numbers s such that $f(s) = 0$. This tree is not well-founded if and only if $\exists g \forall x f(g \mid x) = 0$, where for any type 1 function g, $g \mid x$ is the number s of the finite sequence $g_0, \ldots, g(x-1)$. The Suslin-Kleene operator is the associated type 2 choice functional μ_1, obtained by taking the left-most descending branch in Tree(f) if that tree is not well-founded. It satisfies the axiom

$$(\mu_1) \quad \forall x f(g \mid x) = 0 \to \forall x f((\mu_1 f) \mid x) = 0$$

which may be re-expressed in QF form using the μ operator. From the work of Feferman (1977) and Feferman and Jäger (1983) one then obtains characterizations of the proof-theoretical strength of the semi-constructive systems HA$^\omega$ + (AC) + (MP) + (μ) + (μ_1), its restricted version, and its extension under the Bar-Rule, in a form analogous to Theorem 3. For example, in analogy to part (ii) of that theorem, the strength of HA$^\omega$ + (AC) + (MP) + (μ) + (μ_1) is characterized as that of the iterated Π^1_1-CA systems up to ϵ_0, which is the same as that of (Σ^1_2-DC). See Avigad and Feferman (1998) pp. 384-385 for full statement of results and indication of proofs. An alternative characterization may be given in terms of the iterated ID systems up to ϵ_0. And, finally, addition of the NOS comes for free by Theorem 2.

5 The strength of a semi-constructive theory of finite type over the natural numbers and countable tree ordinals.

Here we can draw directly on Avigad and Feferman (1998), sec. 9, which reports the work of an unpublished MS, Feferman (1968). The type structure is expanded by an additional ground type for abstract constructive countable tree ordinals, denoted Ω, and lower case Greek letters $\alpha, \beta, \gamma, \ldots$ are used to range over Ω. But now we use 'N' to denote the type symbol 0. The constants are augmented by 0_Ω of type Ω, Sup of type $(N \to \Omega) \to \Omega$, Sup^{-1} of type $\Omega \to (\Omega \to N)$, and for each σ, $R_{\Omega\sigma}$ of type $(\Omega \to (N \to \sigma) \to \sigma) \to \sigma \to \Omega \to \sigma$. The subscript '$\sigma$' is omitted from the ordinal recursor $R_{\Omega\sigma}$ when there is no ambiguity. The constant 0_Ω represents the one-point tree, and for f of type $(N \to \Omega)$, $\mathrm{Sup}(f)$ represents the tree obtained by joining together the subgrees fn for each natural number n. For $\alpha = \mathrm{Sup}(f)$, $\mathrm{Sup}^{-1}(\alpha) = f$ is the constructor of α; in that case we write α_n for $(\mathrm{Sup}^{-1}(\alpha))n$. For each type σ the ordinal recursor R_Ω works to take an element a of type σ, a functional f of type $(\Omega \to (N \to \sigma) \to \sigma)$, and a tree ordinal α to an element $R_\Omega f a \alpha$ satisfying the recursion equations

$$(R_\Omega) \quad R_\Omega f a 0_\Omega = a, \text{ and for } \alpha \neq 0_\Omega, R_\Omega f a \alpha = f\alpha(\lambda n.R_\Omega f a \alpha_n)$$

We also take the language to include the constant μ. In it, we form three theories of countable tree ordinals of finite type, first a classical theory $\mathrm{CO}^\omega_\Omega + (\mu)$, then a semi-intuitionistic theory $\mathrm{SO}^\omega_\Omega + (\mu)$, both with full quantification at all finite types, and finally a quantifier free theory T_Ω.[4] The basic axioms of $\mathrm{CO}^\omega_\Omega + (\mu)$ and $\mathrm{SO}^\omega_\Omega + (\mu)$ are the same, consisting of the following:

(1) The axioms of $\mathrm{HA}^\omega + (\mu)$, with the induction scheme extended to all formulas of the language;

(2) $\mathrm{Sup}(f) \neq 0_\Omega$ and $\mathrm{Sup}^{-1}(\mathrm{Sup}(f)) = f$, for f of type $N \to \Omega$

(3) $\mathrm{Sup}(\mathrm{Sup}^{-1}(\alpha)) = \alpha$ for $\alpha \neq 0_\Omega$

(4) $(0_\Omega)_x = 0_\Omega$

(5) the (R_Ω) equations

(6) $\phi(0_\Omega) \wedge \forall\alpha[\alpha \neq 0_\Omega \wedge \forall x \phi(\alpha_x) \to \phi(\alpha)] \to \forall\alpha\phi(\alpha)$ for each formula $\phi(\alpha)$

The theory $\mathrm{T}_\Omega + (\mu)$ has as axioms:

$(1)^*$ The axioms of $\mathrm{T} + (\mu)$

$(2)^*$-$(5)^*$ The same as (2)-(5)

$(6)^*$ The rule of induction on ordinals for QF formulas ϕ

[4]In Avigad and Feferman (1998), p. 387, we wrote OR^ω_1 for the system $\mathrm{CO}^\omega_\Omega$ + (QF-AC).

Note that this last is to be expressed in quantifier free form using the μ operator. In the next statement, ID_1 and $\mathrm{ID}_1^{(i)}$ are respectively the classical and intuitionistic theory of non-iterated positive inductive definitions given by arithmetical $\phi(x, P^+)$.

Theorem 4. The following theories are all of the same proof-theoretical strength:

(i) ID_1

(ii) $\mathrm{CO}^\omega_\Omega$ + (μ) + (QF-AC)

(iii) $\mathrm{SO}^\omega_\Omega$ + (μ) + (AC) + (NOS)

(iv) T_Ω + (μ)

(v) $\mathrm{ID}_1^{(i)}$

Proof. It is shown in Avigad and Feferman (1998) pp. 388-389 how to translate ID_1 into $\mathrm{CO}^\omega_\Omega$ + (μ). That system is then carried into $\mathrm{SO}^\omega_\Omega$ + (μ) by the N-translation. By a direct extension of the work described in secs. 2-4 above, we see that $\mathrm{SO}^\omega_\Omega$ + (μ) + (AC) + (NOS) is D-interpreted in T_Ω + (μ); this also verifies the classical (QF-AC) under the ND-interpretation. Next, as in op. cit. pp. 390-391, T_Ω + (μ) has a model in HRO(^{2}E), the indices of operations hereditarily recursive in ^2E in the sense of Kleene (1959), interpreting the type Ω objects as the members of a version O of the Church-Kleene ordinal notations. That model can be formalized in ID_1 so as to reduce T_Ω + (μ) to ID_1. Finally, the reduction of ID_1 to $\mathrm{ID}_1^{(i)}$ is due to Buchholz (1980), in fact to the theory of an accessibility inductive definition.[5]

The language of the theory W of Coquand and Palmgren (2000) is close to that of $\mathrm{SO}^\omega_\Omega$, but does not contain the Sup^{-1} operator or the (μ) operator. Its axioms are essentially the same as those of $\mathrm{SO}^\omega_\Omega$ without the axioms for those two operators. In addition, it has three special choice axiom schemata, unique choice (AC!), countable choice (AC_0) and dependent choice (DC) — all of which follow from (AC) — as well as the Numerical Omniscience Scheme (NOS). Thus W is a subtheory of $\mathrm{SO}^\omega_\Omega$ + (AC) + (NOS), and so the proof-theoretic strength of W is no greater than that of $\mathrm{ID}_1^{(i)}$. Presumably, the latter (at least for accessibility inductive definitions) can be interpreted in W, but I have not checked that. The main part of Coquand and Palmgren (2000) is devoted to producing a constructive sheaf-theoretic model of W in Martin-Löf type theory with generalized inductive definitions; an obvious question is whether their argument provides an alternative reduction of W to $\mathrm{ID}_1^{(i)}$. Finally, as noted in Theorem 2, NOS already follows in their system from LPO from countable choice.

6 Semi-constructive systems of set theory.

The basic idea for semi-constructive systems of set theory was stated in the introduction: each set is considered to be a definite totality, so that the membership relation and bounded quantification are definite, i.e. classical logic apply

[5] Avigad and Towsner (2009) have obtained an interesting alternative proof of the reduction of ID_1 to an accessibility $\mathrm{ID}_1^{(i)}$, using a variant of the functional interpretation method.

to both, while the universe as a whole is considered to be indefinite, so that only intuitionistic logic applies to that. This suggests considering axiomatic systems of set theory based on intuitionistic logic for which it is assumed that classical logic applies to all Δ_0 formulas. The latter is accomplished by assuming the following restricted scheme for the Law of Excluded Middle,

$$(\Delta_0\text{-LEM}) \quad \phi \vee \neg\phi, \text{ for all } \Delta_0 \text{ formulas } \phi$$

In this context, we also take Markov's principle in the form:

$$(\text{MP}) \quad \neg\neg\exists x\phi \rightarrow \exists x\phi, \text{ for all } \Delta_0 \text{ formulas } \phi$$

Let IKPω be the system KP with logic restricted to be intuitionistic. To be more precise, IKPω takes the following as its non-logical axioms:

1. Extensionality
2. Unordered pair
3. Union
4. Infinity, in the specific form that there is a smallest set containing the empty set 0 and closed under the successor operation, $x' = x \cup \{x\}$.
5. Δ_0-Separation
6. Δ_0-Collection
7. The $\in$-Induction Rule

By 7, we mean the rule which allows us to infer $\forall x\psi(x)$ from $\forall x[(\forall y \in x)\psi(y) \rightarrow \psi(x)]$ for any formula $\psi(x)$. This is easily seen to imply the $\in$-Induction Scheme

$$\forall x[(\forall y \in x)\phi(y) \rightarrow \phi(x)] \rightarrow \forall x\phi(x)$$

by taking $\psi(x) = \{\forall z[(\forall y \in z)\phi(y) \rightarrow \phi(z)] \rightarrow \phi(x)\}$.

Some further schemata in the language of set theory shall be added to IKPω, first of all the Bounded Omniscience Scheme:

$$(\text{BOS}) \quad \forall x \in a[\phi(x) \vee \neg\phi(x)] \rightarrow \forall x \in a(\phi(x)) \vee \exists x \in a(\neg\phi(x))$$

for *all* formulas $\phi(x)$. The set-theoretical form of NOS is the special case of this in which $a = \omega$ the unique set specified by Axiom 4. We shall strengthen IKPω by (Δ_0-LEM) and BOS; but we can make a further considerable strengthening by adding the following form of the Axiom of Choice,

$$(\text{AC}_{\text{Set}}) \quad \forall x \in a\exists y\phi(x,y) \rightarrow \exists r[\text{Fun}(r) \wedge \text{Dom}(r) = a \wedge (\forall x \in a)\phi(x, r(x))]$$

for *all* ϕ, where Fun(r) expresses in usual set theoretic form that the binary relation r is a function, and Dom$(r) = a$ expresses that a is the domain of r; both of these may be given as Δ_0 formulas. Note that in the presence of (AC_{Set})

with axioms 1-3 and 5 we can infer Full Collection and Full Replacement i.e. these schemes for arbitrary formulas.

Theorem 5. The semi-constructive theory of sets, SCS = IKPω + (Δ_0-LEM) + (MP) + (BOS) + ($\mathrm{AC}_{\mathrm{Set}}$), is of the same strength as KPω and thence of $\mathrm{ID}_1^{(i)}$. The same holds for a natural finite type extension SCS^ω over the universe of sets.

Proof. The proof is in three parts.

I. First, we show that KPω is interpretable in SCS via the N-translation. Since ϕ^N is provably equivalent to ϕ for every Δ_0 formula f in IKPω + (Δ_0-LEM), one readily checks that the N-translation of each of the axioms 1-5 holds in that subsystem of SCS. In the case of Δ_0-Collection, the N-translation is of the form

$$(\forall x \in a)\neg\neg\exists y\phi(x,y) \to \neg\neg\exists b(\forall x \in a)(\exists y \in b)\phi(x,y)$$

where ϕ is a Δ_0 formula. But then by (MP) this follows from Δ_0-Collection in SCS. Finally, the N-translation of an instance of the $\in$-induction rule is an instance of the same.

II. Next we introduce a new system $\mathrm{T_V}$ and define a D-interpretation of SCS into $\mathrm{T_V}$; by following through the interpretation, one may see what natural finite type extension SCS^ω of SCS is also verified in the process. The language of $\mathrm{T_V}$ is typed, with a ground type V for sets, and function types $\sigma \to \tau$ for each types σ and τ. Variables for sets will be at the beginning or end of the alphabet, while variables for functions will generally be $f, g, h, \ldots$. Capital letters will be used for constants, except for 0 and ω; the constants are 0 (empty set), ω (natural numbers), D (disjunction operator), N (negation operator), E (characteristic function of equality), M (characteristic function of membership), C (bounded choice operator), P (unordered pair function), U (union function), S (separation operator), R^* (range operator) and R_σ (recursion operators). Terms are generated from variables and constants by closure under well-typed .application, ts. Atomic formulas are equations between terms, $s = t$, and membership of terms, $s \in t$. Formulas $\phi, \psi, \ldots$ are generated by closing the atomic formulas under the propositional operations and bounded quantification, $(\forall y \in t)\phi$ and $(\exists y \in t)\phi$. Truth values are represented in V by using 0 for True and any other value for False. The axioms of $\mathrm{T_V}$ fall into three groups (A, B and C), as follows; these also implicitly determine the types of the various constants.

A. Equality and logical operation axioms.

1. (Decidability) $x = y \vee x \neq y$
2. (Equality) $Exy = 0 \leftrightarrow x = y$
3. (Membership) $Mxy = 0 \leftrightarrow x \in y$
4. (Disjunction) $Dxy = 0 \leftrightarrow x = 0 \vee y = 0$
5. (Negation) $Nx = 0 \leftrightarrow x \neq 0$
6. (Bounded choice) $x \in a \wedge fx = 0 \to Caf \in a \wedge f(Caf) = 0$

Note by 6 that $(\exists x \in a) fx = 0 \leftrightarrow Caf \in a \wedge f(Caf) = 0$. The following is then a direct consequence of the group A axioms.

Lemma 1. For each Δ_0 formula ϕ of set theory, all of whose free variables are among the list $\underline{x} = x_1, \ldots x_n$, we have a closed term t_ϕ such that the following is provable in T_V:

$$t_\phi(\underline{x}) = 0 \leftrightarrow \phi(\underline{x})$$

For the next group of axioms we write $a \subseteq b$ for $(\forall x \in a)(x \in b)$.

B. Set theoretic axioms.

7. (Extensionality) $a \subseteq b \wedge b \subseteq a \rightarrow a = b$
8. (Empty set) $\neg(x \in 0)$
9. (Unordered pair) $x \in Pab \leftrightarrow x = a \vee x = b$
10. (Union) $x \in Ua \leftrightarrow (\exists y \in a)(x \in y)$
11. (Infinity) (i) $0 \in \omega \wedge (\forall x \in \omega)(x' \in \omega)$
    (ii) $0 \in a \wedge (\forall x \in a)(x' \in a) \rightarrow \omega \subseteq a$
12. (Separation) $x \in Saf \leftrightarrow x \in a \wedge fx = 0$
13. (Range) $y \in R^*af \leftrightarrow (\exists x \in a)(fx = y)$

As usual, for Axiom 11 in the preceding, we write $\{x, y\}$ for Pxy, $\{x\} = \{x, x\}$, $x \cup y = U\{x, y\}$, and $x' = x \cup \{x\}$. We also define $\langle x, y\rangle = \{\{x\}, \{x, y\}\}$ as usual in set theory, and use it to prove the following:

Lemma 2. There is a closed term Grph such that T_V proves

$$z \in (\text{Grph})af \leftrightarrow (\exists x \in a)(\exists y \in R^*af)[z = \langle x, y\rangle \wedge fx = y]$$

Proof. $(\text{Grph})af$ is the graph of f restricted to a, considered as a set; it is formed by separation from the Cartesian product $a \times (R^*af)$. This depends on the proof in general of the existence of Cartesian products $a \times b$, as follows. First let g be such that for each x, y, $gxy = \langle x, y\rangle$, so that gx is $\lambda y.\langle x, y\rangle$. Then for $x \in a$, $gx : b \rightarrow \{x\} \times b$ and $R^*(b, gx) = \{x\} \times b$. Finally, take $h = \lambda x.R^*(b, gx)$ so that $a \times b = U(R^*(a, h))$. □

In the following I shall write $f|a$ for $(\text{Grph})af$.

The final group of axioms is for recursion and induction. The latter is formulated as a rule in a way specifically to enable the D-interpretation of the $\in$-Induction scheme in KPω.

C. Recursion Axiom and Induction Rule.

14. (Recursion) For each type σ, R_σ is of type $(V \rightarrow V \rightarrow \sigma) \rightarrow (V \rightarrow \sigma)$. Then for f a variable of type $(V \rightarrow V \rightarrow \sigma)$ and x of type V and for $g = R_\sigma f$ we have the equation

$$gx = f(g|x)x$$

15. (Induction) Suppose that $\theta(x, g, u)$ is a formula and that G and Z are closed terms for which the following has been inferred:

$$(\forall y \in x)\theta(y, Gy, Zxu) \to \theta(x, Gx, u)$$

Then we may infer $\theta(x, Gx, u)$.

NB. In the preceding, g and u may be sequences of variables (possibly empty) of arbitrary type, while x is of type V.

This completes our description of the system $\mathrm{T_V}$.

Lemma 3. SCS is D-interpreted in T_V.

Proof. The D-interpretation of each of the axioms 1-6 of IKPω in $\mathrm{T_V}$ is straightforward. Furthermore, by the general facts about the D-interpretation established in section 2 above, we obtain without further work the D-interpretations of (Δ_0-LEM), (MP), and (AC), this last in the functional form $\forall x \exists y \phi(x, y) \to \exists f \forall x \phi(x, fx)$, where ϕ is an arbitrary formula. To obtain the D-interpretation of ($\mathrm{AC_{Set}}$) from this, suppose $(\forall x \in a)\exists y \phi(x, y)$. Then under the D-interpretation, we also have $\forall x \exists y(x \in a \to \phi(x, y))$, so there exists an f such that $(\forall x \in a)\phi(x, fx)$. Let $r = f|a$; then by Lemma 2, Fun(r) and Dom(r) $= a$ and $(\forall x \in a)\phi(x, r(x))$, as required by ($\mathrm{AC_{Set}}$). To prove the D-interpretation of BOS, we argue just as for Theorem 2 in the proof of NOS, but now combining AC with the bounded choice operator C instead of the operator μ.

So the only thing left to deal with is the D-interpretation of the $\in$-Induction Rule 7 of IKPω. For that, let $\psi(x)^D = \exists g \forall u \psi_D(x, g, u)$, where g, u are sequences of variables (possibly empty) of various types. We write θ for ψ_D. Then to form the D-interpretion of the hypothesis of the $\in$-Induction Rule, we pass through the following sequence of formulas

$$\forall x\{(\forall y \in x)\exists h \forall w \theta(y, h, w) \to \exists f \forall u \theta(x, f, u)\},$$

$$\forall x\{\exists g \forall w \forall y[y \in x \to \theta(y, gy, w)] \to \exists f \forall u \theta(x, f, u)\},$$

$$\forall g, x \exists f \forall u \exists w, y\{[y \in x \to \theta(y, gy, w)] \to \theta(x, f, u)\},$$

$$\exists f', y', w' \forall g, x, u\{[y'gxu \in x \to \theta(y'gxu, g(y'gxu), w'gxu)] \to \theta(x, f'gx, u)\}.$$

So finally, by induction hypothesis we have closed terms F, Y, W, such that the following is provable in $\mathrm{T_V}$:

$$[Ygxu \in x \to \theta(Ygxu, g(Ygxu), Wgxu)] \to \theta(x, Fgx, u).$$

Then the following is also provable in $\mathrm{T_V}$:

$$(\forall y \in x)\theta(y, gy, Wgxu) \to \theta(x, Fgx, u).$$

Now apply the Recursion Axiom of $\mathrm{T_V}$ to obtain G satisfying the equation $Gx = F(G|x)x$. Substituting $G|x$ for g throughout the preceding, and taking $Zxu = W(G|x)u$, it follows that $(\forall y \in x)\theta(y, Gy, Zxu)$ has been inferred. Hence by the Induction Rule 15 of $\mathrm{T_V}$ we may infer $\theta(x, Gx, u)$, which is the D-interpretation of $\forall x \psi(x)$.

III. To complete the proof of Theorem 5, we need to interpret $\mathrm{T_V}$ in a system of strength KPω. This is provided by the system of Operational Set Theory, OST, for a type-free applicative structure over set theory introduced in Feferman (2001a); see also Feferman (2006) and Jäger (2007). The language of OST extends the language L of set theory by a binary operation symbol A for application, a unary relation symbol $\downarrow$ for definedness and various constants. The terms $r, s, t, \ldots$ of the extended language are generated from the variables $a, b, c, \ldots, f, g, h, \ldots, x, y, z$ and constants by closing under application, $A(s, t)$. We write st for $A(s, t)$, and think of s as a partial function (coded as a set) whose value at t exists if $(st)\downarrow$ holds; this allows interpretation of a partial combinatory type-free calculus in OST. The logic of OST is the classical logic of partial terms due to Beeson.[6] The axioms of OST come in four groups:

(1) Axioms for the applicative structure given by the (partial) combinators k, s.

(2) Axioms for logical operations for negation, disjunction and bounded quantification, along with the characteristic function for membership, as in $\mathrm{T_V}$.

(3) Basic set-theoretic axioms for extensionality, empty set, unordered pair, union, infinity and the $\in$-Induction Scheme, as in KPω.

(4) Operational set-theoretic axioms for Separation, Range (or Replacement) as in $\mathrm{T_V}$; in addition there is a Universal Choice operator C satisfying $\exists x(fx = 0) \to (Cf)\downarrow \wedge f(Cf) = 0$.

For each type symbol σ of $\mathrm{T_V}$, we define $M_\sigma(x)$ inductively as follows to express in the language of OST that x is an object of type σ:

(i) $M_V(x)$ is $(x = x)$

(ii) $M_{\sigma\to\tau}(x)$ is $\forall y[M_\sigma(y) \to xy\downarrow \wedge M_\tau(xy)]$

We may treat the predicates M_σ as classes and write $f : M_\sigma \to M_\tau$ for $M_{\sigma\to\tau}(f)$. The translation of the constants of $\mathrm{T_V}$ into those of OST except for the recursors is immediate; for each of these we may check that if the constant is of type σ then its translation is a closed term of OST that is provably in M_σ.

So now consider any recursor R_σ; this is of type $(V \to V \to \sigma) \to (V \to \sigma)$. As its interpretation we make use of the type-free form of the recursion theorem that is a consequence of the applicative axioms of OST; this provides a closed term **rec** such that for any f, $\mathbf{rec}f\downarrow$ and for $g = \mathbf{rec}f$ and any x, we have $gx \simeq fgx$, i.e. either both sides are defined and equal or both are undefined. We also make use of Lemma 5 of Feferman (2006), according to which there is a closed term **fun** such that for any f, x such that $(\forall y \in x)fy\downarrow$ we have $\mathbf{fun}fx\downarrow$, and $\mathbf{fun}fx$ is the graph of f restricted to x considered as a set; in other words **fun** may be taken as the interpretation of Grph and we also write

[6] See Troelstra and van Dalen (1988) pp. 50-51, where Et is written for $t\downarrow$ and the logic of partial terms is called E-logic.

$f|x$ for $\mathbf{fun}fx$. Finally, using the recursor **rec**, we obtain a closed term $\mathbf{r}$ such that $\mathbf{r}f\downarrow$ for all f, and for $g = \mathbf{r}f$, the following is provable:

$$gx \simeq f(g|x)x$$

We claim each R_σ can be translated by this same term $\mathbf{r}$. That is, no matter what σ we take, we have

$$\mathbf{r} : (V \to V \to M_\sigma) \to (V \to M_\sigma)$$

For, suppose given any $f : (V \to V \to M_\sigma)$; and let $g = \mathbf{r}f$. It is to be shown that $g : (V \to M_\sigma)$, i.e. that for each x, $gx\downarrow$ and gx is in M_σ. This is proved by $\in$-induction on x; if it holds for all $y \in x$, then $\mathbf{fun}gx\downarrow$, i.e. $g|x$ is in V, so by assumption, $f(g|x)x$ is in M_σ, and hence the same holds for gx. QED

Lemma 4. Under this translation, T_V is interpreted in OST.

Proof. The verification of all the axioms of $\mathrm{T_V}$ by the corresponding axioms of OST up to those for Recursion and Induction are immediate. The Recursion axiom is taken care of in the way just described, so it is only left to check the Induction Rule. So suppose that $\theta(x, v, u)$ is a formula which is a translation of a formula of $\mathrm{T_V}$ for which v is a sequence of variables of type $\sigma = \sigma_1, \ldots, \sigma_n$ and u is a sequence of variables of type $\tau = \tau_1, \ldots, \tau_m$; we write $M_\sigma(v)$ for the conjunction of statements $M_{\sigma_i}(v_i)$ and similarly for $M_\tau(u)$. Suppose further that G and Z are closed terms of OST for which the following has been inferred:

$$\forall x \forall u\{M_\sigma(Gx) \wedge [M_\tau(u) \to M_\tau(Zxu)] \wedge [(\forall y \in x)\theta(y, Gy, Zxu) \to \theta(x, Gx, u)]\}$$

Then we conclude

$$\forall x\{(\forall y \in x)\forall u[M_\tau(u) \to \theta(y, Gy, u)] \to \forall u[M_\tau(u) \to \theta(x, Gx, u)\}$$

Thus by the Induction Scheme in OST we conclude $\forall x \forall u[M_\tau(u) \to \theta(x, Gx, u)]$, which verifies the translation of the conclusion of the Induction Rule in $\mathrm{T_V}$. □

We may now complete the proof of Theorem 5 by means of the fact established in Feferman (2006) (and in another way in Jäger (2007)) that OST is of the same proof-theoretical strength as KPω. Finally, the fact that KPω is of the same proof-theoretical strength of ID_1 is due to Jäger (1982); it is then of the same strength as $\mathrm{ID}_1^{(i)}$ by Buchholz (1980). □

If the power set operation is considered as a definite operation, which is suggested by one philosophical view of set theory which still regards the universe of all sets as an indefinite totality, we are led to a semi-constructive system for which we can prove the following theorem in the same way as was done for Theorem 5.

Theorem 6. The system IKPω + (Pow) + (Δ_0-LEM) + (MP) + (BOS) + (AC) has proof-theoretical strength between the classical systems KPω + (Pow) and KPω + (Pow) + (V=L).

This makes use of the result proved in Jäger (2007) that the proof-theoretical strength of OST + (Pow) is bounded by that of KPω + (Pow) + (V=L). It is conjectured but it is not known whether the strength of the latter is the same as that of KPω + (Pow); the standard argument to eliminate the Axiom of Constructibility does not apply in any obvious way.

7 A miscellany of related work and questions

7.1 Kohlenbach's "Lesser" NOS

Kohlenbach (2001) considers the following weakening of NOS that he calls the Lesser Numerical Omniscience Scheme:

$$\text{(LNOS)} \quad \forall n[(\phi(n) \vee \neg\phi(n)) \wedge (\psi(n) \vee \neg\psi(n))] \wedge \neg\exists n\phi(n) \wedge \neg\exists n\psi(n)$$

$$\to \forall n\neg\phi(n) \vee \forall n\neg\psi(n)$$

His main result is that the semi-constructive system Res-HA$^\omega$ + (AC) + (MP) + (LNOS) is conservative over PRA for Π^0_2 sentences. The proof is by means of functional interpretation combined with the method of majorization. Kohlenbach also shows that the system in question proves WKL, i.e. König's Lemma for binary trees ("weak König's Lemma"). Ferreira and Oliva (2005) have introduced another method, called *bounded functional interpretation*, which they show may be used to obtain the same results in a simpler way. It would be interesting to see if their majorization and/or bounding techniques can be used to amplify the results of the present paper.

Kohlenbach (2008), p. 154, has also observed that WKL implies KL over Res-HA$^\omega$ + AC$_{0,0}$, so in such contexts, the difference between "weak" and "usual" König's Lemma disappears; this is in accord with the advantage of beefing up constructive and semi-constructive systems stressed here.

Since PRA is considered by many to be the limit of finitism, it would also be interesting to produce a natural semi-constructive system of finite type over the natural numbers for which all bounded formulas are decidable and whose proof-theoretical strength is equal to that of PRA. Finally, one may speculate that there are suitable such systems equivalent in strength to feasible arithmetic.

7.2 Friedman's system ALPO

Friedman (1980) introduced a semi-constructive system ALPO (for "Analysis with the Limited Principle of Omniscience") in the language of set theory with the natural numbers as a set of urelements, for which the main result is conservation of ALPO over PA for all arithmetic sentences.[7] For comparison with the system SCS = IKP$_\omega$ + (Δ_0-LEM) + (MP) + (BOS) + (AC) of Theorem 5 above, here are the axioms of ALPO: A. Ontological (urelements and sets), B. Urelement extensionality, C. Successor axioms, D. Infinity, E. Sequential induction, F. Sequential recursion, G. Pairing (unordered), H. Union, I. Exponentiation, J. Countable choice, K. Δ_0-separation, L. Strong collection, M. Limited principle of omniscience. By E is meant that any sequence (i.e. function) a of natural numbers which is such that $a(0) = 0 \wedge \forall n(a(n) = 0 \to a(n') = 0)$ then $\forall n(a(n) = 0)$. Axiom E guarantees definition by primitive recursion. The axiom J is of course a consequence of AC$_{\text{Set}}$ in our system, as is the strong collection axiom L (i.e. collection applied to arbitrary formulas). Other than Axiom I, all of these are thus derivable in SCS. That axiom asserts the existence for any sets a, b, of the set of all functions from a to b, which is not a consequence of SCS or even of its finite type extension SCS$^\omega$ (at least not

[7] I was reminded of Friedman's work by Jeremy Avigad.

in any obvious way). In his paper, Friedman makes use of a special model-theoretic argument in order to eliminate Axiom I before completing the proof that ALPO is conservative over PA. It would be of great interest to see whether the methods of functional interpretation employed here can be adapted to prove the same. Note that Axiom I does follow from the power set axiom used in the extension of SCS for Theorem 6.

7.3 Burr's interpretation of KPω

A useful variant functional interpretation due to Shoenfield (1967) sec. 8.3 in $\forall\exists$ form that is sometimes used applies directly to a classical system without requiring initial passage through the N-translation. The straightforward attempt to give such an interpretation of KPω meets an immediate obstacle if the constant 0 is to be part of the language; namely, it follows from provability of $\forall x \exists y (x \neq 0 \to y \in x)$ that one must have a term $t(x)$ such that $x \neq 0 \to t(x) \in x$ is provable in the target QF system. In other words one must have a non-constructive ("choice") operator for bounded quantification (of the sort provided in the system T_V by the bounded choice operator C). In order to avoid this, Burr (1998, 2000) gives a further Diller-Nahm (1974) $\forall\exists$-variant interpretation of KPω in a QF theory of primitive recursive set functionals of finite type. It is quite different from the interpretation given here in sec. 6, but there may be interesting relationships that are worth pursuing.

7.4 Some systems of semi-intuitionistic set theory with the power set axiom

The study of such subsystems of ZF formulated in intuitionistic logic with LEM for bounded formulas was apparently initiated by Poszgay (1971, 1972) and then studied more systematically by Tharp (1971), Friedman (1973) and Wolf (1974).[8] Poszgay had conjectured that his system is as strong as ZF, but Tharp and Friedman proved its consistency in ZF using a modification of Kleenes method of realizability. Wolf established the equivalence in strength of several related systems. The first is K_1, a system with axioms of Extensionality, Pairing, Union, Infinity and Power Set, the full Induction Scheme, and with Replacement restricted to formulas in which all quantifiers are bounded or subset bounded. K_2 is K_1 + LEM, and K_3 is K_1 plus a certain strong axiom scheme of Transfinite Recursive Definitions which implies the Full Replacement and Collection axiom schemes; finally K_3^* is K_3 + MP. (In this notation, what Tharp and Friedman proved is consistency in ZF of an extension of K_1 plus Full Replacement and the usual Axiom of Choice.) Wolf's main results include equiconsistency of K_1, K_2 + V=L, and K_3^*. The system K_3^* is close in many respects to the system IKPω + (Pow) + (Δ_0-LEM) + (MP) + (BOS) + (AC_{Set}) dealt with here in Theorem 6, except for BOS and AC_{Set} (Full Axiom of Choice scheme), and which also implies Full Replacement and Collection. It should be of interest to make a detailed comparison between these systems and of the methods involved.

[8] I am indebted to Harvey Friedman and Robert Wolf for bringing this work to my attention, after the body of this paper was completed.

7.5 Mathematics in semi-constructive systems

Coquand and Palmgren (2000) give a couple of examples of mathematical theorems in their semi-constructive system for countable tree ordinals (described in sec. 5 above) that can be provided with a constructive foundation via their constructive sheaf-theoretic model of the system. The first is König's Lemma for binary trees; but in fact, as shown by Kohlenbach in the work described in 7.1 above, a much, much weaker system (conservative over PRA) suffices to do the same. The second is Dickson's Lemma, according to which if $u : \mathbb{N} \to \mathbb{N}$ and $v : \mathbb{N} \to \mathbb{N}$ are two sequences of natural numbers then there exist $p < q$ such that $u(p) \leq u(q)$ and $v(p) \leq v(q)$. That follows from a prior lemma, that for any sequence $u : \mathbb{N} \to \mathbb{N}$ of natural numbers, there exists a sequence $n_0 < n_1 < n_2 < \ldots$ such that $u(n_0) \leq u(n_1) \leq u(n_2) \leq \ldots$. To obtain Dickson's Lemma from this, one first finds a strictly increasing sequence of natural numbers on which u is increasing, and then a strictly increasing subsequence of that on which v is increasing, to get a sequence $n_0 < n_1 < n_2 < \ldots$ on which both u and v are increasing. We may then take $p = n_0$ and $q = n_1$. Again, what is needed can be proved in a much, much weaker system, namely that of Theorem 3(i) conservative over PA. The truth of Dickson's Lemma implies that we can obtain p, q as recursive functionals of u and v; simply search for the first p, q which make it true. The constructive model of Coquand and Palmgren can hardly be expected to provide more useful information about the complexity of that functional.

More generally, all of the semi-constructive systems treated in Theorem 3 are candidates of potential interest in which to carry out predicative mathematics. The actual pursuit of that part of mathematics in various classical systems of explicit mathematics, as described, e.g., in (Feferman 1975 and Feferman and Jäger 1993, 1996) as well as in theories of finite type over the natural numbers (Feferman 1977, 1979) make systematic use of explicit witnessing data. For example, a uniformly continuous function on a closed interval of real numbers is treated as a pair consisting of a function of real numbers on that interval and a uniform modulus of continuity functions. As pointed out by Friedman at the outset of his (1980) article, such padding is unnecessary in semi-constructive systems in which the Axiom of Choice holds in sufficiently strong form, as it does in ALPO and in the various systems considered here. How far this freedom takes us is another matter, but the actual development of predicative mathematics in these systems should certainly be revisited in that light. In addition, one should see how much mathematics can be conveniently carried out in the impredicative semi-constructive systems of secs. 5 and 6. Finally, it would be worth pursuing the formulation and determination of the proof-theoretical strength of semi-constructive systems of explicit mathematics and operational set theory, neither of which has been directly handled here, and in both of which mathematics can in general be carried out in a more flexible manner than in typed systems or even in set theoretical systems.

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