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Persistent and invariant formulas for outer extensions

Dedicated to A. Heyting on the occasion of his 70^{th} birthday

by

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1. Introduction

1(a) *Summary.* The notion of *outer extension* for structures which is dealt with here is a generalization of the notions of *end extension* for ordered structures and *ε-extension* for models of type theory and set theory. The principal results (2.4, 4.1, 4.2) characterize those formulas from a wide class of languages which are *persistent*, resp. *invariant*, for outer extensions. These incorporate results announced by G. Kreisel and the author in [8]. The various methods of proof mentioned in [8] are superseded here by a single new method of a basically proof-theoretical nature. This leads directly to theorems (4.4, 4.5) which are syntactic reformulations of the main results in terms of notions of *provable* persistence and invariance. In this form they are of particular interest for investigations into subsystems of classical analysis and set theory having a predicative interpretation.

1(b) *Background.* In the usual first-order model theory of languages with finite formulas a formula ϕ is said to be *persistent for* (or *preserved under*) *extensions* relative to a set S of sentences if whenever (i) $\mathcal{M}$, $\mathcal{M}'$ are models of S, (ii) $\mathcal{M}$ is an extension of $\mathcal{M}'$, and (iii) ϕ is satisfied by some elements of $\mathcal{M}'$ in $\mathcal{M}'$, then it is satisfied by the same elements in $\mathcal{M}$. ϕ is said to be *invariant for extensions* relative to S if both ϕ and $\sim \phi$ are persistent relative to S. It is a familiar result of Tarski's [20] that ϕ is persistent for extensions rel. to S if and only if there is a purely existential formula θ with $S \vdash (\phi \leftrightarrow \theta)$. If ϕ is also invariant there must then also be a purely universal formula ψ with $S \vdash (\phi \leftrightarrow \psi)$. When S

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itself consists of purely universal sentences, this can be strengthened to $S \vdash (\phi \leftrightarrow \theta)$ for some quantifier-free θ ([18]).

Now consider languages with a binary relation symbol ε and certain variables which are interpreted as ranging over sets, for which ε is interpreted as the membership relation. We have in mind both languages for simple type theory, in which case one of the kinds of variables ranges over individuals, and usual set theory where we have just one kind of variable ranging over sets. Natural models for the latter are $\mathscr{M} = (M, E_M)$ where M is a collection of sets and E_M is the $\in$-relation restricted to M. Regarding sets as completed totalities, the natural notion of extension for these is: $\mathscr{M} = (M, E_M)$ is an *$\in$-extension* of $\mathscr{M}' = (M', E_{M'})$ if $\mathscr{M}$ is an extension of $\mathscr{M}'$ in the usual sense such that for all $x \in M'$ and $y \in M$, yE_Mx always implies $y \in M'$. It is seen that this notion makes perfectly good sense even when applied to arbitrary structures $\mathscr{M} = (M, E_M, \cdots)$ with a binary relation E_M. Note that this coincides with the notion of *end extension* used by several authors when E_M is taken to be an ordering of M. In the case of type theory, where models are many-sorted structures, one domain of which is a set M_0 of individuals, we must consider two notions of $\in$-extension, one as before and the other where we demand that $M_0 = M'_0$, i.e. that the set of individuals is kept fixed or, as we shall say below, is *stationary*.

1(c) *Notions and results.* These considerations lead us to define a more general notion of *outer extension* for arbitrary many-sorted relational structures $\mathscr{M}$ (of a given similarity class) including a binary relation E_M, and where certain of the domains may be stationary. We shall write $\mathscr{M}' \leqq \mathscr{M}$ if $\mathscr{M}$ is such an extension of $\mathscr{M}'$; the exact definition is given in 2.1. Furthermore, we can consider properties of such structures expressed by formulas of any language $\mathscr{L}$, finitary or infinitary, appropriate to the given similarity class. The model-theoretic problem then is to characterize those formulas ϕ of $\mathscr{L}$ which are $\leqq$-persistent, resp. $\leqq$-invariant (i.e. for outer extensions) relative to a set S of $\mathscr{L}$-sentences.

This problem is completely solved here for a wide class of sublanguages $\mathscr{L}_A$ of the language $\mathscr{L}_{HC}$ with countably long conjunctions $\prod$ and disjunctions $\sum$ and finite quantification (HC = hereditarily countable sets), and for a wide class of S in $\mathscr{L}_A$. These are classes $\mathscr{L}_A$ and S satisfying certain conditions found by Barwise [1] for completeness of logic in $\mathscr{L}_A$, and A-compactness of S; we give a resumé of Barwise's work in § 2(c) below. Of special interest are the cases of the usual finitary language, corresponding to $A = HF$ = hereditarily finite sets (where S can be arbitrary), and of the language with hyperarithmetic formulas, $A = HH$ = hereditarily hyperarithmetic sets.

The solution is as follows. By *restricted quantification* we mean formation of $\bigvee y(y \,\varepsilon\, t \wedge \phi)$ or $\bigwedge y(y \,\varepsilon\, t \rightarrow \phi)$ from a formula ϕ, where t is a term not containing y. A formula is said to be *essentially existential* (resp. *universal*) if it is built up from atomic formulas and their negations by means of (i) arbitrary $\prod$ and $\sum$, (ii) arbitrary restricted quantification, (iii) arbitrary quantification with respect to variables of stationary sort, and (iv) only existential (resp. universal) quantification otherwise. ϕ is said to be *essentially restricted* if it is in both forms, i.e. is built up using only (i)—(iii). Then for the $\mathscr{L}_A$ and S satisfying the Barwise conditions, Theorems 2.4 and 4.1 below give: *ϕ is $\leqq$-persistent relative to S if and only if there is an essentially existential θ such that $S \vdash (\phi \leftrightarrow \theta)$.* Again one can get a sharper result for $\leqq$-invariant ϕ relative to special S, namely (4.2): *if every sentence in S is essentially universal and ϕ is $\leqq$-invariant relative to S then we can find an essentially restricted θ such that $S \vdash (\phi \leftrightarrow \theta)$.*

Theorem 4.3 below expresses that the model-theoretic condition of $\leqq$-persistence is equivalent to a syntactic (derivability) condition, which we call *provable* $\leqq$-persistence. Then 4.1 and 4.2 can be reformulated as certain proof theoretic results 4.4 and 4.5. These include as particular cases the assertions of [5] p. 489 and [6] Theorem 2.3, which are thus established by the work here. The purpose of these is to determine a general collection of instances of the comprehension axiom in analysis and of the separation and replacement axioms in set theory which can be given a predicative interpretation, namely those given by provably $\leqq$-invariant formulas (relative to a set S with prior predicative interpretation).

1(d) *Relations to earlier work; methods of proof.* Tarski's proof in [20] of the result characterizing formulas of $\mathscr{L}_{HF}$ persistent under extensions made use of the method of diagrams and the compactness theorem. These general model-theoretic arguments do not extend to $\mathscr{L}_{HC}$ (Scott [19], p. 333). Malitz [15] obtained the characterization result for $\mathscr{L}_{HC}$ (with S countable) by means of a special interpolation theorem for universal formulas; the proof of the latter made use of the completeness of a cut-free formalization of logic in $\mathscr{L}_{HC}$ from Lopez-Escobar [13]. According to an abstract [2], the characterization result for $\mathscr{L}_{HC}$ (with S empty) as well as for some languages of inaccessible cardinality was independently found by G. V. Choodnovsky, making use of a special compactness theorem for sets of universal sentences. Barwise [1] extended Malitz' result and argument to arbitrary $\mathscr{L}_A$ with $A \neq HF$ and S satisfying his conditions for completeness and compactness. I then found a uniform proof of the characterization result for all of Barwise's cases together with the case $A = HF$, allowing some domains to be stationary as well. This was based on a combination of my interpolation theorem for many-sorted logic (announced in [8], theorem 4) with a theorem of Barwise's relating forms of quantifier occurrences in an interpolant to forms in the hypothesis and conclusion; details of this proof are given in [7], §§ 4, 5.

Early work on extensions in higher order model theory, for example [16], [17], dealt with rather special situations, even just for $A = HF$. The work by Kreisel and myself on $\leqq$-extensions announced in [8] solved the basic problems for $A = HF$, $A = HH$, $A = HC$, with no stationary domains. It also made use of a variety of proofs, according to whether $A = HF$ or not. In the case $A \neq HF$ we developed a variant of Malitz' argument by means of an interpolation theorem for essentially universal formulas. In the case $A = HF$ an even more special proof-theoretic argument was employed. However, once I found the uniform methods of [7] for persistence under ordinary extensions, I was also able to develop the uniform methods now presented here to treat outer extensions quite generally. The novel idea beyond [7] is the use of a language and logical calculus in which restricted quantifiers are given independent status. The core of the argument below is to show that suitably modified versions (3.1, 3.2) of the interpolation theorems of [7] hold for this calculus. Only the new points in the verification of these are given below. Nevertheless, the line of argument should be understandable without a prior reading of [7], so that this paper may be read independently.

It would be of great interest if the results concerning ordinary and $\leqq$-persistence for the various $\mathscr{L}_A$ could be recaptured by strictly model-theoretic methods. In any case, we see once more the fruitfulness of interpolation theorems in model theory originally realized by Craig [3] and Lyndon [14].

1(e) *Remarks on terminology.* There is no generally accepted terminology for what we call *outer extension,* which is used here for the first time. The abbreviation $\leqq$ was used in [8] without giving a name to this relation. Kreisel and I have both given "$\in$-extension" some currency via seminars and talks. In addition, Kreisel used it in his abstract [11]; Kunen used the terminology *transitive extension* in his paper [12] in the same connection. *End extension* is in common usage for ordered structures, cf. eg. [9] "Outer" was chosen as so to be neutral between these situations. While we have not done so, this might suggest calling $\mathscr{M}'$ an *inner substructure* of $\mathscr{M}$ if $\mathscr{M}' \leqq \mathscr{M}$; this would include the common use of "inner model" in connection with various set theories.

The terminology "essentially existential (universal)" was introduced in [4]. I used "essentially $\sum$ ($\prod$), (with and without subscript "1") in [5] and [6] to indicate the connection with familiar analytic and set-theoretical hierarchies (cf. also § 2(c) below). The earlier terminology is used here in order to avoid confusion with infinite disjunctions and conjunctions.

2. Preliminaries

2(a) *Structures and extensions.* The similarity class of a many-sorted structure

$$\mathscr{M} = (\langle M_j\rangle_{j\in J}, \langle R_i\rangle_{i\in I_0}, \langle a_i\rangle_{i\in I_1}, \langle F_i\rangle_{i\in I_2})$$

is specified by means of a *signature*

$$\sigma = (J, I_0, I_1, I_2, \langle k_i\rangle_{i\in I_0}, \langle l_i\rangle_{i\in I_1}, \langle m_i\rangle_{i\in I_2}, \langle \varphi_i\rangle_{i\in I_2})$$

where $(\mathrm{i})_\sigma\ J \neq 0$, $(\mathrm{ii})_\sigma$ for $i \in I_0$, $0 < k_i < \omega$, $(\mathrm{iii})_\sigma$ $l_i \in J$ *for* $i \in I_1$, $(\mathrm{iv})_\sigma$ *for* $i \in I_2$, $0 < m_i < \omega$ *and* φ_i *is a non-empty partial function with* $\mathscr{D}(\varphi_i) \subseteqq J^{m_i}$ and $\mathscr{R}(\varphi_i) \subseteqq J$. For $\mathscr{M}$ to be of signature σ, the following conditions must be satisfied:

(i) $M_j \neq 0$ *for each* $j \in J$;
(ii) $R_i \subseteqq (\bigcup M_j [j \in J])^{k_i}$ *for each* $i \in I_0$;
(iii) $a_i \in M_{l_i}$ *for each* $i \in I_1$; *and*
(iv) $F_i : (M_{j_1} \times \cdots \times M_{j_{m_i}}) \to M_{\varphi_i(j_1,\cdots,j_{m_i})}$ *for each* $i \in I_2$ *and* $(j_1, \cdots, j_{m_i}) \in \mathscr{D}(\varphi_i)$.

J is called the collection of *sorts* and M_j the set of elements of sort j in $\mathscr{M}$. We write $x \in \mathscr{M}$ for $x \in \bigcup M_j[j \in J]$.

For the special purposes here we shall restrict attention to signatures σ satisfying: $(\mathrm{v})_\sigma$ *$0 \in I_0$ and $k_0 = 2$.* In other words, R_0 is a binary relation. Given $\mathscr{M}$, we shall write E for R_0 and xEy for $(x, y) \in E$.

To simplify some details of syntax below it is convenient to assume further that $(\mathrm{vi})_\sigma$ *J, I_0, I_1, I_2 are countable and σ is recursively given.*

As an illustration of these notions a signature σ suitable for simple type theory over the natural numbers is given by $J = \omega$, $I_0 = I_1 = I_2 = \{0\}$, $k_0 = 2$, $l_0 = 0$, $m_0 = 1$, and $\mathscr{D}(\varphi_0) = \{0\}$, $\varphi_0(0) = 0$. Interesting examples of structures $\mathscr{M}$ with this signature are provided by $M_0 = \omega$, $M_{j+1} \subseteqq \mathscr{P}(M_j)$, $R_0 = \{(x, y) : \text{for some } j, x \in M_j, y \in M_{j+1} \text{ and } x \in y\}$, $a_0 = 0$, $F_0(x) = x+1$ for $x \in M_0$. ($\mathscr{P}(X) =$ set of all subsets of X.) To treat cumulative type theory, where one is interested in the cases $M_{j+1} \subseteqq M_j \cup \mathscr{P}(M_j)$ one would take instead

$$R_0 = \{(x, y) : x, y \in \bigcup M_j[j < \omega] \text{ and } x \in y\}.$$

A signature σ and any structure $\mathscr{M}$ of signature σ are said to be *relational* if $I_2 = 0$. In general, we can associate with each σ and $\mathscr{M}$ corresponding relational σ_{Rel} and $\mathscr{M}_{Rel}$ in the obvious way, having the same domains and individuals and having $\langle R'_i \rangle_{i \in I_0 \cup I_2}$ with $R'_i = R_i$ for $i \in I_0$ and $R'_i = F_i$ for $i \in I_2$. (First disjointing I_0 and I_2.)

Let $J_0 \subseteqq J$; the elements of J_0 will be called the *stationary* sorts. It is assumed that σ and J_0 are fixed throughout the following and that $\mathscr{M}$ and $\mathscr{M}' = (\langle M'_j \rangle_{j \in J}, \langle R'_i \rangle_{i \in I_0}, \langle a'_i \rangle_{i \in I_1}, \langle F'_i \rangle_{i \in I_2})$ are any structures of signature σ, and $E = R_0$, $E' = R'_0$.

2.1 Definition. *$\mathscr{M}$ is an outer extension of $\mathscr{M}'$, in symbols $\mathscr{M}' \leqq \mathscr{M}$, if the following conditions hold:*

(i) *$M'_j = M_j$ for $j \in J_0$;*

(ii) *$M'_j \subseteqq M_j$ for $j \in J - J_0$;*

(iii) *for each $i \in I_0$ and for each $x_1, \cdots, x_{k_i} \in \mathscr{M}'$,*
$(x_1, \cdots, x_{k_i}) \in R_i$ if and only if $(x_1, \cdots, x_{k_i}) \in R'_i$;

(iv) *$a_i = a'_i$ for each $i \in I_1$;*

(v) *for each $i \in I_2$ and $(j_1, \cdots, j_{m_i}) \in \mathscr{D}(\varphi_i)$ and for each $(x_1, \cdots, x_{m_i}) \in M'_{j_1} \times \cdots \times M'_{j_{m_i}}$,*
$F_i(x_1, \cdots, x_{m_i}) = F'_i(x_1, \cdots, x_{m_i})$; and

(vi) *if $x \in \mathscr{M}'$, $y \in \mathscr{M}$ and yEx then $y \in \mathscr{M}'$.*

A still more general notion is discussed at the conclusion of the paper.

2(b) *The language $\mathscr{L}_{HC}$ with restricted quantification.* The first-order countably infinitary language $\mathscr{L}_{HC}$ appropriate to structures of signature σ is described as follows. (i) For each $j \in J$, we have variables $v_{j,0}, \cdots, v_{j,i}, \cdots$ of sort j. There are infinitely many variables of each sort, and variables of distinct sorts are distinct. (In the following we shall use $u, w, u_1, w_1, \cdots$ to range over variables of arbitrary sort.) There are (ii) a symbol for equality, $=$, (iii) for each $i \in I_0$, a k_i-ary relation symbol, r_i, (iv) for each $i \in I_1$ a constant symbol c_i and (v) for each $i \in I_2$ an m_i-ary function symbol, f_i. The *terms* and their corresponding sorts are defined inductively by: (i) each variable of sort j is a term of sort j; (ii) each constant c_i is a term of sort l_i; (iii) if $(j_1, \cdots, j_{m_i}) \in \mathscr{D}(\varphi_i)$ and $t_1, \cdots, t_{m_i}$ are terms of sort $j_1, \cdots, j_{m_i}$ resp. then $f_i(t_1, \cdots, t_{m_i})$ is a term of sort $\varphi_i(j_1, \cdots, j_{m_i})$. The *atomic formulas* are the formulas $(t_1 = t_2)$ for *any* terms t_1, t_2 and $r_i(t_1, \cdots, t_{k_i})$ for any terms $t_1, \cdots, t_{k_i}$. We shall write $(t_1 \in t_2)$ instead of $r_0(t_1, t_2)$.

Arbitrary *formulas* are built up according to the following procedures: (i) every atomic formula is a formula; (ii) if ϕ is a formula, so is $\sim \phi$; (iii) if K is any set of formulas with $0 < \bar{K} \leqq \omega$ and there are altogether only finitely many free variables among the ϕ in K then $\sum_{\phi \in K} \phi$ and $\prod_{\phi \in K} \phi$ are formulas. (iv) if ϕ is a formula and u is any variable then $\bigvee u \phi$ and $\bigwedge u \phi$ are formulas; and (v) if ϕ is a formula, u is any variable, t is any term in which u does not occur, then $\bigvee(u, t)\phi$ and $\bigwedge(u, t)\phi$ are formulas. The operations in (v) are called operations of *restricted existential and universal quantification.* The ordinary operations of existential and universal quantification of (iv) are said to be *unrestricted.* A formula is said to be *standard* if it is built up without the useof restricted quantification. A *sequent* is a pair (Γ, Δ) where Γ, Δ are finite sequences (possibly empty) of formulas; a *standard sequent* is one all of whose formulas are standard. It is assumed from now on that no variable occurs both free and bound in any formula or sequent considered.

We shall write: $(\phi_0 \vee \phi_1)$ for $\sum_{\phi \in \{\phi_0, \phi_1\}} \phi$, $(\phi_0 \wedge \phi_1)$ for $\prod_{\phi \in \{\phi_0, \phi_1\}} \phi$, $(\phi_0 \rightarrow \phi_1)$ for $(\sim \phi_0 \vee \phi_1)$, $(\phi_0 \leftrightarrow \phi_1)$ for $(\phi_0 \rightarrow \phi_1) \wedge (\phi_1 \rightarrow \phi_0)$, $\bigvee u \,\varepsilon\, t\phi$ for $\bigvee(u, t)\phi$, $\bigwedge u \,\varepsilon\, t\phi$ for $\bigwedge(u, t)\phi$, and $(\Gamma \supset \Delta)$ for (Γ, Δ). By a *sentence* we mean a formula without free variables.

The standard formula ϕ^* associated with a formula ϕ is found as follows: * preserves atomic formulas and the standard operations, and

$$(\bigvee u \,\varepsilon\, t\phi)^* = \bigvee u(u \,\varepsilon\, t \wedge \phi^*), \ (\bigwedge u \,\varepsilon\, t\phi)^* = \bigwedge u(u \,\varepsilon\, t \rightarrow \phi^*).$$

If Γ is a sequence $(\phi_1, \cdots, \phi_k)$ put $\Gamma^* = (\phi_1^*, \cdots, \phi_k^*)$.

We now single out the sorts of variables in a formula ϕ which appear in an essentially universal, resp. existential, unrestricted quantification.

2.2 DEFINITION. *$Un(\phi)$ and $Ex(\phi)$ are defined inductively by:*

(i) $Un(\phi) = Ex(\phi) = 0$ *for* ϕ *atomic;*

(ii) $Un(\sim \phi) = Ex(\phi)$, $Ex(\sim \phi) = Un(\phi)$;

(iii) $Un(\sum_{\phi \in K}\phi) = Un(\prod_{\phi \in K}\phi) = \bigcup Un(\phi)[\phi \in K]$ *and* $Ex(\sum_{\phi \in K}\phi) = Ex(\prod_{\phi \in K}\phi) = \bigcup Ex(\phi)[\phi \in K]$;

(iv) $Un(\bigvee u\phi) = Un(\phi)$ *and* $Ex(\bigvee u\phi) = Ex(\phi) \cup \{j\}$ *if* u *is of sort* j;
$Ex(\bigwedge u\phi) = Ex(\phi)$ *and* $Un(\bigwedge u\phi) = Un(\phi) \cup \{j\}$ *if* u *is of sort* j;

(v) $Un(\phi) = Un(\psi)$ *and* $Ex(\phi) = Ex(\psi)$ *when* ϕ *is* $\bigvee u \,\varepsilon\, t\psi$ *or* $\bigwedge u \,\varepsilon\, t\psi$;

(vi) *Un and Ex are defined for sequents by*
$Un(\Gamma \supset \Delta) = \bigcup Ex(\phi)[\phi \text{ in } \Gamma] \cup \bigcup Un(\psi)[\psi \text{ in } \Delta]$ *and*
$Ex(\Gamma \supset \Delta) = \bigcup Un(\phi)[\phi \text{ in } \Gamma] \cup \bigcup Ex(\psi)[\psi \text{ in } \Delta]$.

ϕ is said to be essentially existential (universal) if

$$Un(\phi) \subseteqq J_0 (Ex(\phi) \subseteqq J_0)$$

and essentially restricted if both

$$Un(\phi) \subseteqq J_0,\ Ex(\phi) \subseteqq J_0.$$

The notions of *value* of a term, *satisfaction* of a formula, and *truth* in a structure $\mathscr{M}$ are defined in the expected way, taking the range of variables of sort j to be M_j. If $\phi(u_1, \cdots, u_t)$ has free variables $u_1, \cdots, u_t$ of sorts $j_1, \cdots, j_t$ resp. and the sequence $(x_1, \cdots, x_t) \in M_{j_1} \times \cdots \times M_{j_t}$ satisfies ϕ in $\mathscr{M}$ we write $\models_{\mathscr{M}} \phi[x_1, \cdots, x_t]$. If this holds for all sequences we write $\models_{\mathscr{M}} \phi$ and say ϕ is *valid* in $\mathscr{M}$. A sequent $\Gamma \supset \Delta$ is taken to be valid in $\mathscr{M}$ if every sequence of elements of $\mathscr{M}$ satisfying each member of Γ satisfies some member of Δ. If S is a set of sentences, a formula ϕ, resp. sequent $\Gamma \supset \Delta$, is said to be a *consequence* of S if it is valid in every model of S; we write $S \vdash \phi$, resp. $S \vdash (\Gamma \supset \Delta)$. When S is countable these are equivalent, respectively, to $\vdash (\prod_{\psi \in S}\psi \supset \phi)$ and $\vdash (\Gamma, \prod_{\psi \in S}\psi \supset \Delta)$.

2.3 Definition. *A formula $\phi(u_1, \cdots, u_t)$ is said to be persistent for outer extensions relative to S (more briefly: $\leqq$-persistent rel. to S) if whenever $\mathscr{M}$, $\mathscr{M}'$ are models of S with $\mathscr{M}' \leqq \mathscr{M}$ and $\models_{\mathscr{M}'} \phi[x_1, \cdots, x_t]$ then $\models_{\mathscr{M}} \phi[x_1, \cdots, x_t]$. ϕ is invariant for outer extensions relative to S (more briefly $\leqq$-invariant rel. to S) if both it and $\sim \phi$ are $\leqq$-persistent rel. to S.*

2.4 Theorem. *If $S \vdash (\phi \leftrightarrow \theta)$ where θ is essentially existential then ϕ is $\leqq$-persistent rel. to S.*

Proof. It is easily proved by induction that if a formula is ess. ex. then it is $\leqq$-persistent and if it is ess. un. then its negation is $\leqq$-persistent.

To prove the converse of 2.4 below we make use of a syntactic equivalent of the condition of $\leqq$-persistence, made possible by a complete deductive system for the valid sequents of $\mathscr{L}_{HC}$. The obvious extension of the Gentzen direct (cut-free) rules to the standard sequents in the single-sorted case, together with direct rules for equality, has been shown complete by Lopez-Escobar [13]. The further obvious extension to the many-sorted case is easily seen complete by the same argument (cf. alternatively, the proof in [7]). We call this the *standard (deductive) G-system* here.

We consider now a system of rules for deriving arbitrary valid sequents in the syntax treated here. This consists of the foregoing rules for the standard operations together with the following rules for restricted quantification:

$$(\bigvee \varepsilon \supset) \frac{\Gamma, w \,\varepsilon\, t, \phi(w) \supset \Delta}{\Gamma, \bigvee u \,\varepsilon\, t\, \phi(u) \supset \Delta}$$

$$(\supset \bigwedge \varepsilon) \frac{\Gamma, w \,\varepsilon\, t \supset \Delta, \phi(w)}{\Gamma \supset \Delta, \bigwedge u \,\varepsilon\, t\, \phi(u)}$$

$$(\bigwedge \varepsilon \supset)_1 \frac{\Gamma \supset \Delta, t_1 \,\varepsilon\, t;\ \Gamma, \phi(t_1) \supset \Delta}{\Gamma, \bigwedge u \,\varepsilon\, t\, \phi(u) \supset \Delta}$$

$$(\supset \bigvee \varepsilon)_1 \frac{\Gamma \supset \Delta, t_1 \,\varepsilon\, t;\ \Gamma \supset \Delta, \phi(t_1)}{\Gamma \supset \Delta, \bigvee u \,\varepsilon\, t\, \phi(u)}$$

subject to: w is not free in Γ, Δ or t in $(\bigvee \varepsilon \supset)$ and $(\supset \bigwedge \varepsilon)$ and u is of the same sort as w, and u is of the same sort as t_1 in $(\supset \bigvee \varepsilon)_1$ and $(\bigwedge \varepsilon \supset)_1$. We call this the *first (deductive) G_ε-system*, or G^1_ε-system.

The above mentioned arguments are easily adapted to show the completeness of the first G-system. A direct reduction is also possible by the following:

2.5 LEMMA. *If* $(\Gamma^* \supset \Delta^*)$ *is derivable in the standard G-system then* $(\Gamma \supset \Delta)$ *is derivable in the first* G_ε*-system.*

While the first G_ε-system permits a translation of certain model-theoretic conditions into syntactic form, I have not been able to use it directly to obtain the basic interpolation theorems of § 3. These are obtained instead by using a modified deductive system with the following new $(\bigwedge \varepsilon \supset)$ and $(\supset \bigvee \varepsilon)$ rules:

$$(\bigwedge \varepsilon \supset) \; \frac{\Gamma, t_1 \,\varepsilon\, t, \phi(t_1) \supset \Delta}{\Gamma, t_1 \,\varepsilon\, t, \bigwedge u \,\varepsilon\, t\phi(u) \supset \Delta}$$

$$(\supset \bigvee \varepsilon) \; \frac{\Gamma, t_1 \,\varepsilon\, t \supset \Delta, \phi(t_1)}{\Gamma, t_1 \,\varepsilon\, t \supset \Delta, \bigvee u \,\varepsilon\, t\phi(u)}$$

(u of the same sort as t_1).

With all other rules taken as in the first G_ε-system, this is called simply the G_ε*-system.*

2.6 THEOREM. (i) *The* G_ε*-system is closed under the cut-rule.*

(ii) *The same sequents are derivable in the* G^1_ε *and* G_ε*-systems.*

(iii) $(\Gamma \supset \Delta)$ *is valid if and only if it is derivable in the* G_ε*-system.*

PROOF. The proof of (i) is by an extension of usual cut-elimination arguments; cf. [7] for the argument in the standard case. The new rules are handled just as directly. (ii) follows immediately, since the differing rules of the two systems are equivalent when cut is permitted. Then (iii) follows using 2.5 and the completeness of the standard G-system.

2(c) *The languages* $\mathscr{L}_A$. To get a strong converse to 2.4 (for suitable S) and a uniform proof for the finitary and infinitary cases, we now consider the spectrum of sub-languages $\mathscr{L}_A$ of $\mathscr{L}_{HC}$ dealt with by Barwise in [1]. We give a brief resumé of his notions and results, which have not yet been published.

Identify terms and formulas with sets in any natural way, so that the logical operations appear as operations on sets, and sub-formulas of a formula belong to its transitive closure. Specific such identifications are given in [1], [7]; for example, $\sim \phi$ is taken as $(5, \phi)$ and $\sum_{\phi \in K} \phi$ as $(6, K)$ in [7]. Any collection A of sets closed under these operations has associated with it a language $\mathscr{L}_A$ whose formulas are just those ϕ which belong to A. By an

A-formula, resp. *A-sequent*, we mean one which belongs to A. With the identifications mentioned it is sufficient to assume that $(\text{i})_A$ $0 \in A$, and $(\text{ii})_A$ *$x, y \in A$ implies $\{x, y\} \in A$*. Then the collection of hereditarily finite sets $HF \subseteqq A$. For simplicity it is also assumed that $(\text{iii})_A$ *A is transitive.* Furthermore we shall only be concerned here with the cases that $(\text{iv})_A$ *$A \subseteqq HC$ = collection of hereditarily countable sets*. Thus the A-formulas are always formulas of $\mathscr{L}_{HC}$.

Derivation trees in any of the deductive systems considered above can also be identified with sets in a natural way; it is assumed that all relevant information is encoded in these. Barwise considers in [1] a more general notion of derivation in $\mathscr{L}_{HC}$ than the obvious one; he calls the latter notion that of a *single-valued* derivation. The difference is illustrated by a derivation $\mathscr{D}$ of a sequent $(\Gamma \supset \Delta)$ whose last step is an inference with one hypothesis,

$$\frac{\Gamma_0 \supset \Delta_0}{\Gamma \supset \Delta}.$$

In the single-valued case, the derivation $\mathscr{D}$ has encoded within it a single derivation $\mathscr{D}_0$ of $(\Gamma_0 \supset \Delta_0)$; in the more general case, $\mathscr{D}$ has encoded within it a non-empty *set* of derivations $(\Gamma_0 \supset \Delta_0)$.

When dealing with a language $\mathscr{L}_A$, we are particularly interested in derivations which belong to A. We cannot insure that for every derivation in A there is a single-valued derivation in A of the same conclusion without assuming some sort of choice hypothesis on A. For simplicity, we shall always take "derivation" here in its single-valued sense; the work below is easily extended to the more general case with weaker assumptions on A by using Barwise' methods and results.

$\mathscr{D}$ is called a *G-derivation* or *G_ε-derivation* if it is a derivation in the standard G-system or in the G_ε-system, resp. Following [1], A is called *v-admissible* ("v" for "validity") if every standard valid A-sequent has a G-derivation in A. A is called *v_ε-admissible* if every valid A-sequent has a G_ε-derivation in A. Barwise found rather general sufficient conditions in [1] for A to be v-admissible. These are formulated in terms of the usual finitary language of set theory (single-sorted, with basic symbols $=$ and ε). A *restricted*, or *Δ_0, formula* is one which is essentially restricted for this language (no stationary sorts). A *$\sum_1$-formula* is one of the form $\bigvee u \phi$ where ϕ is Δ_0. Formally define pairs (u, v) and the notion of function, $Fn(u)$, in the usual way. The *Δ_0-separation axiom* consists of all instances of $\bigwedge u \bigvee v \bigwedge w [w \,\varepsilon\, v \leftrightarrow w \,\varepsilon\, u \wedge \phi]$ for ϕ a

Δ_0-formula (v not in ϕ). The $\sum_1$*-axiom of choice* consists of all instances of

$$\bigwedge v \in u \bigvee w\psi \rightarrow \bigvee z(Fn(z) \wedge \bigwedge v \in u \bigvee w(\psi \wedge (v, w) \in z)],$$

with ψ a $\sum_1$-formula (z not in ψ). For any x, let $TC(x)$ be the transitive closure of x. The following is assumed from now on: $(\mathrm{v})_A$ *A satisfies the Δ_0-separation axiom and the $\sum_1$-axiom of choice, and A is closed under TC.*

Barwise's completeness theorem [1], § 2, is that under the assumptions $(\mathrm{i})_A$-$(\mathrm{v})_A$, A is v-admissible. Since his arguments work for any reasonable deductive system with cut-free rules, combining these with the result 2.6 (iii) gives us the following completeness theorem.

2.7 THEOREM. *Under the operating assumptions* $(\mathrm{i})_A$-$(\mathrm{v})_A$, *A is v_ε-admissible.*

Let S be any set of A-sentences. S is called *A-compact* if every A-sequent which is a consequence of S is also a consequence of some subset S' of S with $S' \in A$. In case $A = HF$, any S is A-compact, but this is far from true in general. S is said to be *A-recursively enumerable* (A-r.e.) if it is definable from some elements of A by a $\sum_1$-formula with quantifiers restricted to A. The following is *Barwise's compactness theorem* [1], § 2, again easily extended from the standard language to that considered here.

2.8 THEOREM. *Under the operating assumptions* $(\mathrm{i})_A$-$(\mathrm{v})_A$, *if S is countable and A*-r.e. *then S is A-compact.*

Various special cases of A satisfying $(\mathrm{i})_A$-$(\mathrm{v})_A$ are studied in [1]. Among these are $A = HF$, $A = HH =$ the collection of hereditarily hyperarithmetic sets and, more generally, $A = L_\alpha =$ the collection of sets constructible before α, for any recursively regular ordinal $\alpha \leqq \aleph_1$, and finally, of course, $A = HC$. Regarding compactness: any countable set S of HC-sentences is trivially HC-compact since $S \in HC$. For the case, $A = HH$, a set S of HF-sentences is HH-r.e. if and only if it is $\prod_1^1$ (by theorems of Kleene and Spector); this can be generalized suitably. This is of significance for applications to *generalized ω-models*, i.e. structures satisfying the set S consisting of $\bigwedge u \sum_{n<\omega} u = t_n$, u of a certain sort, and $\langle t_n \rangle_{n<\omega}$ a certain sequence of closed terms. For usual ω-models, we have a language containing that of number theory and $\langle t_n \rangle_{n<\omega}$ is the recursive sequence of numerals, so that $\bigwedge u \sum_{n<\omega}(u = t_n)$ is certainly in HH. In practice, this sentence is supplemented by finitely many further axioms in HF concerning 0, successor, etc. (depending on what functions or relations are taken as basic).

3. Interpolation theorems involving restricted quantification

For any formula ϕ, let $Sort\ (\phi)$ be the set of j such that a variable of sort j occurs free or bound in ϕ, $Fr\ (\phi)$ the set of variables free in ϕ, and $Rel\ (\phi)$ the set of relation symbols in ϕ. The following is the analogue of the interpolation theorem 4.2 of [7] for the calculus with restricted quantification.

3.1 THEOREM. *Assume σ is relational. Suppose $(\phi \to \psi)$ has a G_ε-derivation in A and that*

$$Sort\ (\phi) \cap Ex\ (\psi) \neq 0 \ \text{or}\ Sort\ (\psi) \cap Un\ (\phi) \neq 0.$$

Then we can find an A-formula θ and G_ε-derivations in A of $(\phi \to \theta)$ and $(\theta \to \psi)$, with θ satisfying the following conditions:

(i) $Fr\ (\theta) \subseteqq Fr\ (\phi) \cap Fr\ (\psi)$;

(ii) $Rel\ (\theta) \subseteqq Rel\ (\phi) \cap Rel\ (\psi)$;

(iii) $Sort\ (\theta) \subseteqq Sort\ (\phi) \cap Sort\ (\psi)$;

(iv) $Un\ (\theta) \subseteqq Un\ (\phi)$ *and* $Ex\ (\theta) \subseteqq Ex\ (\psi)$.

PROOF. As with theorem 4.2 of [7], this is obtained from a more general statement about derivable sequents. Given Γ and Γ', by a *mesh* of Γ, Γ' we mean any sequence having the terms of Γ and Γ' arbitrarily interspersed, but otherwise maintaining the original order of terms; $\Gamma \cdot \Gamma'$ is taken to be any mesh of Γ, Γ' in the following.

The more general statement is that for any G_ε-derivation in A of $(\Gamma \cdot \Gamma' \supset \Delta \cdot \Delta')$ and any var. w_0 not free in $\mathscr{D}$, of sort j_0, we can find an A-formula θ and G_ε-derivations in A of $(\Gamma \supset \Delta, \theta)$ and $(\theta, \Gamma' \supset \Delta')$ satisfying:

(i) $Fr\ (\theta) \subseteqq Fr\ (\Gamma \supset \Delta) \cap Fr\ (\Gamma' \supset \Delta') \cup \{w_0\}$;

(ii) $Rel\ (\theta) \subseteqq Rel\ (\Gamma \supset \Delta) \cap Rel\ (\Gamma' \supset \Delta)$;

(iii) $Sort\ (\theta) \subseteqq Sort\ (\Gamma \supset \Delta) \cap Sort\ (\Gamma' \supset \Delta') \cup \{j_0\}$;

(iv) $Un\ (\theta) \subseteqq Ex\ (\Gamma \supset \Delta)$ and $Ex\ (\theta) \subseteqq Ex\ (\Gamma' \supset \Delta')$.

This is proved by induction on $\mathscr{D}$. It is taken that no variable occurs both free and bound in $\mathscr{D}$. The argument is just as in [7] when $\Gamma \cdot \Gamma' \supset \Delta \cdot \Delta'$ is an axiom or inferred from a previous sequent or sequents by one of the structural rules or rules for $\sim$,

$\sum$, $\prod$, $\bigvee$, and $\bigwedge$. Thus we need only consider the rules for restricted quantification. We just consider the rules $(\bigvee \varepsilon \supset)$ and $(\supset \bigvee \varepsilon)$, since the arguments for $(\supset \bigwedge \varepsilon)$ and $(\bigwedge \varepsilon \supset)$ are dual.

Case 1. The last step in $\mathscr{D}$ is an application of $(\bigvee \varepsilon \supset)$ of the form

$$\frac{\Gamma \cdot \Gamma_1',\, w\, \varepsilon\, t,\, \chi(w) \supset \Delta \cdot \Delta'}{\Gamma \cdot \Gamma_1',\, \bigvee u\, \varepsilon\, t\, \chi(u) \supset \Delta \cdot \Delta'}$$

where $\bigvee u\, \varepsilon\, t\, \chi(u)$ is a part of Γ', so $\Gamma' = \Gamma_1', \bigvee u\, \varepsilon\, t\, \chi(u)$. It is required that w not be free in Γ, Γ', Δ, Δ' or t. By inductive hypothesis we get an A-formula θ and G_ε-derivations of $(\Gamma \supset \Delta,\, \theta)$ and $(\theta,\, \Gamma_1',\, w\, \varepsilon\, t,\, \chi(w) \supset \Delta')$ satisfying (i)-(iv) with Γ, Δ and Γ_1', $w\, \varepsilon\, t$, $\chi(w)$, Δ'. Then w is not free in θ and we can make the inference

$$\frac{\theta,\, \Gamma_1',\, w\, \varepsilon\, t,\, \chi(w) \supset \Delta'}{\theta,\, \Gamma_1',\, \bigvee u\, \varepsilon\, t\, \chi(u) \supset \Delta'}$$

Thus the same θ continues to work.

Case 2. Last step $(\bigvee \varepsilon \supset)$ with $\bigvee u\, \varepsilon\, t\, \chi(u)$ in Γ. The argument is similar.

Case 3. The last step in $\mathscr{D}$ is

$$\frac{\Gamma \cdot \Gamma_1',\, t_1\, \varepsilon\, t \supset \Delta \cdot \Delta_1',\, \chi(t_1)}{\Gamma \cdot \Gamma_1',\, t_1\, \varepsilon\, t \supset \Delta \cdot \Delta_1',\, \bigvee u\, \varepsilon\, t\, \chi(u)}$$

where $\Gamma' = \Gamma_1', t_1\, \varepsilon\, t$ and $\Delta' = \Delta_1', \bigvee u\, \varepsilon\, t\, \chi(u)$. Inductive hypothesis gives an A-formula θ and G_ε-derivations of $(\Gamma \supset \Delta,\, \theta)$ and $(\theta,\, \Gamma_1',\, t_1\, \varepsilon\, t \supset \Delta_1',\, \chi(t_1))$ satisfying the additional conditions (i)-(iv). Then we can infer θ, Γ_1', $t_1\, \varepsilon\, t \supset \Delta_1'$, $\bigvee u\, \varepsilon\, t\; \chi(u)$, and the same θ continues to satisfy (i)-(iv).

Case 4. The last step in $\mathscr{D}$ is

$$\frac{\Gamma_1 \cdot \Gamma',\, t_1\, \varepsilon\, t \supset \Delta \cdot \Delta_1',\, \chi(t_1)}{\Gamma_1 \cdot \Gamma',\, t_1\, \varepsilon\, t \supset \Delta \cdot \Delta_1',\, \bigvee u\, \varepsilon\, t\, \chi(u)}$$

where $\Gamma = \Gamma_1, t_1\, \varepsilon\, t$ and $\Delta' = \Delta_1', \bigvee u\, \varepsilon\, t\, \chi(u)$. Inductive hypothesis gives an A-formula θ and G_ε-derivations of

$$(\Gamma_1,\, t_1\, \varepsilon\, t \supset \Delta,\, \theta) \text{ and } (\theta,\, \Gamma' \supset \Delta_1',\, \chi(t_1))$$

satisfying (i)-(iv). We consider two subcases.

Sub-case 4(a). t_1 is a constant or a variable free in Γ', Δ_1', $\bigvee u\, \varepsilon\, t\, \chi(u)$. Using $\Gamma_1,\, t_1\, \varepsilon\, t \supset \Delta,\, t_1\, \varepsilon\, t$ we can infer

$$\Gamma_1,\, t_1\, \varepsilon\, t \supset \Delta,\, (t_1\, \varepsilon\, t \wedge \theta).$$

Also we make the steps

$$\frac{\dfrac{\dfrac{\theta,\, \Gamma' \supset \Delta_1,\, \chi(t_1)}{\theta,\, \Gamma',\, t_1\, \varepsilon\, t \supset \Delta_1',\, \chi(t_1)}}{\theta,\, \Gamma',\, t_1\, \varepsilon\, t \supset \Delta_1',\, \bigvee u\, \varepsilon\, t\, \chi(u)}}{(t_1\, \varepsilon\, t \wedge \theta),\, \Gamma' \supset \Delta_1',\, \bigvee u\, \varepsilon\, t\, \chi(u)}.$$

Hence a suitable interpolant here will be $(t_1\, \varepsilon\, t \wedge \theta(t_1))$.

Sub-case 4(b). t_1 is a variable w which is not free in Γ', Δ_1', $\bigvee u\, \varepsilon\, t\, \chi(u)$. Write $\theta = \theta(w)$ in this case (there is no harm if w is not actually free in θ). Now we make the steps:

$$\frac{\Gamma_1,\, w\, \varepsilon\, t \supset \Delta,\, \theta(w)}{\Gamma_1,\, w\, \varepsilon\, t \supset \Delta,\, \bigvee u\, \varepsilon\, t\, \theta(u)} \qquad \frac{\dfrac{\dfrac{\theta(w),\, \Gamma' \supset \Delta_1',\, \chi(w)}{\theta(w),\, \Gamma',\, w\, \varepsilon\, t \supset \Delta_1',\, \chi(w)}}{\theta(w),\, \Gamma',\, w\, \varepsilon\, t \supset \Delta_1',\, \bigvee u\, \varepsilon\, t\, \chi(u)}}{\bigvee u\, \varepsilon\, t\, \theta(u),\, \Gamma' \supset \Delta_1',\, \bigvee u\, \varepsilon\, t\, \chi(u).}$$

The last step on the right involves a permitted application of $(\bigvee \in \supset)$ by hypothesis on w. Then $\bigvee u\, \varepsilon\, t\, \theta(u)$ will satisfy the conditions (i)-(iv) for the final sequents.

Case 5. $(t_1\, \varepsilon\, t)$ in Γ and $(\bigvee u\, \varepsilon\, t\, \chi(u))$ in Δ; treated like Case 3.

Case 6. $(t_1\, \varepsilon\, t)$ in Γ' and $(\bigvee u\, \varepsilon\, t\, \chi(u))$ in Δ.

Sub-case 6(a). t_1 a constant or a variable free in $\Gamma, \Delta, \bigvee u\, \varepsilon\, t\, \chi(u)$; treated like sub-case 4(a).

Sub-case 6(b). t_1 a variable w not free in $\Gamma, \Delta, \bigvee u\, \varepsilon\, t\, \chi(u)$. Here we have

$$\frac{\dfrac{\dfrac{\Gamma \supset \Delta_1,\, \chi(w),\, \theta(w)}{\Gamma,\, w\, \varepsilon\, t \supset \Delta_1,\, \chi(w),\, \theta(w)}}{\Gamma,\, w\, \varepsilon\, t \supset \Delta_1,\, \bigvee u\, \varepsilon\, t\, \chi(u),\, \theta(w)}}{\Gamma \supset \Delta_1,\, \bigvee u\, \varepsilon\, t\, \chi(u),\, \bigwedge u\, \varepsilon\, t\, \theta(u)} \qquad \frac{\theta(w),\, \Gamma_1',\, w\, \varepsilon\, t \supset \Delta'}{\bigwedge u\, \varepsilon\, t\, \theta(u), \Gamma_1',\, w\, \varepsilon\, t \supset \Delta'}$$

so $\bigwedge u\, \varepsilon\, t\, \theta(u)$ works in this case.

It is thus seen that the restricted quantifier steps never require the introduction of unrestricted quantifiers in the interpolant. When it is necessary to introduce unrestricted quantifiers in the interpolant, this can be arranged to satisfy (iv) just in the same way as in the standard case.

The following is the analogue of the interpolation theorem 4.4 of [7] for the calculus with restricted quantification.

3.2 **Theorem.** *Suppose ϕ is essentially universal and ψ is essentially existential and that $(\phi \to \psi)$ has a G_ε-derivation in A. Then we can find an essentially restricted A-formula θ and G_ε-derivations in A of $(\phi \to \theta)$ and $(\theta \to \psi)$.*

Proof. This again is obtained from a more general statement about sequents: if Γ, Γ' consist of essentially universal formulas, Δ, Δ' of essentially existential formulas and $\Gamma \cdot \Gamma' \supset \Delta \cdot \Delta'$ has a G_ε-derivation $\mathscr{D}$ in A then we can find an essentially restricted A-formula θ and G_ε-derivations in A of $\Gamma \supset \Delta, \theta$ and $\theta, \Gamma' \supset \Delta'$.

This is proved by induction on $\mathscr{D}$, which is possible to apply because if the conclusion of a rule consists of ess. un. formulas in its antecedent and ess. ex. formulas in its consequent, the same holds for each of the hypotheses of the rule.

The argument takes a little different tack from that of 3.1 since now we have no hypothesis on the free variables of θ; but it is just this which permits the stronger conclusion on the structure of θ. The only cases where it may be necessary to introduce quantifiers in the interpolant is in the quantifier rules.

Consider, for example the rules for $\bigvee$. If the last step in $\mathscr{D}$ is

$$\frac{\Gamma \cdot \Gamma' \supset \Delta \cdot \Delta_1', \chi(t)}{\Gamma \cdot \Gamma' \supset \Delta \cdot \Delta_1', \bigvee u\, \chi(u)}$$

where $\Delta' = \Delta_1', \bigvee u\, \chi(u)$, use the inductive hypothesis to find ess. res. θ and derivations of

$$\Gamma \supset \Delta, \theta \text{ and } \theta, \Gamma' \supset \Delta_1', \chi(t).$$

Then we can infer $\theta, \Gamma' \supset \Delta_1', \bigvee u\, \chi(u)$, so the same θ will work. The argument is similar if $\bigvee u\, \chi(u)$ is in Δ. If the last step in $\mathscr{D}$ is

$$\frac{\Gamma \cdot \Gamma_1', \chi(w) \supset \Delta \cdot \Delta'}{\Gamma \cdot \Gamma_1', \bigvee u\, \chi(u) \supset \Delta \cdot \Delta'}$$

where $\Gamma' = \Gamma_1', \bigvee u\, \chi(u)$ and w is not free in $\Gamma, \Gamma_1', \Delta, \Delta'$ then u must be a variable of stationary sort since $\bigvee u\, \chi(u)$ is assumed ess. un. Now inductive hypothesis gives an ess. res. θ and derivations of

$$\Gamma \supset \Delta, \theta \text{ and } \theta, \Gamma_1', \chi(w) \supset \Delta'.$$

Write $\theta = \theta(w)$. Since w is not free in Γ, Δ we can make the inferences

$$\frac{\Gamma \supset \Delta, \theta(w)}{\Gamma \supset \Delta, \bigwedge u\, \theta(u)} \qquad \frac{\dfrac{\theta(w), \Gamma_1', \chi(w) \supset \Delta'}{\bigwedge u\, \theta(u), \Gamma_1', \chi(w) \supset \Delta'}}{\bigwedge u\, \theta(u), \Gamma_1', \bigvee u\, \chi(u) \supset \Delta'.}$$

The argument is similar if $\bigvee u\ \chi(u)$ is in Γ, but in that case we must take $\bigvee u\ \theta(u)$ as the new interpolant. Universal quantifiers are handled dually.

Consider, finally, restricted existential quantification. Again the rule $(\supset \bigvee \varepsilon)$ requires no change. Suppose the last step in $\mathscr{D}$ is

$$\frac{\Gamma \cdot \Gamma_1',\ w\ \varepsilon\ t,\ \chi(w) \supset \Delta'}{\Gamma \cdot \Gamma_1',\ \bigvee u\ \varepsilon\ t\ \chi(u) \supset \Delta'}$$

where w is not free in $\Gamma, \Gamma_1', \Delta, \Delta', t$ and $\Gamma' = \Gamma_1', \bigvee u\ \varepsilon\ t\ \chi(u)$. Applying inductive hypothesis gives $\theta = \theta(w)$ and derivations we extend as follows:

$$\cfrac{\cfrac{\Gamma \supset \Delta,\ \theta(w)}{\Gamma,\ w\ \varepsilon\ t \supset \Delta,\ \theta(w)}}{\Gamma \supset \Delta,\ \bigwedge u\ \varepsilon\ t\ \theta(u)} \qquad \cfrac{\cfrac{\theta(w),\ \Gamma_1',\ w\ \varepsilon\ t,\ \chi(w) \supset \Delta'}{\bigwedge u\ \varepsilon\ t\ \theta(u),\ \Gamma_1',\ w\ \varepsilon\ t,\ \chi(w) \supset \Delta'}}{\bigwedge u\ \varepsilon\ t\ \theta(u),\ \Gamma_1',\ \bigvee u\ \varepsilon\ t\ \chi(u) \supset \Delta'}$$

The other cases are just as easy to handle.

It should be noted that we can obtain the stronger conclusion in **3.2** that θ can be taken to have free variables among those common to ϕ and ψ provided there is a closed term of each sort in the language. For, simply substitute such a term for each variable of θ not held in common, according to sort. Simple examples show that the stronger conclusion does not hold without such an hypothesis. Note that it is not necessary to assume that σ is relational in this theorem.

4. The characterization theorems

We continue $(\mathrm{i})_A$—$(\mathrm{v})_A$; also *assume now that σ is relational.*

4.1 Theorem. *Suppose, in case $A \neq HF$, that S is countable and A-r.e., and suppose that the A-formula ϕ is $\leqq$-persistent relative to S. Then there is an essentially existential A-formula θ with the same free variables as ϕ such that $S \vdash (\phi \leftrightarrow \theta)$.*

Proof. In order to express the $\leqq$-persistence hypothesis syntactically, we consider an extended language $\mathscr{L}_A^+$ in which we can express relations between pairs of structures $\mathscr{M}, \mathscr{M}'$. Let J', I_0', I_1' be in 1—1 correspondence with and disjoint from J, I_0, I_1, respectively; for each $j \in J$, let j' be the corresponding element of J', and similarly for i in I_0 (or I_1) and i'. Let $J^+ = J \cup J'$, $I_0^+ = I_0 \cup I_0'$, $I_1^+ = I_1 \cup I_1'$, and

$$\sigma^+ = (J^+, I_0^+, I_1^+, \langle k_i^+ \rangle_{i \in I_0^+}, \langle l_i^+ \rangle_{i \in I_1^+})$$

(a relational signature) where $k_i^+ = k_{i'}^+ = k_i$ for $i \in I_0$, $l_i^+ = l_{i'}^+ = l_i$ for $i \in I_1$. Given any $\mathscr{M}$, $\mathscr{M}'$ of signature σ, let $[\mathscr{M}, \mathscr{M}']$ be the structure $\mathscr{M}^+ = (\langle M_j^+ \rangle_{j \in J^+}, \langle R_i^+ \rangle_{i \in I_0^+}, \langle a_i^+ \rangle_{i \in I_1^+})$ of signature σ^+ with $M_j^+ = M_j$ and $M_{j'}^+ = M_j'$ for $j \in J$, $R_i^+ = R_i$ and $R_{i'}^+ = R_i'$ for $i \in I_0$, and $a_i^+ = a_i$ and $a_{i'}^+ = a_i'$ for $i \in I_1$. Given $\mathscr{M}^+$ of signature σ^+, let $M_J^+ = \bigcup M_j^+ [j \in J]$, $M_{J'}^+ = \bigcup M_j^+ [j \in J']$. Then take

$$\mathscr{M}_J^+ = (\langle M_J^+ \rangle_{j \in J}, \langle R_i^+ \cap (M_J^+)^{k_i} \rangle_{i \in I_0}, \langle a_i^+ \rangle_{i \in I_1})$$

and $$\mathscr{M}_{J'}^+ = (\langle M_j^+ \rangle_{j \in J}, \langle R_{i'}^+ \cap (M_{J'}^+)^{k_i} \rangle_{i \in I_0}, \langle a_{i'}^+ \rangle_{i \in I_1}).$$

$\mathscr{M}^+$ is any σ^+ structure below. While $\mathscr{M}^+$ need not be the same as $[\mathscr{M}_J^+, \mathscr{M}_{J'}^+]$, it has the same elementary properties that concern us, as we shall see in (2) below.

$\mathscr{L}_A^+$ is taken to be the language of σ^+-structures; it contains the language $\mathscr{L}_A$ of σ-structures directly as a sublanguage. Given a variable u of the form $v_{j,n}$ with $j \in J_0$ we shall write u' for the variable $v_{j',n}$. In the following, $u, w, u_1, w_1, \cdots$ range only over variables of $\mathscr{L}_A$. We write c_i' for the constants $c_{i'}$, $i \in I_1$, and r_i' for the relations $r_{i'}$, $i \in I_0$. Given any formula ψ of $\mathscr{L}_A$, let ψ' be the formula obtained by replacing each constant, *bound* variable, or relation symbol s by the corresponding symbol s'. The free variables of ψ' are the same as those of ψ. S' is the set of sentences ψ' for $\psi \in S$.

We now define *Ext* to be the set consisting of all sentences of the following form:

(i) $\bigwedge u' \bigvee u \, (u' = u)$;

(ii) $\bigwedge u \bigvee u' \, (u = u')$, *for each variable u of stationary sort;*

(iii) $\bigwedge u_i', \cdots, \bigwedge u_{k_i}' [r_i(u_1', \cdots, u_{k_i}') \leftrightarrow r_i'(u_1', \cdots, u_{k_i}')]$ *for each* $i \in I_0$;

(iv) $c_i = c_i'$ *for each* $i \in I_1$;

(v) $\bigwedge u' \bigwedge w \,\varepsilon\, u' \bigvee w' \,\varepsilon\, u' \, (w = w')$.

Then we easily obtain the following:

(1) $\models_{\mathscr{M}^+} Ext$ *iff* $\mathscr{M}_{J'}^+ \leqq \mathscr{M}_J^+$.

(2) *If ψ is an A-formula with free variables $u_1, \cdots, u_t$ of sorts $j_1, \cdots, j_t$ resp. and*

$$(x_1, \cdots, x_t) \in M_{j_1}^+ \times \cdots \times M_{j_t}^+$$

and $$(y_1, \cdots, y_t) \in M_{j_1'}^+ \times \cdots \times M_{j_t'}^+$$

then $$\models_{\mathscr{M}^+} \psi[x_1, \cdots, x_t] \ \textit{iff} \ \models_{\mathscr{M}_J^+} \psi[x_1, \cdots, x_t],$$

and $$\models_{\mathscr{M}^+} \psi'[y_1, \cdots, y_t] \ \textit{iff} \ \models_{\mathscr{M}_{J'}^+} \psi[y_1, \cdots, y_t].$$

Let $u_1, \cdots, u_t$ be the free vars. of ϕ. It follows immediately that ϕ is persistent for outer extensions relative to S if and only if

$$(3)\ Ext \cup S \cup S' \cup \{\bigvee u'_i\, (u'_i = u_i) : i = 1, \cdots, t\} \vdash (\phi' \rightarrow \phi).$$

Now $Ext \cup S \cup S'$ is also A-r.e. if $A \neq HF$. Hence by the extension 2.8 of Barwise's compactness theorem and the completeness theorem 2.7, there are subsets Ext_1 of Ext and S_1 of S which belong to A for which we have an A-derivation of

$$(4)\ (\prod_{\chi \in Ext_1} \chi \wedge \prod_{\psi \in S_1} \psi' \wedge \prod_{i=1}^{t} \bigvee u'_i\, (u'_i = u_i) \wedge \phi) \rightarrow (\prod_{\psi \in S_1} \psi \rightarrow \phi).$$

We now apply the interpolation theorem 3.1. An interpolating A-formula θ can be chosen which is, by 3.1 (i) a formula with just $u_1, \cdots, u_t$ free, containing only relation symbols r_i by 3.1 (ii), and only bound variables of the language $\mathscr{L}_A$ by 3.1 (iii). We can also arrange that only constants of the form c_i appear in θ by using the sentences (iv) of Ext and taking Ext_1 large enough. Hence θ is just a formula of the language $\mathscr{L}_A$. Furthermore, θ is essentially existential by (3.1) (iv). For if there were any non-stationary variable u having an essentially universal occurrence in θ there would have to be a variable of the same sort with an essentially universal occurrence in some sentence of Ext; but this is false by inspection. Now from derivability of

$$(5)\qquad (\prod_{\chi \in Ext_1} \chi \wedge \prod_{\psi \in S_1} \psi' \wedge \prod_{i=1}^{t} \bigvee u'_i\, (u'_i = u_i) \wedge \phi') \rightarrow \theta$$

$$\text{and}\qquad \theta \rightarrow (\prod_{\psi \in S_1} \psi \rightarrow \phi)$$

we easily get a derivation of $\prod_{\psi \in S_1} \psi \rightarrow (\phi \leftrightarrow \theta)$ and hence $S \vdash (\phi \leftrightarrow \theta)$.

4.2 Theorem. *Suppose S satisfies the hypothesis of* 4.1 *and that, in addition, it is a set of essentially universal sentences. Suppose the A-formula ϕ is $\leqq$-invariant relative to S. Then there is an essentially restricted A-formula θ with $S \vdash (\phi \leftrightarrow \theta)$. θ can be chosen to have the same free variables as ϕ provided there is a closed term of each sort.*

Proof. By the preceding theorem we can find ess. un. ϕ_1 and ess. ex. ϕ_2 such that $S \vdash (\phi \leftrightarrow \phi_1)$ and $S \vdash (\phi \leftrightarrow \phi_2)$. By compactness we get $S_1 \subseteqq S$ with $S_1 \in A$ and an A-derivation of

$$(\prod_{\psi \in S_1} \psi \wedge \phi_1) \rightarrow \phi_2.$$

The hypothesis of this implication is also ess. un. Hence by

Theorem 3.2 we can find an essentially restricted A-formula θ as interpolant; then θ satisfies the conclusion of the theorem.

REMARK 4(a). The following example shows that it is not possible to weaken the hypothesis on S in 4.2. It applies to the finitary language of structures consisting of two sorts M_0, M_1 and just a binary relation E. Let $\psi(u_1)$ be

$$\bigvee u_0 \,\varepsilon\, u_1 \bigwedge w_0 \,\varepsilon\, u_1\, (u_0 = w_0), \text{ and}$$
$$S = \{\bigvee !\, u_1 \psi(u_1), \bigwedge u_0 \bigwedge u_1 \sim (u_1 \,\varepsilon\, u_0)\}.$$

Then let $\phi(u_0)$ be $\bigvee u_1(\psi(u_1) \wedge u_0 \,\varepsilon\, u_1)$; ϕ is invariant for outer extensions relative to S, since also $S \vdash \phi\, (u_0) \leftrightarrow \bigwedge u_1[\psi(u_1) \rightarrow u_0 \,\varepsilon\, u_1]$. On the other hand there is no essentially restricted θ with $S \vdash (\phi \leftrightarrow \theta)$.

There are, however, special circumstances in which the strong syntactic hypothesis on S in 4.2 can be omitted. These are stated in [8], using a certain notion of $\cap$-*invariance* relative to S as an intermediate. For simplicity, consider the usual language of set theory, with one sort of variable and ε as the only relation symbol besides $=$. Theorem 2 of [8] states that if (i) ε-closure is definable in S, then ϕ is $\cap$-*invariant* relative to S iff there is an essentially restricted formula θ such that $S \vdash (\phi \leftrightarrow \theta)$.[2] Furthermore, the following are sufficient conditions for showing that ϕ is $\cap$-invariant if it is $\leqq$-invariant (rel. to S): (ii) all models of S are well-founded and extensional, and (iii) the intersection of any two transitive models of S is again a model of S. Hypothesis (i) can be insured in finite type theory or with $A \neq HF$. The hypothesis (ii) can be insured by bounding the ranks of elements of models of S, with varying degrees of freedom according to the language $\mathscr{L}_A$ which is used; (iii) holds whenever the existential axioms of S have a predicative character, such as those considered in [5]. We believe these circumstances (i)—(iii) are too special to merit detailed consideration here. We simply remark that a more general version of Theorem 2 of [8] can be obtained for arbitrary $\mathscr{L}_A$ and S satisfying the hypotheses of 4.1, using a line of attack related to that in the proof of 4.1.

REMARK 4(b). The hypothesis that σ is a relational signature can be omitted from 4.1. To see this, take the hypotheses of 4.1 without this assumption. Now simply apply the standard proce-

[2] Due to an oversight, the following condition was omitted in [8] p. 482 from the explanation of when $\chi(u, w)$ defines ε-closure in S: if θ is a restricted formula then $\bigvee w[\chi(u, w) \wedge \theta]$ should be equivalent in S to a restricted formula.

dure for eliminating function symbols in favor of relation symbols. Then, of course, the statements of functionality of these relations must be adjoined to S, giving an S_{Rel}. By 4.1 we can get ess. ex. θ with $S_{Rel} \vdash (\phi_{Rel} \leftrightarrow \theta)$. θ may involve the new relation symbols; however, these can now be rewritten in terms of the original function symbols without introducing new quantifiers, giving $S \vdash (\phi \leftrightarrow \theta_1)$ for ess. ex. θ_1. Since it is not assumed in 3.2 that σ is relational, the same argument as for 4.2 above can be applied to obtain 4.2 in general from the general form of 4.1.

4(c) *Provably $\leqq$-persistent and $\leqq$-invariant formulas.*

For any of the systems of deduction considered here (and in particular, the system G_ε) we can introduce the notion, *$\mathscr{D}$ is a derivation of $(\Gamma \supset \Delta)$ from S*, meaning: the initial sequents in $\mathscr{D}$ are the logical axioms of the system or sequents ($\supset \psi$) for $\psi \in S$, and the rules of inference of $\mathscr{D}$ are those of the system together with the cut-rule, and $\mathscr{D}$ ends in $(\Gamma \supset \Delta)$. Write $S \vdash_A (\Gamma \supset \Delta)$ if there is a derivation $\mathscr{D}$ in A of $(\Gamma \supset \Delta)$ from S, and $S \vdash_A \phi$ if $S \vdash_A (\supset \phi)$. Then (for the systems considered) if $S \vdash_A (\Gamma \supset \Delta)$ we have $S \vdash (\Gamma \supset \Delta)$. On the other hand, if S is A-compact and $S \vdash (\Gamma \supset \Delta)$ then for some $S_1 \subseteqq S$, $S_1 \in A$ we have $\vdash (\prod_{\psi \in S_1} \psi, \Gamma \supset \Delta)$ and hence $S_1 \vdash_A (\Gamma \supset \Delta)$. Finally, it can be seen that if, in case $A \neq HF$, S is countable and A-r.e. and $\mathscr{D}$ is a derivation in A of $(\Gamma \supset \Delta)$ from S then the set S_1 of ψ in S with ($\supset \psi$) an initial sequent of $\mathscr{D}$ belongs to A and $S_1 \vdash_A (\Gamma \supset \Delta)$. In fact, this is the avenue which Barwise took in [1] to establish his compactness theorem.

Let us call ϕ (with free variables $u_1, \cdots, u_t$) *A-provably $\leqq$-persistent relative to S* if, in the notation of the proof of Theorem 4.1 we have

$$Ext \cup S \cup S' \cup \{\bigvee u_i' \, (u_i' = u_i) : i = 1, \cdots, t\} \vdash_A (\phi' \rightarrow \phi).$$

In these terms, the argument for 4.1 establishes two theorems, *under the hypothesis in case $A \neq HF$, that S is countable and A-r.e.*:

4.3 THEOREM. *Suppose that the A-formula ϕ is $\leqq$-persistent relative to S. Then ϕ is A-provably $\leqq$-persistent relative to S.*

4.4 THEOREM. *If ϕ is A-provably $\leqq$-persistent relative to S then there is an essentially existential A-formula θ with the same free variables as ϕ such that $S \vdash_A (\phi \leftrightarrow \theta)$.*

The argument for 4.4 simply consists in an application of the interpolation theorem 3.1 as in the part of the proof of 4.1 from (4) on.

Call ϕ *A-provably $\leqq$-invariant relative to S* if both it and $\sim \phi$ are A-provably $\leqq$-persistent rel. to S. Then what is essential in the proof of Theorem 4.2 is that it establishes the following syntactic version.

4.5 THEOREM. *The Theorem* 4.2 *remains true if we replace* "$\leqq$-invariant" *by* "*A-provably $\leqq$-invariant*" *and* "$\vdash$" *by* "$\vdash_A$".

The results 4.4 and 4.5 are generalizations of those announced in [5], p. 489, and [6], theorem 2.3 and § 5.1. For applications to formal systems in the usual sense, one takes $A = HF$. In this case, the arguments for the interpolation theorems 3.1 and 3.2 and for 4.4 and 4.5 are completely finitistic.

The main notions and results of this paper can be generalized still further, beginning with the notion of outer extension as follows. Let $I_0^{\#}$ be a subset of I_0 and for each $i \in I_0^{\#}$, let $k_i \geqq 2$ and $0 < n_i < k_i$. Relative to this choice, we modify Definition 2.1 (vi) of $\mathscr{M}' \leqq \mathscr{M}$ to read:

(iv)$^{\#}$ *for each* $i \in I_0^{\#}$, $y_1, \cdots, y_{n_i}$ *in* $\mathscr{M}$ *and* $x_1, \cdots, x_{k_i-n_i}$ *in* $\mathscr{M}'$, *if* $(y_1, \cdots, y_{n_i}, x_1, \cdots, x_{k_i-n_i}) \in R_i$ *then* $y_1, \cdots, y_{n_i}$ *in* $\mathscr{M}'$.

The earlier condition (iv) is the special case of this where $I_0^{\#} = \{0\}$, $k_0 = 2$ and $n_0 = 1$ (and ordinary extension simply the case where $I_0^{\#} = 0$). To emphasize the similarity, write $(y_1, \cdots, y_{n_i} E_i x_1, \cdots, x_{k_i-n_i})$ for $(y_1, \cdots, y_{n_i}, x_1, \cdots, x_{k_i-n_i}) \in R_i$. The associated language with restricted quantification would have "quantifiers" $Q(u_1, \cdots, u_{n_i} \varepsilon_i t_i, \cdots, t_{k_i-n_i})\phi$, $Q = \bigvee$ and $\bigwedge$. A deductive calculus can be set up for these very similar to the G_ε-system. Then, with the various definitions appropriately modified, it can be seen that all of the results of this paper continue to hold for these more general notions of outer extension and restricted quantification. The argument is only slightly more delicate in the case of the interpolation theorem 3.1, where more subcases like 4(a), 4(b), 6(a), 6(b) must be considered. This generalization is of particular interest for applications to higher type theories of relations as well as sets, where $n_i = k_i + 1$ and $(y_1, \cdots, y_{n_i} E_i x)$ holds in standard models if $(y_1, \cdots, y_{n_i})$ is an element of x.

Added in proof: The text [21] of Kreisel-Krivine contains a detailed treatment of finite type theory of relations in $\mathscr{L}_{HF}$ and $\mathscr{L}_{HC}$ and in particular of results on $\cap$-invariance for the case $A = HF$ (l.c., pp. 119, 129).

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