The Operational Perspective: Three Routes

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For Gerhard Jäger, in honor of his 60th birthday.

Let me begin with a few personal words of appreciation, since Gerhard Jäger is one of my most valued friends and long time collaborators. It's my pleasure to add my tribute to him for his outstanding achievements and leadership over the years, and most of all for having such a wonderful open spirit and being such a fine person.

I first met Gerhard at the 1978 logic colloquium meeting in Mons, Belgium. He was attending that with Wolfram Pohlers and Wilfried Buchholz, with both of whom I had long enjoyed a stimulating working relationship on theories of iterated inductive definitions. From our casual conversations there, it was clear that Gerhard was already someone with great promise in proof theory. But things really took off between us a year later when we both visited Oxford University for the academic year 1979-1980. Gerhard had just finished his doctoral dissertation with Kurt Schütte and Wolfram Pohlers. I remember that we did a lot of walking and talking together, though I had to walk twice as fast to keep up with him. We talked a lot about proof theory and in particular about my explicit mathematics program that I had introduced in 1975 and had expanded on in my Mons lectures; Gerhard was quick to take up all my questions and to deal with them effectively. Since then, as hardly needs saying, he became a leader in the development of the proof theory of systems of explicit mathematics and related systems in the applicative/operational framework (among many contributions to a number of other areas), and he went on to establish in Bern a world center for studies in these subjects.

It was also through Gerhard that I was able to come in useful contact with a number of his students and members of his group, and most particularly with Thomas Strahm, who then became a second very important collaborator of mine, both on explicit mathematics and the unfolding program, of which I'll say something below. Since some time now, Gerhard and Thomas have been working with me on a book on the foundations of explicit mathematics, and in the last few years we have made great progress on that with the assistance of my former student Ulrik Buchholtz, for which I'm very grateful.[1] Finally on the personal front I want to add that my wife, Anita, and I have had the pleasure over the years of visiting Gerhard and his family, first in Zürich, and then in Bern, and we want to thank him and his wife Corinna for their generous, ever-ready welcome and hospitality.

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This article consists of four sections, beginning in sec. 1 with an explanation of the general features of the operational perspective. That is then illustrated in the remaining three sections by the explicit mathematics program, operational set theory, and the unfolding program, resp. The material of this article is by no means exhaustive of the work carried out under the operational perspective; instead, it concentrates on those areas with which I have been personally involved and that I thus know best, but references with a wider scope are given where relevant. I have two readers in mind: the general reader with a background in logic on the one hand and the expert in applicative theories on the other. For the former I have emphasized the aims of the work and filled in its background. For the latter, I have added new points toward the end of each of secs. 2-4 that I hope will be worthy of attention. In particular, sec. 2 has material on the development of constructive and predicative mathematics in systems of explicit mathematics, sec. 3 deals with problems that arose in my development of OST and sketches Gerhard Jäger's solution to them, and sec. 4 concludes with new conjectures on the unfolding of systems of operational set theory.

1. The operational perspective. Operations are ubiquitous in mathematics but not adequately accounted for in current global (or universal) foundational schemes. In particular, the only operations that have a direct explanation in set theory are those represented by functions *qua* many-one relations, so cannot explain operations such as union and power set that are supposed to be applicable to arbitrary sets. The attempt of

[1] Let me also take this opportunity to thank Thomas Strahm and Thomas Studer for organizing the December 2014 meeting in honor of Gerhard Jäger, and for arranging for me to participate via Skype since I was unable to attend in person.

Church (1932, 1933) to provide a foundation of mathematics in purely operational terms that would be an alternative to set theory was shown to be inconsistent, and later efforts at similar programs such as that of Fitch (1963) have had only very limited success. In any case, one should not expect a "one size fits all" theory of operations; witness the great conceptual variety of computational, algebraic, analytic and logical operations among others. Nevertheless, there is a core theory of operations that can readily be adapted to a number of local purposes by suitable expansions in each case. This has the following features:

(i) Operations are in general allowed to be partial. For example, the operations of division in algebra and integration and differentiation in analysis are not everywhere defined.

(ii) Operations may be applied to operations. For example, one has the operation of composition of two operations, the operation of n-times iteration of a given operation, the "do…until…" operation of indefinite iteration, etc.

(iii) In consequence of (ii), a generally adaptable theory of operations is type-free.

(iv) Extensionality is not assumed for operations. For example, the theory should allow the indices of partial recursive functions to appear as one model.

(v) The language of the theory is at least as expressive as the untyped lambda-calculus and the untyped combinatory calculus.

(vi) Though logical operations of various kinds on propositions and predicates may appear in particular applications, first-order classical or intuitionistic predicate logic is taken as given.

These features form the general operational perspective.

In accordance with (i)-(iii), in any particular expansion of the basic theory, we will want some way of introducing *application terms* s, t, u,…, generated from variables and constants by closure under application, written $s(t)$ or st, and then to express that a term t is defined, in symbols $t\downarrow$. In the original operational approach that I took in my article (F 1975) on explicit mathematics, the basic relations included a three place relation App(x, y, z), informally read as expressing that the operation x applied to y has the value z. Then application (pseudo-)terms were introduced contextually, first in (pseudo-)formulas $t \simeq z$ expressing that t is defined and has the value z: for t a variable or constant, this is simply taken to be ordinary equality, while for t of the form $t_1 t_2$, this is taken to be $\exists x,y(t_1 \simeq x \wedge t_2 \simeq y \wedge \mathrm{App}(x, y, z))$. Finally, $t\downarrow$ is defined to be $\exists z(t \simeq z)$ and $t_1 \simeq t_2$ is defined to be $\forall z(t_1 \simeq z \leftrightarrow t_2 \simeq z)$. An elegant alternative to this approach was provided by Beeson in 1981 (cf. (Beeson 1985) Ch. VI.1) under the rubric, the Logic of Partial Terms (LPT). In LPT, terms are now first class citizens and the expressions $t\downarrow$ are taken to be basic formulas governed by a few simple axioms and rules, among which we have suitable restrictions on universal and existential instantiation. In this system, $t_1 \simeq t_2$ is defined by the formula $t_1\downarrow \vee\, t_2\downarrow \rightarrow t_1 = t_2$. Most of the work after 1981 on the systems within the operational perspective has taken LPT as basic, but we will see in sec. 3 below that there may still be cases where the original approach is advantageous. Note that LPT contains the "strictness" axioms that if a relation $\mathrm{R}(t_1,\ldots,t_n)$ holds then $t_i\downarrow$ holds for each i; in particular, that is the case for the equality relation. By $t_1 t_2 \ldots t_n$ we mean the result of successive application by association to the left.

The minimal theory we use has two basic constants **k** and **s** (corresponding to Curry's combinators **K** and **S**) with axioms: (i) $\mathbf{k}xy = y$, and (ii) $\mathbf{s}xy\downarrow \rightarrow \mathbf{s}xyz \simeq xz(yz)$. These serve to imply that with any term $t(x,\ldots)$ we may associate a term $\lambda x.t(x,\ldots)$ in which the variable x is not free, and is such that $(\lambda x.t(x,\ldots))y \simeq t(y,\ldots)$. Moreover we can construct a universal recursor or fixed point operator **r** (sometimes denoted **rec**), i.e. a term for which $\mathbf{r}f\downarrow \wedge \mathbf{r}fx \simeq f(\mathbf{r}f)x$ is provable. In all the applications of the operational perspective below, axioms (i) and (ii) are further supplemented by suitable axioms for pairing and projection operations $\mathbf{p}$, $\mathbf{p}_0$, and $\mathbf{p}_1$, and definition by cases $\mathbf{d}$.[2] For the purposes below, let's call these *the basic operational axioms*, whether with respect to the App formulation or that in LPT.

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[2] There are several possible formulations of the definition by cases operator. In the one originally taken in (F 1975), sometimes called definition by cases on V, this takes the form $\mathbf{d}xyuv = (x \text{ if } u = v, \text{ else } y)$. However, when added to the axioms for **k** and **s**, extensionality is inconsistent for operations. More restrictive versions have subsequently been used, mainly definition by cases on the natural numbers, allowing both extensionality and totality of operations; cf. Jäger and Strahm (1995).

2. Explicit Mathematics. My work with the operational approach began with the explicit mathematics program in (F 1975). Here is what led me to that. In the years following the characterization (F 1964, Schütte 1965) of predicative analysis in terms of an autonomous progression of ramified analytic systems whose limit is at the ordinal Γ_0, I had explored various ways to simplify conceptually the formal treatment of predicativity via unramified systems.[3] Moreover, that would be important to see which parts of mathematical practice could be accounted for on predicative grounds going beyond (Weyl 1918). Independently of that work, in (F 1971) I had made use of extensions of Gödel's functional ("Dialectica") interpretation to determine the proof-theoretical strength of various subsystems of analysis by the adjunction of the unbounded minimum operator as well as the Suslin-Kleene operator. The two pursuits came closer together in the article "Theories of finite type related to mathematical practice" (F 1977) for the *Handbook of Mathematical Logic*. As with Gödel's interpretation, that made use of the functional finite type structure over the natural numbers.

Meanwhile, Errett Bishop's novel informal approach to constructive analysis (Bishop 1967) had made a big impression on me and I was interested in seeing what kind of more or less direct axiomatic foundation could be given for it that would explain how it managed to look so much like classical analysis in practice while admitting a constructive interpretation. Closer inspection showed that this depended on dealing with all kinds of objects (numbers, functions, sets, etc.) needed for analysis as if they are given by explicit presentations, each kind with an appropriate "equality" relation, and that operations on them are conceived to lead from and to such presentations preserving the given equality relations. In other words, the objects are conceived of as given *intensionally*, while a classical reading is obtained by instead working *extensionally* with the equivalence classes with respect to the given equality relations. Another aspect of Bishop's work that was more specific to its success was his systematic use of witnessing data as part of what constitutes a given object, such as modulus of convergence for a real number and modulus of (uniform) continuity for a function of real numbers. Finally, his

[3] First steps in that direction had already been made in (F 1964) via the system IR. For subsequent explorations cf. (F 1968, 1974 and 1979a).

development did not require restriction to intuitionistic logic (though Bishop himself abjured the Law of Excluded Middle).

Stripped to its core, the ontology of Bishop's work is given by a universe of objects, each conceived to be given explicitly, among which are operations and classes (*qua* classifications). This led to my initial formulation of a system T_0 of explicit mathematics in (F 1975) in which that approach to constructive mathematics could be directly formalized. In addition, I introduced a second system T_1, obtained by the adjunction of the unbounded minimum operator so as to include a foundation of predicative mathematics. The theory T_0 was formulated in a single sorted language with basic relations =, App, Cl, and η where Cl(x) expresses that x is a class(ification) and $y \eta x$ expresses that y has the property given by x when Cl(x) holds. Variables A, B, C,… X, Y, Z, are introduced to range over the objects satisfying Cl, and $y \in X$ is also written for $y \eta x$ where $x = X$. The basic logic of T_0 is the classical first-order predicate calculus.[4] The axioms of T_0 include the basic operational axioms, and the remaining axioms are operationally given class existence axioms. For example, we have an operation **prod** which takes any pair X, Y of classes to produce their cartesian product, $X \times Y$ and another operation **exp** which takes X, Y to the cartesian power Y^X, also written $X \rightarrow Y$. The formation of such classes is governed by an *Elementary Comprehension Axiom* scheme (ECA) that tells which properties determine classes in a uniform way from given classes. These are given by formulas φ in which classes may be used as parameters to the right of the membership relation and in which we do not quantify over classes, and the uniformity is provided by operations $\mathbf{c}_\varphi$ applied to the parameters of φ.[5] But to form general products we need further notions and an additional axiom. Given a class I, by an I-termed sequence of classes is meant an operation f with domain I such that for each $i \in I$ the value of $f(i)$ is a class X_i; one wishes to use this to define $\Pi X_i[i \in I]$. It turns out that in combination with ECA a more basic operation is that of forming the join (or disjoint sum) $\sum X_i[i \in I]$ whose members are all pairs (i, y) such that $y \in X_i$; an additional *Join*

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4 In (F 1979) I also examined T_0 within intuitionistic logic.

5 The scheme ECA can be finitely axiomatized by adding constants for the identity relation, the first-order logical operations for negation, conjunction, existential quantification, and inverse image of a class under an operation.

axiom (J) is needed to assure existence of the join as given by an operation $\mathbf{j}(I, f)$. Finally, we have an operation $\mathbf{i}(A, R)$ and associated axiom (IG) for *Inductive Generation* which produces the class of objects accessible under the relation R (a class of ordered pairs) hereditarily within the class A. In particular, IG may be used to produce the class **N** of natural numbers, then the class **O** of countable tree ordinals, and so on.

In later expositions of systems of explicit mathematics, the language of LPT was used instead of the App relation for the operational basis, and the natural numbers **N** were taken to be a basic class for which several forms of the principle of induction were distinguished for proof theoretic purposes, as will be explained below. Also, in the approach to the formalization of Explicit Mathematics due to Jäger (1988), it turned out to be more convenient to treat classes extensionally but each with many possible representations within the universe **V** of individuals.[6] Membership has its usual meaning, but a new basic relation is needed, namely that an object x *names* or *represents* the class X, written $\mathfrak{R}(x, X)$. In these terms, for example, one has operations **prod** and **exp** such that whenever $\mathfrak{R}(x, X)$ and $\mathfrak{R}(y, Y)$ hold then $\mathfrak{R}(\mathbf{prod}(x, y), X \times Y)$ and $\mathfrak{R}(\boldsymbol{exp}(x, y), X \to Y)$ hold.

The only difference of T_1 from T_0 lies in the adjunction of a numerical choice operator $\boldsymbol{\mu}$ as a basic constant, together with the axiom:

(μ) $\quad f \in (\mathbf{N} \to \mathbf{N}) \to \boldsymbol{\mu} f \in \mathbf{N} \wedge [\exists x(fx = 0) \to f(\boldsymbol{\mu} f) = 0]$,[7]

from which the unbounded least number operator can be defined. This is equivalent to assumption of the operator $\mathbf{E}_0$ for quantification over **N**. Later on a third system T_2 was introduced by adjoining a constant for the Suslin-Kleene operator $\mathbf{E_1}$ for choosing a descending sequences g from a non-well-founded tree in the natural numbers represented by an operation f. Models of the basic operational axioms of T_0 are provided in the natural numbers by taking App(x, y, z) to be the relation $\{x\}(y) \simeq z$; thus the extensions of the total operations are just the recursive functions. Similarly, a model of the operational

[6] There is a difference in terminology, though: Jäger used 'types' for our classes.
[7] In certain subsystems of T_1 with restricted induction we need to add to the (μ) axiom that if $\boldsymbol{\mu} f \in \mathbf{N}$ then $f \in (\mathbf{N} \to \mathbf{N})$.

part of T_1 is given by indices of partial $\prod^1_1$ functions, so in this case the extensions of the total operations are just the hyperarithmetic functions. Finally, in the case of T_2, one uses indices of the functions partial recursive in the E_1 functional. In general, given any model (A, App,…) of the operational axioms with or without these special operators, one obtains a model of the class construction axioms by a transfinite inductive definition of names of classes with suitable codes for the operations on classes.[8]

The proof-theoretical study of subsystems of T_0 began in (F 1975) and was continued in (F 1979). Since then the proof theory of subsystems of the T_i (either given directly or by interpretation) has greatly proliferated and has become the dominant part of research in explicit mathematics, continuing until this day. The paper Jäger, Kahle and Strahm (1999) provides a useful survey of a considerable part of such work that begins with a relatively weak theory BON (Basic Theory of Operations and Numbers). That adds to the applicative language the constants **N, 0**, and **sc** as well as a constant $\mathbf{r_N}$ for primitive recursion on **N**. Over the basic operational theory, BON has the usual axioms for 0 and successor; for primitive recursion, we have an axiom which asserts that for arbitrary f and g, total on **N** and $\mathbf{N}^3$ (each to **N**), resp., the operation $h = \mathbf{r_N} fg$ is total on $\mathbf{N}^2$ to **N**, and satisfies $hx\mathbf{0} = fx$ and $hx(\mathbf{sc}(y)) = gxy(hxy)$. Several forms of induction are considered over BON; the full scheme, called *formula induction*, (F-I_N) is of the usual form for each formula $\varphi(x)$ in the language, namely $\varphi(\mathbf{0}) \wedge (\forall x \in \mathbf{N})(\varphi(x) \rightarrow \varphi(\mathbf{sc}(x)) \rightarrow (\forall x \in \mathbf{N})\varphi(x)$. The single special case of this for $\varphi(x)$ of the form $fx = \mathbf{0}$ (where f is a variable) is called *operation induction* (O-I_N), and when f is further assumed to be total from **N** to $\{\mathbf{0}, \mathbf{1}\}$ that is called *set induction* (S-I_N). Finally, the case for $\varphi(x)$ of the form $fx \in \mathbf{N}$ is called **N**-*induction* (N-I_N). The paper Jäger, Kahle and Strahm (1999) summarizes the proof-theoretical strength of all combinations of these with BON and then with BON plus the

[8] Parts of T_0 relate to Aczel's Frege structures and Martin-Löf's constructive theory of types; cf. for example, Beeson (1985), Chs. XI and XVII. But neither of these approaches goes on to the adjunction of non-constructive functional operators like μ (or E_0) and E_1.

axioms for $\boldsymbol{\mu}$ and $\mathbf{E}_0$.[9] For reasons to be seen in a moment, let me single out only two of their theorems:

(i) BON + (F-I_N) $\equiv$ PA, and

(ii) BON(μ) + (S-I_N) $\equiv$ PA,

where $\equiv$ is the relation of proof-theoretical equivalence; we also have conservation of the l.h.s. over the r.h.s in each of these results. (BON(μ) is BON plus the (μ) axiom.) The result (i) is part of the folklore of the subject, and (ii) was established in Feferman and Jäger (1993).

Let us now look at these systems within the language of T_0 and T_1, resp. There it is natural to also consider *class induction* (C-I_N), i.e. the case of the induction scheme where $\varphi(x)$ is of the form $x \in X$. Under the Elementary Comprehension Axiom scheme (ECA), that implies (F-I_N) for the formulas of the language of BON. Moreover, under the assumption of the (μ) axioms we can alternatively use set induction to obtain all those instances. Put in these terms it turns out that we have the following from Feferman and Jäger (1996):

(iii) BON +ECA + (C-I_N) $\equiv$ PA, and

(iv) BON(μ)+ ECA + (S-I_N) $\equiv$ PA.

These results are of significance with respect to the question: what parts of mathematics are accounted for in different parts of the explicit mathematics systems? The results (iii) and (iv) are relevant to constructive and predicative mathematics, resp., as follows.

A careful examination of Bishop and Bridges (1985)—a reworking and expansion of Bishop (1967)—shows that all its work in constructive analysis can be formalized in the theory BON + ECA + (C-I_N), hence requires no more principles for its justification than given by Peano Arithmetic. The typical choice of notions and style of argument is presented in (F 1979), pp. 176ff. Closer inspection shows that much of Bishop and Bridges (1985) can already be carried out in BON + ECA + (S-I_N), which is equivalent in

[9] For the proof theory of systems of explicit mathematics with $\mathbf{E}_1$ see Jäger and Strahm (2002) and Jäger and Probst (2011).

strength to PRA. Although the theory of measure and integration presented in Bishop (1967) made use of Borel sets, and thus of the countable join of classes and the countable tree ordinals, Bishop and Bridges (1985) substituted for that an approach to measure and integration that does not require J or IG at all.

Turning now to predicative mathematics, it is easily seen that all the redevelopment of 19th century analysis on those grounds as sketched in (Weyl 1918) can be carried out in the system BON(μ)+ ECA + (S-I_N) of (iv) above. The natural question to be raised is how much of modern analysis can be carried out in that system. In that respect we can make use of extensive detailed notes that I prepared in the period 1977-1981 but never published at the time; a scanned copy of those notes with an up-to-date introduction is now available in (F 2013). That work supports my conjecture (F 1988, 1993) that all scientifically applicable mathematics can be formalized in a system conservative over PA, namely BON(μ) + ECA + (S-I_N). To carry this out in the case of 19th c. analysis, systematic use is made of Cauchy completeness rather than the impredicative l.u.b. principle, and sequential compactness is used in place of the Heine-Borel theorem. Then for 20th c. analysis, Lebesgue measurable sets and functions are introduced directly via the Daniell approach without first going through the impredicative operation of outer measure; the existence of non-measurable sets cannot be proved in the system. Moving on to functional analysis, again the "positive" theory can be developed, at least for separable Banach and Hilbert spaces, and can be applied to various L_p spaces as principal examples. Among the general results that are obtained are usable forms of the Riesz Representation Theorem, the Hahn-Banach Theorem, the Uniform Boundedness Theorem, and the Open Mapping Theorem. The notes conclude with the spectral theory for compact self-adjoint operators on a separable Hilbert space.

This of course invites comparison with the work of Simpson (1988) that examines various parts of mathematics from the standpoint of the Reverse Mathematics program initiated by Harvey Friedman. That centers on five subsystems of second order arithmetic: RCA_0, WKL_0, ACA_0, ATR_0 and Π^1_1-CA_0. Each of these beyond the first is given by a single second-order axiom scheme in addition to the induction axiom for N in the form (C-I_N). In contrast to our work, which permits the free representation of practice in the full variable finite type structure over N, all mathematical notions considered by

Simpson are represented in the second-order language by means of considerable coding. The main aim of the Reverse Mathematics program is to show that for a substantial part of practice, if a given mathematical theorem follows from a suitable one of the five axioms above then it is equivalent to it, i.e. the implication can be reversed. For comparison with our work, much of predicative analysis falls under these kinds of results obtained for WKL_0 and ACA_0, of proof-theoretical strength PRA and PA respectively. Thus, on the one hand Simpson's results are more informative than ours, since the strength of various individual theorems of analysis is sharply determined. On the other hand, the exposition for the work in WKL_0 and ACA_0 is not easily read as a systematic development of predicative analysis, as it is in our notes. Still, the Simpson book is recommended as a rich resource of other interesting results that could be incorporated into our approach through explicit mathematics.

Of course, predicative analysis—as measured by the explication of (F 1964) and Schütte (1965)—in principle goes far beyond what can be reduced to PA. First of all, there are a number of interesting subsystems of second-order arithmetic that are of the same proof-theoretical strength as the union of the ramified analytic systems up to Γ_0. Among these we have the system Σ^1_1-DC + BR, where BR is the Bar Rule; the proof-theoretical equivalence in this case was first established in Feferman (1979a) and later (as a special case of a more general statement) in Feferman and Jäger (1983). In the latter publication, another system of this type is formulated as the autonomous iteration of the Π^0_1 comprehension axiom. Finally, Friedman, McAloon and Simpson (1982) showed by model-theoretic methods that the system ATR_0 is also of the same strength as full predicative analysis.[10] Since that may be given by a single axiom over ACA_0 (cf., *ibid.* p. 204), it follows that results in analysis and other parts of mathematics that are provably equivalent to (that axiom of) ATR_0 are impredicative. Simpson (1998, 2010) gives a number of examples of theorems from descriptive set theory that are equivalent to ATR_0, such as that every uncountable closed (or analytic) set contains a perfect subset.

By contrast to these systems of second order arithmetic, in (F 1975) I conjectured

[10] Then Jäger (1984) and Avigad (1996) showed that they are of the same proof-theoretical strength, by proof-theoretical methods, the first via theories of iterated admissible sets without foundation and the second via fixed point theories.

that a certain subsystem $T_1^{(N)}$ of T_1 is equivalent in strength to predicative analysis; in the notation here, that system may be written as BON(μ) + ECA + (F-I_N) + J. However, Glass and Strahm (1996) showed that $T_1^{(N)}$ is proof-theoretically equivalent to the iteration of Π^0_1-CA through all $\alpha < \varphi\varepsilon_0 0$, hence still far below Γ_0. It is an open question whether there is a natural subsystem of T_1 of strength full predicative analysis. Marzetta and Strahm (1998) give a partial answer to this question by the employment of axioms for universes.[11]

Finally, let us turn to the evaluation of the proof-theoretical strength of the full systems T_0 and T_1 of explicit mathematics. In unpublished notes from 1976, I showed how to interpret T_0 in Δ^1_2-CA + BI (cf. (F 1979a) p. 218). Then Jäger and Pohlers (1982) determined an upper bound for the proof-theoretic ordinal—call it κ—of the latter system, and Jäger (1983) gave a proof in T_0 of transfinite induction on α for each $\alpha < \kappa$, thus closing the circle.[12] One of the main results of Glass and Strahm (1996) is that proof-theoretically, T_1 is no stronger than T_0. In a personal communication, Dieter Probst has sketched arguments to show that also T_2 is no stronger than T_0, but natural variants of E_1 lead to stronger systems.[13]

NB. Currently, work is well advanced on a book being coauthored with Gerhard Jäger and Thomas Strahm with the assistance of Ulrik Buchholtz in which much of the foundations of explicit mathematics will be exposited in a systematic way. In the meantime, Buchholtz has set up an online bibliography of explicit mathematics and closely related topics at http://www.iam.unibe.ch/~til/em_bibliography/ that can be

[11] In sec. 4 below I conjecture that the unfolding of a suitable subsystem of T_0 is equivalent in strength to predicative analysis.

[12] Recently, Sato (2014) has shown how to establish the reduction of Δ^1_2-CA + BI to T_0 without going through the ordinal notation system for κ.

[13] Another interesting group of questions concerns the strength over T_0 (or its restricted version $T_0\restriction$) of the principle MID that I introduced in (F1982). That expresses that if *f* is any monotone operation from classes to classes then *f* has a least fixed point. Takahashi (1989) showed that T_0 + MID is interpretable in Π^1_2-CA + BI, and then Rathjen (1996) showed that it is much stronger than T_0. Next, exact strength of $T_0\restriction$ + MID was determined by Glass, Rathjen, and Schlüter (1997). A series of further results by Rathjen for the strength of T_0 + MID and T_0 + UMID, where UMID is a natural uniform version of the principle, are surveyed in the paper Rathjen (1998).

searched chronologically or by author and by title. The plans are to maintain this independently of the publication of the book. At the time of writing it consists of 127 items; readers are encouraged to let us know if there are further items that should be added.

3. Operational set theory. I introduced operational set theory in notes (F 2001), eventually published in detail in (F 2009).[14] This is an applicative based reformulation and extension of some systems of classical set theory ranging in strength from KP to ZFC and beyond. For the system of strength ZFC, this goes back in spirit to the theory of sets and functions due to von Neumann (1925), but Beeson (1988) is a direct predecessor as an operational theory. In addition to extending the range of this approach through intermediate systems down to KP, my work differs from both of these in its primary concern, namely to state various large cardinal notions in general applicative terms and, among other things, use those to explain admissible analogues in the literature. Significant further work on the strength of various systems of operational set theory has been carried out by Jäger (2007, 2009, 2009a, 2013) and Jäger and Zumbrunnen (2012).[15] The last of these is of particular significance for the following, since it shows that one of the main conjectures of (F 2009) is wrong and that it (and related other conjectures) need to be modified in order to obtain the intended consequences; Gerhard Jäger has suggested two ways to do that that will be described later in this section.

The system OST allows us to explain in uniform operational terms the informal idea from Zermelo (1908) that any definite property of elements of a set determines the subset of that set separated by the given property. Namely, represent the truth values "truth" and "falsity" by 1 and 0, resp., and let B = {0, 1}. In the applicative extension of the usual set-theoretical language, write $f : a \to b$ for $\forall x(x \in a \to fx \in b)$, and $f : a \to \mathrm{V}$ for $\forall x(x \in a \to fx\downarrow)$. Then definite properties of subsets of a set a may be identified with

[14] To explain some anomalies of the dates of subsequent work on this subject, it should be noted that my 2009 paper was submitted to the journal *Information and Control* in December 2006 and in revised form in April 2008. In the meantime, Jäger (2007) had appeared and so I could refer to it in that revised version.
[15] Further important work is contained in Zumbrunnen (2013) and Sato and Zumbrunnen (2014). Constructive operational theories of sets have been treated by Cantini and Crosilla (2008, 2010) and Cantini (2011).

operations $f: a \to$ B, and the uniform separation principle is given by an operation S such that for each such a and f, S(f, a) is defined and exists as a set, $\{x\colon x \in a \wedge fx = 1\}$. Furthermore, it allows us to explain in uniform operational terms the idea from von Neumann (1925) that if a is a set and f is an operation from a into the universe of sets then the range of f on a is a set. Namely, we have an operation R such that for each $f: a \to$ V, R(f, a) is defined and exists as a set, $\{y\colon \exists x(x \in a \wedge y = fx)\}$. Finally, OST allows us to express a uniform form of global choice by means of an operation C such that for each f, $\exists x(fx = 1)$ then Cf is defined and f(Cf) = 1.

In a little more detail, here is a description of the theory OST essentially as presented in (F 2009). Its language is an expansion of the language of usual set theory by the atomic applicative formulas, together with a number of constants to be specified along with their axioms. The basic logic is LPT, the logic of partial terms. The axioms of OST fall into five groups. Group 1 axioms are those for **k** and **s**. Group 2 axioms consist of extensionality and the existence of the empty set **0**, unordered pair, union and $\boldsymbol{\omega}$; we take **1** = {**0**} and **B** = {**0**, **1**}. Group 3 axioms are for the logical operations **el**, **cnj**, **neg** and $\mathbf{uni}_b$ (bounded universal quantification), with the obvious intended axioms, the last of which is that if $f: a \to \mathbf{B}$ then $\mathbf{uni}_b(f, a) \in$ B and $\mathbf{uni}_b(f, a) = \mathbf{1} \leftrightarrow (\forall x \in a)(fx = \mathbf{1})$. The Group 4 axioms are for **S** (Separation), **R** (Replacement) and **C** (Choice), as described above. Finally, Group 5 is the scheme of set induction for *all* formulas in the language of OST; when that is restricted to the formula $x \in a$ we write OST↾ for the system.

From the Group 3 (logical) axioms it is shown that one can associate with each Δ_0 formula $\varphi(\underline{x})$ in the language of ordinary set theory (where $\underline{x} = x_1,\ldots,x_n$) a closed term t_φ that is defined and maps $\mathbf{V}^n$ into **B**, and which satisfies $\forall\underline{x}(t_\varphi(\underline{x}) = 1 \leftrightarrow \varphi(\underline{x}))$. Thus OST satisfies the separation axiom for bounded formulas. Then using Replacement and Choice, one obtains the Δ_0-Collection axiom. Hence KP (here taken to include the axiom of infinity) is contained in OST. This leads us to the ≥ direction of the following.

(i) OST ≡ KP.

My proof in (F 2009) for the ≤ direction went via an interpretation of OST in KP + V= L, beginning with an interpretation of the operations as those given by codes for the partial functions that are Σ_1 definable in parameters. An alternative proof of that bound was given by Jäger (2007) using his theories of numbers and ordinals for transfinite inductive definitions.

To obtain systems of strength full set theory and beyond, one adds constants **uni** for unbounded universal quantification with axiom (Uni) like that for $\mathbf{uni}_b$, and **P** for the power set operation, the latter with axiom (Pow) which states that for each x, $\mathbf{P}x$ is defined and $\forall y(y \in \mathbf{P}x \leftrightarrow y \subseteq x)$.[16] Then we have:

(ii) $\quad$ OST↾ + (Pow) + (Uni) ≡ ZFC.

The proof that ZFC is contained in the left side is easy, using that every formula is now represented by a definite operation. The proof that OST↾ + (Pow) + (Uni) can be reduced to ZFC + (V=L) was given by Jäger (2007). On the other hand, as shown in Jäger (2009) and Jäger and Krähenbuhl (2010), the unrestricted system OST + (Pow) + (Uni) is of the same strength as NBG extended by a suitable form of Σ^1_1-AC.

Next, in view of (i) and (ii) it is natural to ask what the strength is of OST + (Pow). Of course we have that this contains KP + (Pow), where that is formulated with an additional constant **P** as above. The problem concerns the other direction. In this case Jäger (2007) showed that

(iii) $\quad$ OST + (Pow) is interpretable in KP + (Pow) + V = L.

But the latter theory is much stronger than KP + (Pow) as shown by Rathjen (2014),[17] so one can't use (iii) to determine the strength of OST + (Pow). Nevertheless, Rathjen (2014a) has been able to establish the following there, using novel means:

(iv) $\quad$ OST + (Pow) ≡ KP + (Pow).

Let's turn now to the problematic conjectures of (F 2009). These concern the natural formulation in operational terms of an ordinal κ being regular, inaccessible, and Mahlo, resp., as well as a notion of being 2-regular due to Aczel and Richter (1972) that is equivalent to being Π^1_1-indescribable (cf. Richter and Aczel (1974) pp. 329-331).

[16] Rathjen (2014) uses (Pow(**P**)) for our formulation in the language of KP as well, in order to distinguish it from the usual power set axiom formulated without the additional constant. It would have been better to do that in (F 2009), but not having done so I here follow the notation from there.

[17] An earlier such result for the system with a restricted form of set induction is due to Mathias (2001).

Using lower case Greek letters to range over ordinals, the first of these is defined in the language of OST by

$$\mathrm{Reg}(\kappa) := (\kappa > 0) \wedge \forall \alpha, f\,[\alpha < \kappa \wedge (f : \alpha \to \kappa) \to \exists \beta < \kappa\,(f : \alpha \to \beta)].$$

Then being inaccessible is defined by

$$\mathrm{Inacc}(\kappa) := \mathrm{Reg}(\kappa) \wedge \forall \alpha < \kappa\, \exists \beta < \kappa\, [\mathrm{Reg}(\beta) \wedge \alpha < \beta].$$

The statements of regularity and inaccessibility of the class $\mathbf{\Omega}$ of ordinals are defined analogously by:

$$\mathrm{Reg} := \forall \alpha, f\,[(f : \alpha \to \mathbf{\Omega}) \to \exists \beta (f : \alpha \to \beta)].$$

$$\mathrm{Inacc} := \mathrm{Reg} \wedge \forall \alpha\, \exists \beta\, [\mathrm{Reg}(\beta) \wedge \alpha < \beta].$$

Let Fun(a) be the usual set-theoretical formula expressing that the set a is a function, i.e. a many-one binary relation; for x in dom(a), $a(x)$ is the unique y with $\langle x, y \rangle \in a$. Then among the immediate consequences of the OST axioms are, first, that there is a closed term **op** such that for each set a, $\mathbf{op}a\downarrow$ and if Fun(a) and $f = \mathbf{op}a$ then for each $x \in$ dom(a), $fx = a(x)$ and, second, there is a closed term **fun** such that for each f, a, if $f : a \to \mathbf{V}$ then $\mathbf{fun}(f, a)\downarrow$ and if $c = \mathbf{fun}(f, a)$ then Fun(c) and for each $x \in$ dom(c), $c(x) = fx$. Thus the above notions and statements of regularity and inaccessibility can be read as usual in the ordinary language of set theory. That led me mistakenly to assert in Theorem 10 of (F 2009) that OST + (Inacc) is interpretable in KPi + V = L, and to conjecture that OST + (Inacc) is equivalent in strength to KPi.[18] This has been proved to be wrong by Jäger and Zumbrunnen (2012), who show that OST + (Inacc) is equivalent in strength to the extension KPS of KP by the statement Inacc when read in ordinary set-theoretical terms, denoted SLim for "strong limit axiom." KPS proves that for any κ that satisfies Reg(κ), L_κ is a standard model of ZFC without the power set axiom; hence KPS is much stronger than second-order arithmetic.

My mistake was that the notion of regularity here—while natural in the context of ordinary set theory—does not correspond to that used in KP viewed as a theory for admissible sets. Namely, as presented in Jäger (1986), that is given by the additional

[18] However, I did say that I had not checked the details. In fact, I hadn't thought them through at all.

predicate Ad(x) expressing that x is admissible, with the appropriate axioms. Then KPi asserts the unboundedness of the admissibles, in the sense that $\forall x\exists y[\mathrm{Ad}(y) \wedge x \in y]$. So the question arises as to whether there is a natural extension of OST that is equivalent in strength to KPi. My first thought was that there should be some notion of universe, Uni(x), formulated in the language of OST, that is analogous to the notion of admissibility, Ad(x), in the language of KP, such that when we extend OST by the statement $\forall x\exists y[\mathrm{Uni}(y) \wedge x \in y]$ we obtain a system of the same strength as KPi. In e-mail exchanges with Gerhard Jäger early in the summer of 2014, I made several proposals for the definition of Uni(x) to do just that, but each proved to be defective. One of these proposals was to say that u is a universe if it is a transitive set that contains all the constants of OST, is closed under application, satisfies the basic set-theoretic axioms of OST and the axioms for **S**, **R** and **C** under the hypotheses suitably relativized to u. But Jäger pointed out that the system OST + $\forall x\exists u[\mathrm{Uni}(u) \wedge x \in u]$ is still stronger than KPi, by the results of his paper Jäger (2013).

In going over this situation, Jäger noted that the applicative structure must also be relativized in explaining the notion of a universe in the language of OST. This first led him to make the following suggestion. Namely, one returns to the formulation of the applicative basis in terms of the ternary App relation, rather than the logic of partial terms. Then a universe is defined to be a pair $\langle u, a\rangle$ such that (i) u is a transitive set with $a \subseteq u^3$, (ii) whenever $(f, x, y) \in a$ then App(f, x, y), (iii) u contains ω and all the constants of OST, and (iv) all the axioms of OST hold when relativized to u provided that the App relation is replaced by the set a. Jäger outlined a proof that OST + $\forall x\exists u,a[\mathrm{Uni}(\langle u, a\rangle)\wedge x \in u]$ is proof-theoretically equivalent to KPi. More recently, in work in progress Jäger (2015), he has proposed another way of modifying the notion of regularity (and thence inaccessibility) so as to stay within the language and logic of partial terms while relativizing it to a universe in the preceding sense. In the new approach one adds a predicate Reg(u, a) to the language satisfying certain axioms similar to (i)-(iv) and in addition an assumption of the linear ordering of those pairs $\langle u, a\rangle$ for which Reg(u, a) holds. Then in place of the above condition Inacc on the class of ordinals one can consider the statement Lim-Reg (abbreviated LR), which asserts that $\forall x \exists u, a[\mathrm{Reg}(u, a) \wedge x \in u]$. The main result of Jäger (2015) is that OST + LR is proof-theoretically equivalent to KPi. The advantage of his second approach is that one can re-express further large cardinal notions such as Mahlo, etc., much as before. Assuming this is successful, we can look forward to a reexamination of my original aim to use OST as a vehicle to restate various large cardinal notions in applicative terms in order to explain the existing admissible analogues that are in the literature.

4. The unfolding program. In sec. 2 I spoke of my work on unramified systems of predicative strength[19] as being one precursor to the development of explicit mathematics. That mainly had to do with the potential use of such systems as a means to determine which parts of classical analysis could be justified on predicative grounds. But one of the articles indicated, (F1979a), was concerned with more basic conceptual aims, namely those advanced by Kreisel (1970) who suggested the study of principles of proof and definition that "we recognize as valid once we have understood (or, as one sometimes says, 'accepted') certain given concepts." The two main examples Kreisel gave of this were finitism and predicativity, and in both cases, he advanced for that purpose the use of some form of autonomous transfinite progressions embodying a "high degree of self-reflection." My aim in (F 1979a) was to show in the case of predicativity how that might be generated instead by "a direct finite rather than transfinite reflective process, and without alternative use of the well-foundedness notion in the axioms." The motivation was that if one is to model actual reflective thought then one should not invoke the transfinite in any way. But not long after that work I realized that what is implicit in accepting certain basic principles and concepts can be explained more generally in terms of a notion of reflective closure of schematic systems, where schemata are considered to be open-ended using symbols for free predicate variables P, as in the scheme for induction on the natural numbers. One crucial engine in the process of reflection is the employment of the substitution rule A(P)/A(B), where B(x) is a formula that one has come to recognize as meaningful in the course of reflection, and where by A(B) is meant the result of substituting B(t) for each occurrence P(t) in A(P). I described the notion of

[19] Cf. fn 2.

reflective closure and its application to the characterization of predicativity in a lecture for a meeting in 1979 on the work of Kurt Gödel, but was only to publish that work in the article, "Reflecting on incompleteness" (F 1991). For the technical apparatus I introduced there an axiomatization[20] of the semantic theory of truth of Kripke (1975) in which the truth predicate may consistently be applied to statements within which it appears by treating truth and falsity as partial predicates.

Though the axiomatic theory of truth employed in (F1991) proved to be of independent interest, as an engine for the explanation of reflective closure it still had an air of artificiality about it. I was thus led to reconsidering the entire matter in (F 1996) in which the notion of unfolding of open-ended schematic systems was introduced in close to its present form by means of a basic operational framework. As formulated there, given a schematic system S, the question is: which operations and predicates—and which principles concerning them ought to be accepted if one has accepted S? And under the heading of operations one should consider both operations on the domain D_S of individuals of S and operations on the domain Π of predicates; both domains are included in an overarching domain V. For the underlying general theory of operations applicable to arbitrary members of V, in (F1996) I made use of a type 2 theory of partial functions and (monotone) partial functionals, generated by explicit definition and least fixed point recursion, and that is what was followed in the paper Feferman and Strahm (2000) for the unfolding of a schematic system NFA of non-finitist arithmetic. Later, in order to simplify various matters in the treatment of finitist arithmetic, the work on NFA was reformulated in Feferman and Strahm (2010) so as to use instead the basic operational language and axioms on V essentially as described at the end of sec. 1 above, and that is what has been followed in subsequent work on unfolding.

Here are a few details for the unfolding of NFA, which in many ways is paradigmatic. The axioms of NFA itself are simply the usual ones for 0, sc and pd together with the induction scheme given as $P(0) \wedge \forall x[P(x) \rightarrow P(sc(x))] \rightarrow \forall x(P(x))$ where P is a free predicate variable. The language of the unfolding of NFA adds a

[20] Since referred to as KF in the literature.

number of constants, the predicate symbol $N(x)$, the predicate symbol $\Pi(x)$, and the relation $y \in x$ for x such that $\Pi(x)$. The axioms of the unfolding U(NFA) consist of the following five groups: (I) The axioms of NFA relativized to N. (II) The partial combinatory axioms, with pairing, projections and definition by cases. (III) An axiom for the characteristic function of equality on N. (IV) Axioms for various constants in the domain Π of predicates, namely for the natural numbers, equality, and the free predicate variable P, and for the logical operations $\neg$, $\wedge$, and $\forall$. (V) An axiom for the join of a sequence of predicates, given by $\mathbf{j}(f)$ when $f : N \to \Pi$. The full unfolding U(NFA) is then obtained by applying the substitution rule A(P)/A(B) where B is an arbitrary formula of the unfolding language. A natural subsystem of this called the *operational unfolding* of NFA and denoted U_0(NFA) is obtained by restricting to axiom groups (I)-(III) with the formulas B in the substitution rule restricted accordingly. In U_0(NFA) one successively constructs terms $t(x)$ intended to represent each primitive recursive function, by means of the recursion operator and definition by cases. Applying the substitution rule it is then shown by induction on the formula $t(x)\downarrow$ that each such term defines a total operation on the natural numbers. Thus the language of PA may be interpreted in that of U_0(NFA) and so by application of the substitution rule once more, we have PA itself included in that system. Moving on to U(NFA), the domain of predicates is expanded considerably by use of the join operation. Once one establishes that a primitive recursive ordering $\prec$ satisfies the schematic transfinite induction principle TI($\prec$, P) with the free predicate variable P, one may apply the substitution rule to carry out proofs by induction on $\prec$ with respect to arbitrary formulas. In particular, one may establish existence of a predicate corresponding to the hyperarithmetical hierarchy along such an ordering, relative to any given predicate p in Π. Then by means of the usual arguments, if one has established in U(NFA) the schematic principle of transfinite induction along a standard ordering for an ordinal α, one can establish the same for $\varphi\alpha 0$, hence the same for each ordinal less than Γ_0. Thus U(NFA) contains the union of the ramified analytic systems up to Γ_0. The main results of Feferman and Strahm (2000) are that U_0(NFA) is proof-theoretically equivalent to PA and is conservative over it, and U(NFA) is proof-theoretically equivalent to the union of the ramified analytic systems up to Γ_0 and is conservative over it. In other words, U(NFA) is proof-theoretically equivalent to predicative analysis. In addition we showed that the intermediate system U_1(NFA) without the join axiom (V) is proof-theoretically equivalent to the union of the ramified systems of finite level.

The unfolding of finitist arithmetic was later taken up in Feferman and Strahm (2010); two open-ended schematic systems of finitist arithmetic are treated there, denoted FA and FA + BR, resp. The basic operations on individuals are the same as in NFA together with the characteristic function of equality, while those on predicates are given by $\perp$, $\wedge$, $\vee$, and $\exists$. Reasoning now is applied to sequents $\Gamma \rightarrow A$, and the basic assumptions are the usual ones for 0, sc and pd, and the induction rule in the form: from $\Gamma \rightarrow P(0)$ and $\Gamma, P(x) \rightarrow P(sc(x))$, infer $\Gamma \rightarrow P(x)$, with P a free predicate variable. Now the substitution rule is applied to sequent inference rules of the form $\Sigma_1, \Sigma_2, \ldots, \Sigma_n \Rightarrow \Sigma$; we may substitute for P throughout by a formula B to obtain a new such rule. The first main result of Feferman and Strahm (2010) is that all three unfoldings of FA are equivalent in strength to PRA. That is in accord with the informal analysis of finitism by Tait (1981). On the other hand, Kreisel (1965), pp. 169-172, had sketched an analysis of finitism in terms of a certain autonomous progression and alternatively "for a more attractive formulation" without progressions but with the use of the Bar Rule, BR, that is equivalent to PA. The rule BR allows one to infer from the no-descending sequence property NDS(f, $\prec$) for a decidable ordering $\prec$, where f is a free function variable, the principle of transfinite induction on the ordering TI($\prec$, P), with the free predicate variable P. The second main result of Feferman and Strahm (2010) is that all three unfoldings of FA + BR are equivalent in strength to PA, thus in accord with Kreisel's analysis of finitism.

Extending the unfolding program to still weaker theories, Eberhard and Strahm (2012, 2015) have dealt with three unfolding notions for a basic system FEA of feasible arithmetic. Besides the operational unfolding U_0(FEA) and (full) predicate unfolding U(FEA), they introduced a more general truth unfolding system U_T(FEA) obtained by adding a truth predicate for the language of the predicate unfolding.[21] Their main result is that the provably total functions of binary words for all three systems are exactly those computable in polynomial time.

The most recent result in the unfolding program is due to Buchholtz (2013) who determined the proof-theoretic ordinal of $U(ID_1)$, where the usual system of one arithmetical inductive definition ID_1 is recast in open-ended schematic form. That is taken to expand NFA and for each arithmetical $A(P, x)$ in which P has only positive occurrences one assumes the following principles for the predicate constant P_A associated with A: (i) $\forall x(A(P_A, x) \to P_A(x))$ and (ii) $\forall x(A(P, x) \to P(x)) \to \forall x(P_A(x) \to P(x))$, with P the free predicate variable. The axioms of $U(ID_1)$ are similar to those of U(NFA), except that for Axiom (V), the join operation more generally takes an operation f from a predicate p to predicates $fx = q_x$ to the disjoint sum $\mathbf{j}(f)$ of the q_x's over the x's in p. The main result of Buchholtz (2013) is that $|U(ID_1)| = \psi(\Gamma_{\Omega+1})$ $(= \psi_\Omega(\Gamma_{\Omega+1}))$. This invites comparison with the famous result of Howard (1972) according to which $|ID_1| = \psi(\varepsilon_{\Omega+1})$, previously denoted $\varphi\varepsilon_{\Omega+1}0$.[22] In addition, Buchholtz, Jäger and Strahm (2014) show that a number of proof-theoretic results for systems of strength Γ_0 have direct analogues for suitable systems of strength $\psi(\Gamma_{\Omega+1})$. Finally, Buchholtz (2013) p. 48 presents very plausible conjectures concerning the unfolding of schematic theories of iterated inductive definitions generalizing the results for ID_1.

Readers may already have guessed that the unfolding of NFA and ID_1 can be recast in terms of systems S of explicit mathematics. For that purpose it is simplest to return to the original syntax of those systems and use $Cl(x)$ in place of $\Pi(x)$. Note that

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[21] This follows the proposed formulation of U(NFA) via a truth predicate in Feferman (1996), p. 14.

[22] Ulrik Buchholtz originally thought that $\psi(\Gamma_{\Omega+1})$ is the same as the ordinal H(1) of Bachmann (1950). This seemed to be supported by Aczel (1972) who wrote (p. 36) that H(1) may have proof theoretical significance related to those of the ordinals ε, Γ_0 and $\varphi\varepsilon_{\Omega+1}0$. And Miller (1976) p. 451 had conjectured that "H(1) [is] the proof-theoretic ordinal of ID_1* which is related to ID_1 as predicative analysis ID_0* is to first-order arithmetic ID_0." However, Wilfried Buchholz recently found that the above representation of H(1) in terms of the ψ function is incorrect. This suggests one should revisit the bases of Aczel's and Miller's conjectures.

with the variables X, Y, Z,… taken to range over Cl, every second-order formula over the applicative structure is expressible as a formula of the language. The substitution rule now takes the form $\varphi(X)/\varphi(\{x\colon \psi(x)\}\}$ where ψ is an arbitrary formula of the language, and where in the conclusion of the rule each instance of the form $t \in X$ that occurs in φ is replaced by $\psi(t)$. In place of the operations on predicates in Π we now use the corresponding operations on classes.[23] Thus, in place of U(NFA) we would consider the system U*(S) generated by the substitution rule from the system S = BON + ECA + J + (C-I_{N}), where the class induction axiom on N takes the place of the induction scheme of NFA. So it is reasonable to conjecture that U*(S) in this case is of the same strength as predicative analysis.[24] Similarly, we may obtain an analogue of $\text{U}(\text{ID}_1)$ by making use of the Inductive Generation Axiom IG of T_0. Recall that IG takes the form that we have an operation $\mathbf{i}(A, R)$ that is defined for all classes A and R, and whose value is a class I that satisfies the closure condition

$$\text{Clos}(A,\ R,\ I)\colon = \forall x \in A[(\forall y((y,\ x) \in R \rightarrow\ y \in I) \rightarrow x \in\ I]$$

together with the minimality condition

$$\text{Min}(A,\ R,\ \varphi)\colon = \text{Clos}(A,\ R,\ \varphi) \rightarrow (\forall x \in I)\varphi(x),$$

where φ is an arbitrary formula. In its place the schematic form $\text{IG}\restriction$ restricts the minimality condition to formulas $\varphi(x)$ of the form $x \in X$, i.e.

$$\text{Clos}(A,\ R,\ X) \rightarrow I \subseteq X.$$

Let us denote by IG(O) the instance of IG used to generate the class of countable tree ordinals and by $\text{IG(O)}\restriction$ the same with the restricted minimality condition. Then with S as above, the system $\text{S} + \text{IG(O)}\restriction$ is analogous to ID_1, so we may expect that

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[23] Alternatively, one can of course work with the formulation of explicit mathematics in terms of the representation relation $\mathfrak{R}(x, X)$.
[24] This would provide another answer to the question of finding a system of explicit mathematics of the same strength as predicative analysis.

U*(S + IG(O)↾) is equivalent in strength to U(ID_1) and so its proof theoretic ordinal would be equal to $\psi(\Gamma_{\Omega+1})$. But now we can also form the system S + IG↾ and it is natural to ask what the strength is of *its* unfolding U*(S + IG↾). This would seem to encompass autonomously iterated systems ID_α (cf. Pohlers (1998) p. 332).

One of the motivations for (F 1996) was to give substance to the idea of Gödel (1947) that consideration of axioms for the existence of inaccessible cardinals and the hierarchies of Mahlo cardinals more generally "show clearly, not only that the axiomatic system of set theory as known today is incomplete, but also that it can be supplemented without arbitrariness by new axioms which are only the natural continuation of those set up so far." (Cf. Gödel (1990), p. 182). My idea was that this could be spelled out by the unfolding of a suitable schematic system of set theory, but the details in (F1996) sec. 5 were rather sketchy. These now can be spelled out as follows, using the language of OST as a point of departure. For unfoldings, we could either take the U(·) approach by adding the predicate $\Pi(x)$ or the variant U*(·) approach by adding the predicate Cl(x) as in systems of explicit mathematics. For simplicity I shall follow the latter here. Take S-OST to be the schematic version of OST, which is obtained by replacing the set induction axiom scheme by its class version $\forall x[\forall y(y \in x \rightarrow y \in X) \rightarrow x \in X] \rightarrow \forall x(x \in X)$. Then we can consider U*(S-OST ± Pow ± Uni), where (Pow) is formulated as in sec. 3 above with a symbol **P** for the power set operation, and (Uni) is the axiom for unbounded universal quantification with **uni** as the corresponding basic operation. In particular, we would be interested in characterizing the three systems, U*(S-OST), U*(S-OST + Pow) and, finally U*(OST + Pow + Uni).

Now with OST ≡ KP ≡ ID_1, one may think of S-OST as analogous to schematic ID_1, so that I conjecture that U*(S-OST) ≡ U(ID_1). Secondly, Rathjen (2014a) has studied KP + AC + Pow($\wp$) using relativized ordinal analysis methods, and shown that this system proves the existence of the cumulative hierarchy of V_α's for all $\alpha < \psi(\varepsilon_{\Omega+1})$ and moreover that that is best possible. Thus I conjecture that U*(S-OST + Pow) proves the existence of the cumulative hierarchy of V_α's for all $\alpha < \psi(\Gamma_{\Omega+1})$ and that that is best possible. Finally, I expect that the analogous results for the unfolding of S-OST + Pow + Uni would make use of the ordinal notation system up to $\psi(\Gamma_{\mathrm{ORD}+1})$ in a suitable sense. Thus the unfolding of the system S-OST + (Pow) + (Uni) would be equivalent in strength to the extension of ZFC by a certain range of small large cardinals. It would then be another question to see how far that goes in terms of the standard classifications of such.

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References

P. Aczel (1972), Describing ordinals using functionals of transfinite type, *J. Symbolic Logic* 37, 35-47.

P. Aczel and W. Richter (1972), Inductive definitions and analogues of large cardinals, in *Conference in Mathematical Logic, London '70* (W. Hodges, ed.), Lecture Notes in Mathematics 255, 1-9.

J. Avigad (1996), On the relationship between ATR_0 and $\widehat{ID}_{<\omega}$, *J. Symbolic Logic* 61, 768-779.

H. Bachmann (1950), Die Normalfunktionen und das Problem der ausgezeichneten Folgen von Ordnungszahlen, *Vierteljschr. Naturforsch. Ges. Zürich* 95, 115-147.

M. J. Beeson (1985), *Foundations of constructive mathematics. Metamathematical studies*, Springer-Verlag, Berlin.

M. J. Beeson (1988), Toward a computation system based on set theory, *Theoretical Computer Science* 60, 297–340.

E. Bishop (1967), *Foundations of Constructive Analysis*, McGraw Hill, NY.

E. Bishop and D. Bridges (1985), *Constructive Analysis*, Springer-Verlag, Berlin.

U. Buchholtz (2013), *Unfolding of Systems of Inductive Definitions*, PhD Thesis, Stanford University, http://math.stanford.edu/~utb/thesis.pdf .

U. Buchholtz, G. Jäger and T. Strahm (2014), Theories of proof-theoretic strength $\psi(\Gamma_{\Omega+1})$. [To appear]

A. Cantini (2011), Extending constructive operational set theory by impredicative principles, *Mathematical Logic Quarterly* 57, 299-322.

A. Cantini and L. Crosilla (2008), Constructive set theory with operations, in *Logic Colloquium 2004* (A. Andretta et al., eds.), Lecture Notes in Logic 29, 47–83.

A. Cantini and L. Crosilla (2010), Elementary constructive operational set theory, in *Ways of Proof Theory* (R. Schindler, ed.), Ontos Verlag, Frankfurt, 199-240.

A. Church (1932), A set of postulates for the foundation of logic, *Annals of Mathematics* 33, 346-366.

A. Church (1933), A set of postulates for the foundation of logic (second paper), *Annals of Mathematics* 34, 839-864.

S. Eberhard and T. Strahm (2012), Towards the unfolding of feasible arithmetic (Abstract), *Bull. Symbolic Logic* 18, 474-475.

S. Eberhard and T. Strahm (2015), Unfolding feasible arithmetic and weak truth, in *Unifying the Philosopy of Truth* (T. Achourioti, et al., eds.), Springer-Verlag, Berlin.

F. B. Fitch (1963) The system CΔ of combinatory logic, *J. Symbolic Logic* 28, 87-97.

S. Feferman (1964), Systems of predicative analysis, *J. Symbolic Logic* 29, 1-30.

S. Feferman (1968), Autonomous transfinite progressions and the extent of predicative mathematics, in *Logic, Methodology, and Philosophy of Science* III (B. van Rootselaar and J. F. Staal, eds.) North-Holland, Amsterdam, 121-135.

S. Feferman (1971), Ordinals and functionals in proof theory, in *Actes du Congrès International des Mathématiciens (Nice) 1970,* vol. 1, Gauthier-Villars, Paris, 229-233.

S. Feferman (1974), Predicatively reducible systems of set theory, in *Axiomatic Set Theory* (D. Scott, ed.), Proc. Symposia in Pure Math. XIII, Part 2, Amer. Math. Soc., Providence, 11-32.

S. Feferman (1975), A language and axioms for explicit mathematics, in *Algebra and Logic* (J. Crossley, ed.), Lecture Notes in Mathematics 450, 87-139.

S. Feferman (1977), Theories of finite type related to mathematical practice, in *Handbook of Mathematical Logic* (J. Barwise, ed.), North-Holland, Amsterdam, 913-971.

S. Feferman (1979), Constructive theories of functions and classes, in *Logic Colloquium '78* (M. Boffa, et al., eds.), North-Holland, Amsterdam, 159-224.

S. Feferman (1979a), A more perspicuous formal system for predicativity, in *Konstruktionen versus Positionen* I (K. Lorenz, ed.), Walter de Gruyter, Berlin, 68-93.

S. Feferman (1982), Monotone inductive definitions, in *The L. E. J. Brouwer Centenary Symposium* (A. S. Troelstra and D. van Dalen, eds.), North-Holland, Amsterdam, 77-89.

S. Feferman (1988), Weyl vindicated: *Das Kontinuum* 70 years later, in *Temi e prospettive della logica e della filosofia della scienza contemporanee, vol. I*, pp. 59-93, CLUEB, Bologna, 1988; reprinted in Feferman (1998), 249-283.

S. Feferman (1991), Reflecting on incompleteness, *J. Symbolic Logic* 56, 1-49.

S. Feferman (1993), Why a little bit goes a long way: Logical foundations of scientifically applicable mathematics, in *PSA 1992*, Vol. II, 442-455; reprinted in Feferman (1998), 284-298.

S. Feferman (1996) Gödel's program for new axioms: Why, where, how and what?, in *Gödel '96* (P. Hájek, ed.), Lecture Notes in Logic 6, 3-22.

S. Feferman (1998), *In the Light of Logic*, Oxford University Press, NY.

S. Feferman (2001), Notes on operational set theory I. Generalizations of "small" large cardinals and admissible set theory, http://math.stanford.edu/~feferman/papers/OperationalST-I.pdf.

S. Feferman (2009), Operational set theory and small large cardinals, *Information and Computation* 207, 971-979.

S. Feferman (2013), How a little bit goes a long way: Predicative foundations of analysis, http://math.stanford.edu/~feferman/papers/pfa(1).pdf.

S. Feferman and G. Jäger (1983), Choice principles, the bar rule and autonomously iterated comprehension schemes in analysis, *J. Symbolic Logic* 48, 63-70.

S. Feferman and G. Jäger (1993), Systems of explicit mathematics with non-constructive μ-operator I, *Annals of Pure and Applied Logic* 65, 243-263.

S. Feferman and G. Jäger (1996), Systems of explicit mathematics with non-constructive μ-operator II, *Annals of Pure and Applied Logic* 79, 37-52.

S. Feferman and T. Strahm (2000), The unfolding of non-finitist arithmetic, *Annals of Pure and Applied Logic* 104, 75-96.

S. Feferman and T. Strahm (2010), The unfolding of finitist arithmetic, *Review of Symbolic Logic* 3, 665-689.

H. Friedman, K. McAloon and S. G. Simpson (1982), A finite combinatorial principle which is equivalent to the 1-consistency of predicative analysis, in *Patras Logic Symposion* (G. Metakides, ed.), North-Holland, Amsterdam, 197-230.

T. Glass, M. Rathjen and A. Schlüter (1997) On the proof-theoretic strength of monotone induction in explicit mathematics, *Annals of Pure and Applied Logic* 85, 1–46.

T. Glass and T. Strahm (1996), Systems of explicit mathematics with non-constructive μ-operator and join, *Annals of Pure and Applied Logic* 82, 193–219.

K. Gödel (1947), What is Cantor's continuum problem?, *Amer. Math. Monthly* 54, 515-525; errata 55, 151; reprinted in Gödel (1990), 176-187.

K. Gödel (1990), *Collected Works, Vol. II. Publications 1938-1974* (S. Feferman, et al., eds.), Oxford University Press, NY.

G. Jäger (1983), A well-ordering proof for Feferman's theory T_0, *Archiv für mathematische Logik und Grundlagenforschung* 23, 65–77.

G. Jäger (1984), The strength of admissibility without foundation, *J. Symbolic Logic* 49, 867-879.

G. Jäger (1986), *Theories for admissible sets: A unifying approach to proof theory*, Studies in Proof Theory, vol. 2, Bibliopolis, Naples.

G. Jäger (1988), Induction in the elementary theory of types and names, in *Computer Science Logic '87* (E. Börger, et al., eds.), Lecture Notes in Computer Science 329, 118–128.

G. Jäger (2007), On Feferman's operational set theory OST, *Annals of Pure and Applied Logic* 150, 19–39.

G. Jäger (2009), Full operational set theory with unbounded existential quantification and power set, *Annals of Pure and Applied Logic* 160, 33–52.

G. Jäger (2009a), Operations, sets and classes, *Logic, Methodology and Philosophy of Science: Proceedings of the Thirteenth International Congress* (C. Glymour et al., eds.), College Publications, London, 74–96.

G. Jäger (2013), Operational closure and stability, *Annals of Pure and Applied Logic* 164, 813-821.

G. Jäger (2015), Relativizing operational set theory. (In preparation)

G. Jäger, R. Kahle and T. Strahm (1999), On applicative theories, in *Logic and foundations of mathematics* (A. Cantini, et al., eds.), Kluwer, Dordrecht, 83–92.

G. Jäger and J. Krähenbuhl (2010), Σ^1_1 choice in a theory of sets and classes, in *Ways of Proof* (R. Schindler, ed.), Ontos Verlag, Frankfurt, 283-314.

G. Jäger and W. Pohlers (1982), Eine beweistheoretische Untersuchung von Δ^1_2-CA + BI und verwandter Systeme, *Sitzungsberichte der Bayerischen Akademie der Wissenschaften, Mathematisch-Naturwissenschaftliche Klasse*, 1-28.

G. Jäger and D. Probst (2011), The Suslin operator in applicative theories: its proof-theoretic analysis via ordinal theories, *Annals of Pure and Applied Logic* 162, 647-660.

G. Jäger and T. Strahm (1995), Totality in applicative theories, *Annals of Pure and Applied Logic* 74, 105–120.

G. Jäger and T. Strahm (2002), The proof-theoretic analysis of the Suslin operator in applicative theories, in *Reflections on the Foundations of Mathematics*: *Essays in Honor of Solomon Feferman* (W. Sieg, R. Sommer and C. Talcott, eds.), AK Peters Ltd, Natick MA, 270-292.

G. Jäger and R. Zumbrunnen (2012), About the Strength of Operational Regularity, in *Logic, Construction, Computation* (U. Berger, et al., eds.) Ontos Verlag, Frankfurt, 305–324.

G. Kreisel (1965), Mathematical logic, in *Lectures on Modern Mathematics*, Vol. III (T. L. Saaty, ed.), Wiley, New York, 95-195.

G. Kreisel (1970), Principles of proof and ordinals implicit in given concepts, in *Intuitionism and Proof Theory* (A. Kino, et al., eds.), North-Holland, Amsterdam, 489-516.

S. Kripke (1975). Outline of a theory of truth, *J. of Philosophy* 72, 690-716.

M. Marzetta and T. Strahm (1988), The μ quantification operator in explicit mathematics with universes and iterated fixed point theories with ordinals, *Archive for Mathematical Logic* 37, 391–413.

A. R. D. Mathias (2001), The strength of Mac Lane set theory, *Annals of Pure and Applied Logic* 110, 107-234.

L. W. Miller (1976), Normal functions and constructive ordinal notations, *J. Symbolic Logic* 41, 439-459.

W. Pohlers (1998), Subsystems of set theory and second order number theory, in *Handbook of Proof Theory* (S. Buss, ed.), North-Holland, Amsterdam, 209-335.

M. Rathjen (1996), Monotone inductive definitions in explicit mathematics, *J. Symbolic Logic* 61, 125–146.

M. Rathjen (1998), Explicit mathematics with monotone inductive definitions: a survey, in (W. Sieg, et al., eds.) *Reflections on the Foundations of Mathematics*, 329-346. Second printing, Lecture Notes in Logic 15 (2002).

M. Rathjen (2014), Relativized ordinal analysis. The case of Power Kripke-Platek set theory, *Annals Pure and Applied Logic* 165, 316-339.

M. Rathjen (2014a), Power Kripke-Platek and the axiom of choice, http://www1.maths.leeds.ac.uk/~rathjen/PowerAC.pdf.

W. Richter and P. Aczel (1974), Inductive definitions and reflecting properties of admissible ordinals, in *Generalized Recursion Theory* (J. E. Fenstad and P. G. Hinman, eds.), North-Holland, Amsterdam, 301-381.

K. Sato (2014), A new model construction by making a detour via intuitionistic theories II. Interpretability lower bound of Feferman's Explicit Mathematics T_0. (t.a.)

K. Sato and R. Zumbrunnen (2014), A new model construction by making a detour via intuitionistic theories I. Operational set theory without choice is Π_1 equivalent to KP, http://www.iam.unibe.ch/ltgpub/2014/sz14.pdf.

K. Schütte (1965), Eine Grenze für die Beweisbarkeit der transfiniten Induktion in der verzweigten Typenlogik, *Archiv für mathematische Logik und Grundlagenforschung* 7, 45-60.

S. G. Simpson (1988), *Subsystems of Second Order Arithmetic*, Perspectives in Logic, Springer-Verlag, Berlin; 2nd edn. 2009.

S. G. Simpson (1998), Predicativity: the outer limits, in *Reflections on the Foundations of Mathematics* (W. Sieg, et al., eds.), 130-136; 2nd printing, Lecture Notes in Logic 15 (2002).

S. G. Simpson (2010), The Gödel hierarchy and reverse mathematics, in *Kurt Gödel. Essays for his Centennial* (S. Feferman, et al., eds.), Lecture Notes in Logic 33, 109-127.

S. Takahashi (1989), Monotone inductive definitions in a constructive theory of functions and classes, *Annals of Pure and Applied Logic* 42, 255–297.

W. W. Tait (1981), Constructive reasoning, in *Logic, Methodology and Philosophy of Science* III, North-Holland, Amsterdam, 185-199; reprinted in Tait (2005), 21-42.

W. W. Tait (2005), *The Provenance of Pure Reason*, Oxford University Press, NY.

J. van Heijenoort, ed. (1967), *From Frege to Gödel. A source book in mathematical logic,* Harvard Univ. Press, Cambridge, MA.

J. von Neumann (1925), Eine Axiomatisierung derMengenlehre, *J. für die reine und angewandte Mathematik* 154, 219–240; English translation in van Heijenoort (1967), 393-413.

H. Weyl (1918), *Das Kontinuum. Kritische Untersuchungen über die Grundlagen der Analysis*, Veit, Leipzig.

E. Zermelo (1908), Untersuchungen über die Grundlagen der Mengenlehre I, *Mathematische Annalen* 65, 261-281; English translation in van Heijenoort (1967), 199-215.

R. Zumbrunnen (2013), *Contributions to Operational Set Theory*, PhD Thesis, University of Bern, http://www.iam.unibe.ch/ltgpub/2013/zum13.pdf.