A LANGUAGE AND AXIOMS FOR EXPLICIT MATHEMATICS

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1. *Introduction*

Systematic explicit mathematics (of various kinds, to be described below) deals with *functions* and *classes* only *via certain means of definition or presentation.* The former operational definitions are called here *rules* or *operations*; definitions of the latter are called *classifications.* In the literature one has also used *constructions* for the first and *predicates, properties, types* or *species* for the second. A new language $\mathcal{L}$ is introduced for which such notions of operation and classification are basic.

Two systems of axioms T_0 and T_1 are formulated in $\mathcal{L}$, the first of which is evident when the operations are interpreted to be given by *rules for mechanical computation.* In T_1 these must be understood instead to be given by *definitions admitting quantification over* N (the natural numbers); T_1 is obtained from T_0 by adjoining a single axiom. In both cases, the classifications may be conceived of as *successively explained or generated* from preceding ones. Some variants and extensions of T_0 and T_1 suggested by the same ideas are also considered.

Several metamathematical results (as to models, conservative extensions, etc.) are obtained for these theories. It is also shown

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how to formalize directly in them or treat in terms of their models such enterprises as *constructive, recursive, predicative and hyper-arithmetic mathematics*. This permits a rather clear view of *what portions of mathematics are accounted for by these systematic redevelopments*.

The following are some distinctive features of the notions axiomatized here, in contrast with current set-theoretical conceptions.

(i) The set-theoretical notions of function and class are viewed *extensionally*, e.g. two classes which have the same members are identical. The notions here are viewed *intensionally*, e.g. two essentially distinct rules may well compute the same values at the same arguments.

(ii) The notions of function and set are *interreducible*: functions may be explained in terms of sets of ordered pairs and sets in terms of characteristic functions. In contrast, the characteristic function associated with a classification A is *not* in general given by a rule. (For example, in the constructive interpretation of T_0, there is no rule for telling which sequences of rationals belong to the classification A of being Cauchy.) There is a *significant asymmetry* in the treatment of the basic notions here. Roughly speaking, rules are taken to be of a quite restricted character, while the properties expressed by classifications may be quite rich. Mathematics consists in discovering which such properties are held by given mathematical objects (e.g. numbers, syntactic expressions, operations and classifications themselves).

(iii) *Self-application* is both possible and reasonable for rules and classifications. The identity operation is given by the rule which associates with any object x the value x. The universal classification V holds of all objects. In general though, *operations are partial*, i.e. have domains which may be a proper part of the universe and so need not be self-applicable. (For example, the operation of differentiation is defined only for certain operations from reals to reals.) Further there may be no extension of a rule f to all of V when there is no test for membership in the domain of f.

(iv) *Operations may be applied to classifications as well as operations.* Important examples are the operation c which applies to any A,B to give the *Cartesian product* $A \times B$, and the operation e which applied to any A,B gives the *exponentiation* classification B^A holding of just those f which map A into B. Still further we have a *join operation* j which applies to any A, f for which fx is a classification B_x whenever x belongs to A; this holds exactly of those pairs $z = (x,y)$ for which x belongs to A and y belongs to B_x. These operations are all guaranteed by the axioms of T_0. In addition, *general principles of inductive generation* in T_0 permit their transfinite iteration.

The classifications generated by e applied any finite number of times starting with N are usually called the *finite types*. The objects falling under these classifications are the *functionals of finite type*. The important recognition of this as a constructively admissible notion is due to Gödel [58]. *Constructive theories of transfinite types* have been formulated by Scott [70] and Martin-Löf [prelim.Ms]. The theory T_0 is also constructively justified and is richer than these. Its formulation seems to me to constitute an improvement in other respects as well; however, no detailed comparison is made here[2].

Some ideas for extensions of T_0 are discussed at the conclusion. The interest there is to find much stronger reasonable axioms for classifications; such go beyond current practice if not the needs of explicit mathematics.

2. *The language* $\mathcal{L}$

2.1. *Syntax*

Variables: a,b,c,...,x,y,z

Constants: $\underline{0}$, $\underline{k}$, $\underline{s}$, $\underline{d}$, $\underline{p}$, $\underline{p}_1$, $\underline{p}_2$, $\underline{c}_n(n<\omega)$, $\underline{j}$, $\underline{i}$

Atomic relations: x = y, App(f,x,y), Cl(a), xηa

Atomic formulas: any substitution instance by variables or constants in atomic relations, together with an atomic sentence ⊥

Connectives and quantifiers: ∧, ∨, →, ∀, ∃

Formulas are generated from the atomic formulas by the connectives and quantifiers. ϕ, ψ, θ, ... range over formulas.

2.2. *Informal interpretation of the basic syntax*

The variables will be interpreted as ranging over a universe of mathematical objects among which are rules and classifications.

The meaning of the constants will be explained in connection with the axioms.

⊥ is a false or absurd proposition.

x = y holds when x and y are identical.

App(f,x,y) holds when f is a rule (or operation) which is defined at the argument x and which has value y when applied

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[2] Cf. Scott [70] for an extensive discussion of previous work. (Added in proof: cf. the addenda below.)

to x.

$Cl(x)$ holds when x is a classification.

$x\eta a$ holds when x falls under (belongs to, is in) the classification a.

2.3. *Abbreviations*

$\neg\phi$ for $(\phi \to \bot)$; $(\phi \leftrightarrow \psi)$ for $(\phi \to \psi) \wedge (\psi \to \phi)$;

$\phi(t/x)$ for $Sub(t,x,\phi)$ – this is also written $\phi(t,\ldots)$ when ϕ is written $\phi(x,\ldots)$;

$\exists!x\phi$ for $\exists x[\phi \wedge \forall y(\phi(y/x) \to x = y)]$; $x \neq y$ for $\neg(x=y)$;

$\exists x\eta a\phi$ for $\exists x(x\eta a \wedge \phi)$; $\forall x\eta a\phi$ for $\forall x(x\eta a \to \phi)$.

2.4. *Application terms* or, simply *terms*, are generated as follows:

(i) Each variable and constant is a term;

(ii) if t_1, t_2 are terms then t_1t_2 is a term.

The informal interpretation is that t_1t_2 is the unique value y of t_1 applied to t_2, if t_1 is defined at t_2. In that case we write $t_1t_2 = y$. Since there may be no y with $App(t_1,t_2,y)$, strictly speaking terms cannot be considered part of $\mathcal{L}$. Their use with $\mathcal{L}$ can be established by the following *abbreviations* ($t, t_1, \ldots, t_n$ all terms):

$t \simeq y$ for $t = y$, when t is a variable or constant;

$t_1t_2 \simeq y$ for $\exists x_1,x_2[t_1 \simeq x_1 \wedge t_2 \simeq x_2 \wedge App(x_1,x_2,y)]$;

$(t\downarrow)$ for $\exists y(t \simeq y)$;

$t_1 \simeq t_2$ for $\forall y[t_1 \simeq y \leftrightarrow t_2 \simeq y]$

$\phi(t,\ldots)$ for $\exists y[t \simeq y \wedge \phi(y,\ldots)]$;

$t_1t_2,\ldots,t_n$ for $(\ldots(t_1t_2)\ldots)t_n$.

In each of these abbreviations, the quantified variables on the right are to be distinct and not appear in the expression on the left.

2.5. Classification variables

The classification variables: A,B,C,...,X,Y,Z are introduced by convention to range over the objects for which $Cl(x)$ holds. In other words $\forall X\phi(X)$ is written for $\forall x[Cl(x) \rightarrow \phi(x)]$ and $\exists X\phi(X)$ for $\exists x[Cl(x) \wedge \phi(x)]$. Alternatively, we may consider $\mathcal{L}$ as being expanded to a 2-sorted language $\mathcal{L}^{(2)}$ having this new sort of variable which may be used in any of the atomic formulas. We then take as axioms

(1) $$\exists x(x=X),$$

(2) $$Cl(x) \leftrightarrow \exists X(x=X).$$

When we write $\phi(x_1,\ldots,x_n, X_1,\ldots,X_m)$ we are treating ϕ as a formula of $\mathcal{L}^{(2)}$ (all of whose free variables are among $x_1,\ldots,x_n$, $X_1,\ldots,X_m$).

2.6. Elementary formulas

A formula $\phi(x_1,\ldots,x_n, X_1,\ldots,X_m)$ is said to be *elementary* (with respect to classifications) if

(i) its atomic formulas are all of the form $t_1 = t_2$, $App(t_1,t_2,t_3)$ or $t_1 \eta X_i$ for t_1,t_2,t_3 constants or individual variables, and

(ii) ϕ contains no bound classification variables.

Informally, such ϕ does not refer in any way to the general notion of classification. Any given classifications $X_1,\ldots,X_m$ may be tested only with respect to questions as to which objects belong to them.

Each formula ϕ is assigned a Gödel number $\ulcorner\phi\urcorner$ in a standard way. We shall write $\underline{c}_\phi$ for $\underline{c}_{\ulcorner\phi\urcorner}$ when ϕ is elementary.

2.7. Remarks on the choice of language

1. It might be thought that there should also be a predicate $Op(f)$ expressing that f is an operation. For our purposes this could be introduced instead by definition

$$Op(f) \leftrightarrow \exists x,y(fx \simeq y),$$

since we never really have to deal with completely undefined rules. However, it would not serve our purposes to define $Cl(a)$ as $\exists x(x\eta a)$, since it is important to reserve the possibility that a given classification is empty (a common matter for mathematical investigation).

2. Define $A_1 \subseteq A$ by $\forall x[x\eta A_1 \to x\eta A]$ and $f : A \to B$ by $\forall x[x\eta A \to fx\eta B]$. It follows that if $f : A \to B$ and $A_1 \subseteq A$ then $f : A_1 \to B$. One might prefer to follow the practice of category theory so that for any f there is at most one A (and at most one B) such that $f : A \to B$ holds. The syntax and axioms could easily be modified accordingly if desired. For our purposes it is more convenient not to do this. The algebraic notion of morphism can be explained in T_0 in terms of triples (f,A,B) where $f : A \to B$.

3. The use of the many constants is not essential but is only to simplify statement of the axioms.

3. The theory T_0

3.1. Logical axioms are taken to be those of intuitionistic predicate calculus. Use of classical logic (law of excluded middle) will also be permitted when noted explicitly. (If $\mathcal{L}$ is identified with $\mathcal{L}^{(2)}$ then also axioms 2.5(1), (2) are included.)

3.2. The axioms of T_0 are given in five groups I-V. Some further abbreviations are introduced after II and IV.

I (i) $x = y \vee x \neq y$.

(ii) $fx \simeq y_1 \wedge fx \simeq y_2 \rightarrow y_1 = y_2$.

(iii) $x \eta a \rightarrow Cl(a)$.

II *Basic operations*

(i) (Constant) $\underline{k}xy \simeq x$

(ii) (Substitution) $\underline{s}xy\downarrow \wedge \underline{s}xyz \simeq xz(yz)$

(iii) (Defn. by cases) $(x = y \rightarrow \underline{d}abxy \simeq a) \wedge (x \neq y \rightarrow \underline{d}abxy \simeq b)$

(iv) (Pairing, projection) $\underline{p}x_1x_2\downarrow \wedge \underline{p}_1z\downarrow \wedge \underline{p}_2z\downarrow \wedge \underline{p}_i(\underline{p}x_1x_2) \simeq x_i$

(v) (Zero) $\neg(\underline{p}xy \simeq \underline{0})$

Abbreviations: (x,y) for $\underline{p}xy$

$(x_1,\ldots,x_{n+1})$ for $((x_1,\ldots,x_n), x_{n+1})$

x' for $(x,\underline{0})$; $\underline{1}$ for $\underline{0}'$.

III *Elementary comprehension scheme.* For each elementary $\phi \equiv \phi(x,y_1,\ldots,y_n, A_1,\ldots,A_m)$:

$$\exists C\{\underline{c}_\phi(y_1,\ldots,y_n,A_1,\ldots,A_m) \simeq C \wedge \forall x[x\eta C \leftrightarrow \phi]\}.$$

IV *Join*

$\forall x\eta A\exists X(fx \simeq X) \rightarrow \exists J\{\underline{j}(A,f) \simeq J \wedge \forall z[z\eta J \leftrightarrow \exists x,y(z=(x,y) \wedge x\eta A \wedge y\eta fx)]\}$

V *Inductive generation.* For each formula ψ:

$$\exists I\{\underline{i}(A,R) \simeq I \wedge \forall x\eta A[\forall y((y,x)\eta R \rightarrow y\eta I) \rightarrow x\eta I]$$

$$\{\forall x\eta A[\forall y((y,x)\eta R \rightarrow \psi(y/x)) \rightarrow \psi] \rightarrow \forall x\eta I.\psi\}.$$

These axioms are fairly transparent. Some fine points of meaning will emerge in the next section. The consistency of T_0 and some direct extensions will be established in §4.[2a]

3.3. *Some consequences of the axioms* These are treated informally and only sketched.

(1) *Explicit definition.*

By the usual argument for the combinators $\underline{k}$, $\underline{s}$ we can associate with each (application) term t a new term t* such that vars(t*) $\subseteq$ vars(t)-{x} and

$$t^*\!\downarrow \text{ and } \forall x[t^*x \simeq t].$$

t* is denoted $\lambda x.t$. Informally, it is reasonable that t* have a value no matter what choice of values for its variables, namely, it is the rule which at x follows out the rule given by t. The special case $\lambda z.\underline{s}xyz$ is incorporated in the axiom for $\underline{s}$, according to which $\underline{s}xy\!\downarrow$.

(2) *Pairing, n-tupling.* Note that the projection operations $\underline{p}_i$ are defined for all objects. This is in accord with the informal idea that we can tell whether or not an object z is an ordered pair (x_1,x_2) or not. In the first case take $\underline{p}_i z = x_i$, otherwise $\underline{p}_i z = \underline{0}$ (say). Thus $\exists x,y.z = (x,y) \leftrightarrow \underline{p}(\underline{p}_1 z)(\underline{p}_2 z) \simeq z$. From the pairing axiom we derive $(x_1,x_2) = (y_1,y_2) \rightarrow x_1 = y_1 \wedge x_2 = y_2$.

For each $n \geqslant 2$ and $1 \leqslant i \leqslant n$ we can find p_i^n such that $\forall z.p_i^n z\!\downarrow \wedge \forall x_1,\ldots,x_n.\ p_i^n(x_1,\ldots,x_n) \simeq x_i$. Then given any t we can find t* with vars(t*) $\subseteq$ vars(t) $-\{x_1,\ldots,x_n\}$ such that

$$t^*\!\downarrow \wedge\ t^*(x_1,\ldots,x_n) \simeq t.$$

[2a] Myhill has shown that schema III can be replaced by finitely many axioms, in the expected way.

Namely, t^* is $\lambda z.t[p_1^n z/x_1,\ldots,p_n^n z/x_n]$. We write $t^* = \lambda(x_1,\ldots,x_n).t$.

(3) *The Recursion Theorem.* *(Self-Referential Rules).* The following form

$$\forall f\,\exists g\,\forall y_1,\ldots,y_n[gy_1\ldots y_n \simeq fgy_1\ldots y_n]$$

is proved essentially as in recursion theory: first define the term $s_1 = \lambda y.zxy$ with vars.z,x. Then for all x,y,z:

$$s_1[z,x]\downarrow \wedge (s_1[z,x])y \simeq zxy.$$

Next, form $\lambda x\lambda y.f(s_1[x,x])y$; this exists, call it h. Then also $g = s_1[h,h]$ exists, and for all y

$$gy \simeq fgy,$$

hence also $gy_1\ldots y_n \simeq fgy_1\ldots y_n$.

We can equally well get the theorem in the form

$$\forall f\,\exists g\,\forall y_1,\ldots,y_n[g(y_1,\ldots,y_n) \simeq f(g,y_1,\ldots,y_n)].$$

(4) *Elementary operations on classes.* For each elementary $\phi(x,y_1,\ldots,y_n,A_1,\ldots,A_m)$ write

$$\hat{x}\phi(x,y_1,\ldots,y_n,A_1,\ldots,A_m) \text{ for } \underline{c}_\phi(y_1,\ldots,y_n,A_1,\ldots,A_m).$$

We may then make the following *abbreviations:*

V for $\hat{x}.x=x$;

Λ for $\hat{x}.x\neq x$;

$\{a,b\}$ for $\hat{x}(x=a \vee x=b)$;

$\{a\}$ for $\{a,a\}$;

$-A$ for $\hat{x}.\neg(x\eta A)$;

$A \cap B$ for $\hat{x}(x\eta A \wedge x\eta B)$; $A \cup B$ for $\hat{x}(x\eta A \vee x\eta B)$;

$A \times B$ for $\hat{x}(x = (\underline{p}_1 x, \underline{p}_2 x) \wedge \underline{p}_1 x\eta A \wedge \underline{p}_2 x\eta B)$;

$f : A \to B$ for $\forall x(x\eta A \to fxB)$ [i.e. for $\forall x\eta A\, \exists y(fx \simeq y \wedge y\eta B)$];

$f : A \underset{E_1,E_2}{\to} B$ for $(f : A \to B) \wedge \forall x_1\eta A\, \forall x_2\eta A[(x_1,x_2)\eta E_1$
$\to (fx_1,fx_2)\eta E_2]$;

B^A for $\hat{f}(f:A \to B)$;

$(B,E_2)^{(A,E_1)}$ for $\hat{f}(f : A \underset{E_1,E_2}{\to} B)$;

$\mathfrak{D}(f)$ for $\hat{x}.\exists y(fx \simeq y)$.

(5) *Join and product; union and intersection.* When $\forall x\eta A \exists B(fx \simeq B)$ we write B_x for fx and then $\Sigma_{x\eta A}B_x$ for $\underline{j}(A,f)$. Thus $z\eta(\Sigma_{x\eta A}B_x)$ just in case z has the form (x,y) where $x\eta A$, $y\eta B_x$. Under the same hypothesis we can write

$\Pi_{x\eta A}B_x$ for $\hat{g}.\forall x(x\eta A \to gx\eta B_x)$, [i.e. for $\hat{g}.\forall x\{x\eta A \to (x,gx)\eta\underline{j}(A,f)\}$].

If we take $f = \lambda x.B$ then we write $\Sigma_{x\eta A}B$ and $\Pi_{x\eta A}B$ for these, resp. Thus $z\eta(\Sigma_{x\eta A}B) \leftrightarrow z\eta(A \times B)$ and $z\eta(\Pi_{x\eta A}B) \leftrightarrow z\eta(B^A)$.

Again, under the same hypotheses for A, f and B_x we can introduce $\cup_{x\eta A}B_x$ and $\cap_{x\eta A}B_x$ with the usual definitions.

(6) *The natural numbers.* We defined $x' = (x,\underline{0})$. By the axioms for zero and pairing we have:

(i) $x' \neq \underline{0}$

(ii) $x' = y' \to x = y$

(iii) $x = y' \to y = \underline{p}_1 x$.

Thus we may consider $'$ as the successor operation for generating natural numbers and $\underline{p}_1$ the predecessor operation.

These are now used to set up the inductive generation of N. Let $A = \{\underline{0}\} \cup \hat{x}.\exists y\ (x=y') = \{\underline{0}\} \cup \hat{x}\ (x=(\underline{p}_1 x)')$. Let $R = \hat{z}\exists x,y[z=(y,x) \wedge x=y']$. Define $N = \underline{i}(A,R)$. It is seen that

(iii) $\underline{0}\eta N$

(iv) $x\eta N \rightarrow x'\eta N$

(v) for any $\psi(x,\ldots)$:

$$\psi(\underline{0},\ldots) \wedge \forall x[\psi(x,\ldots) \rightarrow \psi(x',\ldots)] \rightarrow \forall x\eta N.\psi(x,\ldots).$$

Using the Recursion Theorem and definition by cases, we find r_N such that for all x, f, a

$$r_N(x,a,f) \simeq \begin{cases} a & \text{if } x = \underline{0} \\ f(\underline{p}_1 x, r_N(\underline{p}_1(x),a,f)) & \text{if } x \neq \underline{0}. \end{cases}$$

In other words

(vi) $r_N(\underline{0},a,f) \simeq a$

$r_N(x',a,f) \simeq f(x,r_N(x,a,f))$.

Hence for any B,

$$r_N : N \times B \times B^{N\times B} \rightarrow B.$$

r_N is a *recursion operator* for N. Using it we may successively define all primitive recursive functions of natural numbers.

The *bounded minimum operator* $(\mu y \leqslant x)fy \simeq \underline{0}$ and the predicate of *bounded existential quantification* $\exists y \leqslant x.fy \simeq \underline{0}$ are obtained by primitive recursive defining schemes. Applying the recursion theorem we find g such that

(vii) $$g(f,x) \simeq \begin{cases} (\mu y \leqslant x) fy \simeq \underline{0} & \text{if } \exists y \leqslant x . fy \simeq \underline{0} \\ g(f,x') & \text{otherwise.} \end{cases}$$

Let $\mu f \simeq g(f,\underline{0})$; then μf is defined and equal to $\mu x . fx \simeq \underline{0}$ when $\exists x(fx \simeq \underline{0} \wedge \forall y < x . fy\downarrow)$.

Having primitive recursion and μ, we obtain Kleene's enumeration of the partial recursive functions, which associates with each $z \eta N$ a rule $\{z\}$; the total recursive functions are those for which $\{z\} : N \to N$.

Church's Thesis may then be formulated in this language by:

(CT) $$\forall f \eta N^N \, \exists z \eta N \, \forall x \eta N [fx \simeq \{z\}x].$$

(7) *Recursion on inductively generated classifications.* Consider any $\underline{i}(A,R) \simeq I$ in general. By the Recursion Theorem we can find r_I such that

(i) $r_I(x,f) \simeq f(x,r_I)$.

Let $Pd_R(x) = \mathcal{Y}.(y,x)\eta R$. Suppose f is such that

(ii) whenever $x \eta I$ and $g:Pd_R(x) \to V$ then $f(x,g)\downarrow$; then $\forall x \eta I . r_I(x,f)\downarrow$.

r_N is a special case of r_I.

(8) *Tree ordinals.* The countable tree ordinals 0_1 are inductively generated from $\underline{0}$ using successor and N-supremum, where we may identify $\sup_{x \eta N} hx$ with (h,N).

In general, define $\sup_X h$ or $\sup_{x \eta X} hx$ simply as (h,X).

Suppose A consists only of non-empty classifications, i.e. $\forall x\eta A\exists X(x=X \wedge \exists z(z\eta X))$. Thus $z\eta X \wedge X\eta A \leftrightarrow (X,z)\eta J$ where $J = \Sigma_{y\eta A}y$ and $x\eta A \leftrightarrow \exists z.(x,z)\eta J$. We can use the principle of inductive generation to find a classification $I = 0_A$ satisfying the following:

(i) $\underline{0}\eta 0_A$

(ii) $x\eta 0_A \rightarrow x'\eta 0_A$

(iii) $X\eta A \wedge h:(X \rightarrow 0_A) \rightarrow (\sup_X h)\eta 0_A$

(iv) $\psi(\underline{0}) \wedge \forall x[\psi(x) \rightarrow \psi(x')] \wedge \forall X\eta A\forall h[\forall z\eta X\psi(hz) \rightarrow \psi(\sup_X h)] \rightarrow \forall\psi\forall x\eta 0_A.\psi(x)$, for each $\psi(x) = \psi(x,\ldots)$.

Namely 0_A is $\underline{i}(B,R)$ where

$$B = \{\underline{0}\} \cup \hat{x}.\exists y(x=y') \cup \hat{x}.\exists X\eta A\exists h(\forall y\eta X(hy\downarrow) \wedge x = (h,X))$$

and $(y,x)\eta R \leftrightarrow x = y' \vee \exists X,y,z[x = (h,X) \wedge X\eta A \wedge z\eta X \wedge y = hz]$.

When A is empty, 0_A has the same numbers as N. For $A = A_1 = \{N\}$, 0_A is the 0_1 described above. Then for $A = A_2 = \{N,0_1\}$, 0_A is 0_2, etc. We may define A_n and 0_n recursively for $n\eta N$ by: $0_n = 0_{A_n}$ and $A_{n+1} = A_n \cup \{0_n\}$. Then we can pass to transfinite number classes e.g. by taking $A = \cup_{x\eta N}0_x$. More generally, given any C we can associate an 0_x in a natural way with each $x\eta 0_C$, by recursion on 0_C.

(9) *Finite and transfinite types.* Suppose A consists only of non-empty classifications, as in the preceding section. Given a,b,h write $a \dot{\times} b$ for $(\underline{0},a,b)$, $(a \dot{+} b)$ for $(\underline{1},a,b)$, $\dot{\Sigma}_{x\eta a}hx$ for $(2,a,h)$ and $\dot{\Pi}_{x\eta a}hx$ for $(3,a,h)$. We inductively generate a classification Typ_A of *A-ary type symbols*, by which is intended that we can form $\dot{\Sigma}$ and $\dot{\Pi}$ over any X in A.

(i) $\underline{0}\eta\mathrm{Typ}_A$

(ii) $a\eta\mathrm{Typ}_A \wedge b\eta\mathrm{Typ}_A \rightarrow (a \dot{\times} b)\eta\mathrm{Typ}_A \wedge (a \dot{\rightarrow} b)\eta\mathrm{Typ}_A$

(iii) $X\eta A \wedge (h:X \rightarrow \mathrm{Typ}_A) \rightarrow (\dot{\Sigma}_{x\eta X}hx)\eta\mathrm{Typ}_A \wedge (\dot{\Pi}_{x\eta X}hx)\eta\mathrm{Typ}_A$.

We have in addition a corresponding principle of proof by induction on Typ_A for each formula ψ, which permits definition by recursion on A. Greek letters $\sigma, \tau, \ldots$ are used in the following for type symbols.

When $A = \Lambda$ we call Typ_A the *finite type symbols*. Typ_N is written for $\mathrm{Typ}_{\{N\}}$, the N-ary type symbols. For any A we may define by recursion an operation on Typ_A whose value at each σ is denoted by N_σ, satisfying:

(i) $N_0 = N$

(ii) $N_{\sigma\dot{\times}\tau} = N_\sigma \times N_\tau$, $N_{\sigma\dot{\rightarrow}\tau} = N_\tau^{N_\sigma}$

(iii) $N_{\dot{\Sigma}_{x\eta X}hx} = \Sigma_{x\eta X}N_{hx}$, $N_{\dot{\Pi}_{x\eta X}hx} = \Pi_{x\eta X}N_{hx}$.

It is proved by induction that

(iv) $\sigma\eta\mathrm{Typ}_A \rightarrow \mathrm{Cl}(N_\sigma)$.

We call the operation $\sigma \mapsto N_\sigma$, the (*non-extensional*) *A-ary type hierarchy*.

An *extensional A-ary type hierarchy* can also be defined. This determines for each $\sigma \in \mathrm{Typ}_A$ two classifications $\overline{N}_\sigma$ and E_σ, defined simultaneously. We write down the clauses for the finite type symbols only:

(i) $\overline{N}_0 = N$, $E_0 = \hat{z}.\exists x(z=(x,x))$.

(ii) $\overline{N}_{\sigma\dot{\times}\tau} = \overline{N}_{\sigma\times\tau}$, $E_{\sigma\dot{\times}\tau} = \hat{z}.\exists x_1,y_1,x_2,y_2[z = ((x_1,y_1),(x_2,y_2))$
$\wedge (x_1,x_2)\eta E_\sigma \wedge (y_1,y_2)\eta E_\tau]$.

(iii) $\overline{N}_{\sigma\dot{\to}\tau} = (\overline{N}_\tau, E_\tau)^{(\overline{N}_\sigma, E_\sigma)} = \hat{f}\{f:\overline{N}_\sigma \to \overline{N}_\tau \wedge \forall x,y\eta\overline{N}_\sigma[(x,y)\eta E_\sigma \to (fx,fy)\eta E_\tau]\}$

$E_{\sigma\dot{\to}\tau} = \hat{z}.\exists f,g\{z=(f,g) \wedge f,g\eta\overline{N}_{\sigma\dot{\to}\tau} \wedge \forall x\eta\overline{N}_\sigma (fx,gx)\eta E_\tau\}$.

It is seen that:

(iv) for each finite type symbol σ, $Cl(\overline{N}_\sigma) \wedge Cl(E_\sigma)$ and E_σ is an equivalence relation on $\overline{N}_\sigma$.

It is obvious how to extend the definition of $(\overline{N}_\sigma, E_\sigma)$ to all $\sigma\eta Typ_A$.

Inductively generated classifications of trees with prescribed codings and branchings can be treated in a way similarly to that for the 0_A and Typ_A. Infinite formulas and terms may be considered among such.

3.4. Non-extensionality of the basic notions

The classifications $\overline{N}_\sigma$, E_σ introduced in 3.3(9) will be used at various points below to relate certain statements in T_0 to classical mathematical statements concerning extensionally conceived functions and sets. It was stressed in the introduction that for T_0 as a whole the intended conception of the basic notions is intensional. Kreisel has raised the question whether there is an actual *conflict between extensionality and self-application* in this context. He also referred back to a related specific question in Kreisel [71], p.186, as to whether enumeration without repetition conflicts with the axioms for enumerative recursion theory called BRFT in Friedman [71]. As it happens, Friedman stated (loc. cit. p.117) that these are jointly inconsistent. This can be transferred to the present context, since the axioms I(i) and II of T_0 are essentially the same as for an *enumerative system* in Friedman [71], which are in turn equivalent to BRFT. In addition, there is also a conflict of extensionality with self-application for classifications (at least as formulated in T_0). The details are as follows.

The statement of *extensionality for rules* may be considered in either of the following forms:

(i) (a) $\forall f,g[\forall x(fx \simeq gx) \rightarrow f = g]$,

(b) $\forall f,g,h[\forall x(fx \simeq gx) \rightarrow hf \simeq hg]$.

These are equivalent as we see by applying (b) to $h = \lambda x.x$. Similarly the statement of *extensionality for classifications* is considered in the forms:

(ii) (a) $\forall A,B[\forall x(x\eta A \leftrightarrow x\eta B) \rightarrow A = B]$,

(b) $\forall A,B,C[\forall x(x\eta A \leftrightarrow x\eta B) \rightarrow (A\eta C \leftrightarrow B\eta C)]$.

Again these are equivalent by applying (b) to $C = \{A\}$.

Let $Tot(x)$ be the formula $\forall y\exists z(xy \simeq z)$ expressing that x *is a total operation*. Let $e = \lambda x.\lambda y.\underline{d}\underline{0}\underline{0}(xy)\underline{0}$. By 3.3(1) we can prove in T_0 that for all x,y: $ex\downarrow$ and $exy \simeq \underline{d}\underline{0}\underline{0}(xy)\underline{0}$. Thus

(iii) $Tot(e)$, $\forall y(exy\downarrow \leftrightarrow xy\downarrow)$, $\forall y(xy\downarrow \rightarrow exy \simeq \underline{0})$ and $Tot(x) \leftrightarrow Tot(ex)$.

Let $0^* = \lambda x.\underline{0}$. If extensionality held for rules we would have

(iv) $Tot(x) \leftrightarrow ex \simeq 0^*$.

Put $n = \lambda z.\underline{d}\underline{1}\underline{0}\underline{z}\underline{0}$ so

(v) $Tot(n)$ and $\forall z\neg(nz \simeq z)$.

Finally, let $f = \lambda x.\underline{d}(n(xx))\underline{0}\ (ex)0^*$ so that for all x:

$$\text{(vi)} \quad fx \simeq \begin{cases} n(xx) & \text{if } ex \simeq 0^* \\ \underline{0} & \text{otherwise.} \end{cases}$$

Tot(f) because Tot(e) and Tot(x) whenever $ex \simeq 0^*$; hence $ef \simeq 0^*$ and $ff \simeq n(ff)$ which is impossible by (v). It is seen by this argument that *extensionality for rules is inconsistent with axioms* I(i), (ii) *and* II *of* T_0.

Turning to classifications, let $\phi(y,x)$ be $\exists z.xy \simeq z$ and $c = \underline{c}_\phi$. By Axiom III, $cx\downarrow$ for all x, and Cl(cx). Thus

(vii) Tot(c) and $\forall y[y\eta(cx) \leftrightarrow \exists z.xy \simeq z]$, so $\forall y(y\eta(cx)) \leftrightarrow$ Tot(x).

If extensionality held for classifications we would have

(viii) $\text{Tot}(x) \leftrightarrow cx \simeq V$.

In the definition (vi) of f above replace e by c and 0^* by V. It is seen that *extensionality for classifications is inconsistent with axioms* I, II, III *of* T_0.

4. *Metamathematical results concerning* T_0 *and related theories*

4.1. A recursion-theoretic model of T_0. I believe that the general informal interpretation given in 2.2 should be clear enough for one to recognize that the axioms of T_0 are correct, hence consistent. As a particular informal interpretation, V may be taken to consist of all expressions generated from finitely many (> 1) symbols, and $fx \simeq y$ to hold whenever f is a program (represented in V) for a mechanical computation which yields the value y at the argument x. Further, the classifications are taken to be certain finite or infinite formulas ϕ represented in V, successively built up and with meaning explained according to the axioms III-V; $a\eta\phi$ is written when ϕ holds of a. Evidently, (CT) is also correct in this interpretation.

The following serves to establish the consistency of T_0 + (CT) assuming set-theory and classical logic. (It is not excluded that one may accept both this and the preceding.) Here "model" is used in its usual sense so that also the laws of classical logic may be applied in T_0. The proof itself gives set-theoretical form to the informal interpretation just given.

THEOREM 4.1.1

There is a model of T_0 *in which the range of the variables is the set* ω *of natural numbers and* $fx \simeq y$ *is interpreted as* $\{f\}(x) \simeq y$. (CT) *is true in this model.*

Note: We are using ω for the natural numbers, to distinguish it from N which is to be interpreted as a particular element of ω. $\{f\}$ $(f=0,1,2,\ldots)$ is a standard enumeration of the partial recursive functions on ω. Church's Thesis (CT) is formulated in terms of N as in 3.3(6).

The proof is straightforward and will only be sketched. By ordinary recursion theory we may choose numbers k, s, d, p, p_1, p_2 so that the basic operation axioms II are satisfied. (Pairing is chosen so that $(x,y) \neq 0$.) Take $\{c_n\}z \simeq (1,n,z)$, $\{j\}(a,f) \simeq (2,a,f)$, $\{i\}(a,r) \simeq (3,a,r)$.

A set Cl_α is defined by transfinite recursion for each ordinal α; the predicate $a \in \cup_\alpha Cl_\alpha$ will be the interpretation of $Cl(a)$. We also define $\{x : x \in \omega \,\&\, x\eta a\}$ for $a \in Cl_\alpha$ along with Cl_α. Let Cl_0 be empty. Suppose given Cl_α and η restricted to $\omega \times Cl_\alpha$. Then for elementary $\phi(x,y_1,\ldots,y_n,A_1,\ldots,A_m)$, the truth of ϕ as a function is well-determined when we assign to each A_i the value a_i in Cl_α. $Cl_{\alpha+1}$ consists of all numbers b obtained by one of the following clauses (i)-(iv); $\{x : x\eta b\}$ is also defined for each of these:

(i) $b \in Cl_\alpha \Rightarrow b \in Cl_{\alpha+1}$; $x\eta b$ is unchanged.

(ii) If $\phi(x,y_1,\ldots,y_n,A_1,\ldots,A_m)$ is elementary and $a_1,\ldots,a_m \in Cl_\alpha$ and $b = \{c_\phi\}(y_1,\ldots,y_n,a_1,\ldots,a_m)$ then $b \in Cl_{\alpha+1}$;

$$x\eta b \Leftrightarrow \phi(x,y_1,\ldots,y_n,a_1,\ldots,a_m) \text{ is true.}$$

(iii) If $a \in Cl_\alpha$ and $\forall x\eta a[\{f\}(x) \in Cl_\alpha]$ then $b = \{j\}(a,f) \in Cl_{\alpha+1}$;

$$z\eta b \Leftrightarrow \exists x,y[z = (x,y) \wedge y\eta\{f\}(x)].$$

(iv) If $a, r \in Cl_\alpha$ then $b = \{i\}(a,r) \in Cl_{\alpha+1}$;

$$u\eta b \Leftrightarrow u \in \cap X\{X \subseteq \omega \wedge \forall x\eta a[\forall y((y,x)\eta r \Rightarrow y \in X) \Rightarrow x \in X]\}.$$

If α is a limit number take $Cl_\alpha = \cup_{\beta<\alpha} Cl_\alpha$. It is then seen that $Cl = \cup Cl_\alpha[\alpha \text{ countable}]$ provides a model of the axioms III-V. In this model, the interpretation of N (a particular $\{i\}(a,r)$) is such that

$$(\text{iv})^N \qquad u\eta N \Leftrightarrow u \in \cap X\{0 \in X \wedge \forall x(x \in X \Rightarrow (x,0) \in X)\}.$$

Thus N is isomorphic with ω, with the successor operation on ω

corresponding to the operation $x \mapsto (x,0)$ which is x' in $\mathcal{L}$. This recursive isomorphism is used to show that (CT) is true in the model.

(1) *Remarks*. (a) Except for (CT) the same method of proof may be applied to give a model of T_0 on any structure $(M, App, 0, k, s, d, p, p_1, p_2)$ which satisfies the axioms I(ii) and II of T_0. As mentioned in §3.4 above these are essentially the same as the enumerative systems of Friedman [71] (§1). Recursion theory on admissible sets satisfying Σ_1-uniformization gives a wide and familiar class of examples of such structures. The theory of prime computable functions on any structure (Moschovakis [69]) provides still further examples; this simplifies when the structure is on a transitive set closed under pairing.

(b) It might be thought we could just as well get a model with $Cl = \omega$, e.g. simply by taking $x \not\eta a$ for each $a \notin \cup_\alpha Cl_\alpha$ a defined in the proof. However, $T_0 + \forall a Cl(a)$ is inconsistent: for it follows from $\forall a Cl(a)$ that there is a classification $B = \Sigma_{a \eta V} a$; then take $x \eta C \leftrightarrow (x,x) \not\eta B$.

(c) The proof of Theorem 4.1.1 can be formalized in classical 2nd order analysis, by taking Cl and the graph of the characteristic function of η as the least pair of sets satisfying certain (arithmetical) closure conditions[3]. It may be of interest to see if there are some familiar subsystems of analysis (or set theory) which are of the same strength as T_0.

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[3]To be more precise, T_0 + (CT) may be translated into the subsystem $(\Pi^1_2\text{-CA})$ + (BI) of 2nd order analysis.

(2) *Question. Is* $T_0^{(c)}$ *(in classical logic) no stronger than* $T_0^{(i)}$ *(in intuitionistic logic)?*

The usual reduction by Gödel's $\neg\neg$-translation breaks down with iterated inductive definitions. (cf. Zucker [73] for some of the problems involved in comparing classical with intuitionistic theories of such definitions.)

(3) *Explicit definability and disjunction properties.* These have been established for most intuitionistic systems which have been considered in the literature, including various theories of species and sets (cf. (Troelstra [73] for much of this and further references). Contrary to my expectation, Myhill pointed out that they fail for T_0 for the simple reason that though $(a = b \vee a \neq b)$ is an axiom, the theory does not decide which of $(t_1 = t_2)$ or $\neg(t_1 = t_2)$ holds for various closed (and defined) t_1, t_2. He conjectures that the properties in question do hold for some simple extensions of T_0 by such basic sentences. In any case, as Kreisel and Troelstra have both emphasized, the fact that a theory enjoys these properties is neither necessary nor sufficient for its constructivity (cf. Troelstra [73], p.91).

Let FT be the language of finite type theory with induction and recursion over N, e.g. the language $N - HA^{\omega}$ of (Troelstra [73]) I.6, expanded to include product types. We have variables x^{σ}, y^{σ},... and an equality relation $=_{\sigma}$ for each type symbol σ. If θ is a sentence of $\mathcal{L}_{FT}$ let $\theta^{(\overline{N}_{\sigma},E_{\sigma})}$ or simply $\theta^{(\overline{N},E)}$ be its translation into $\mathcal{L}$, taking the variables of type σ to range over the members of $\overline{N}_{\sigma}$ and translating $x =_{\sigma} y$ by $(x,y)\eta E_{\sigma}$.

Note that the interpretation of $(\overline{N}_\sigma, E_\sigma)_\sigma$ in the model $(\omega, Cl, \ldots)$ of Theorem 1 is in 1-1 correspondence with the *hierarchy of hereditarily extensional (recursive) operations* $(HEO_\sigma, \equiv_\sigma)_\sigma$ (due to Kreisel; cf. (Troelstra [73], II.4). This is an isomorphism with respect to pairing, projections and application. $\theta^{(HEO,\equiv)}$ is written for the interpretation of θ in this hierarchy.

COROLLARY 4.1.2

If θ *is a sentence of* $\mathcal{L}_{FT}$ *and* $T_0 \vdash \theta^{(\overline{N},E)}$ *(with classical logic) then* $\theta^{(HEO,\equiv)}$ *is true.*

The definition of $(HEO_\sigma, \equiv_\sigma)$ can be extended in an obvious way to transfinite types, for which the corollary continues to hold.

4.2. Set-theoretical interpretation of T_0. Here we want an interpretation which matches up the extensional finite type hierarchy $(\overline{N}_\sigma, E_\sigma)_\sigma$ with the set-theoretical *maximal type structure* $(M_\sigma)_\sigma$ defined by: $M_0 = N$, $M_{\sigma \dot{\times} \tau} = M_\sigma \times M_\tau$ and

$$M_{\sigma \dot{\to} \tau} = \{F \mid Fun(F) \wedge Dom(F) = M_\sigma \wedge F : M_\sigma \to M_\tau\}.$$

These are defined in Zermelo set-theory (ZS).

To extend the following group of results to transfinite types we need Zermelo-Fraenkel set theory; thus they are stated for ZF instead of ZS.

THEOREM 4.2.1

(i) For any model $\mathfrak{A} = (A, \in)$ *of* ZF *we can associate a model* $\mathfrak{A}^*$ *of* T_0 *in which* $V = A$.

(ii) The interpretation of the $\overline{N}_\sigma, E_\sigma$ *in* $\mathfrak{U}^*$ *is such that* $\overline{N}_\sigma/E_\sigma$ *is in* 1-1 *correspondence with* M_σ *of* $\mathfrak{U}$ *for each finite type* σ. *This correspondence is the identity on* N *and preserves pairing, projections and applications.*

The idea of the proof is to use the theory of prime computable functions (Moschovakis [69]) for the structure $\mathfrak{A}' = (A,\in,F,\langle a\rangle_{a\in A})$, where $F(u,x)\ u(x)$; this gives an enumerative system of functions $PR(\mathfrak{A})$ which includes every constant function[3a]. Take $Cl_0 = \{0\} \times A$; for $x,y \in A$ take $x\eta(0,y) \leftrightarrow \exists z[x=(0,z) \wedge z \in y]$. Then proceed to determine Cl_α and η on Cl_α for $\alpha > 0$ just as in 4.1.1. The definition of Cl_0 gives an injection of $(A,\in)$ in (Cl,η). The correspondence is set up recursively. For example, associate with each f in $\overline{N}_{0\dot{\to}0}$ the function $\lambda x \in N.fx$ in $M_{0\dot{\to}0}$; this association is surjective since every element of $M_{0\dot{\to}0}$ is a partial function in $PR(\mathfrak{A})$ and equivalent members of $\overline{N}_{0\dot{\to}0}$ correspond to the same function.

For θ in L_{FT} let $\theta^{(M)}$ be its interpretation in the maximal type structure, taking $=_\sigma$ to be $=$ for each σ.

COROLLARY 4.2.2

If θ *is a sentence of* $\mathcal{L}_{FT}$ *and* $T_0 \vdash \theta^{(\overline{N},E)}$ *then* $ZF \vdash \theta^{(M)}$.

4.3. Realizing axioms of choice. By the *relative axiom of choice schema* in $\mathcal{L}$ we mean all formulas:

[3a] J. Stavi pointed out to me that my previous formulation of this argument in terms of admissible sets worked only for ZFC and then only with some additional considerations. He suggested the use of Moschovakis [69] instead.

(AC) $\forall x\eta A\ \exists y\ \phi(x,y) \to \exists f\ \forall x\eta A\ \phi(x,fx)$.

For particular A we denote this by (AC_A). This may be analyzed as a consequence of

(AC_V) $\forall x\ \exists y\ \phi(x,y) \to \exists f\ \forall x\ \phi(x,fx)$

and a principle called Independence of Premiss:

(IP) $\forall x\eta A\ \exists y\ \phi(x,y) \to \forall x\ \exists y(x\eta A \to \phi(x,y))$.

It will be shown here that (AC) gives a conservative extension of a certain subtheory $T_0^{(-)}$ of T_0, where the use of the existential quantifier in defining properties is restricted to the cases in which that use can be made explicit. It can be shown that T_0 itself is consistent with some instances of (AC), including (AC_V).

The axiom groups I, II for $T_0^{(-)}$ are the same as for T_0. III-V are modified as follows:

III'. Elementary comprehension schema is restricted to ϕ which do not contain existential quantifiers.

III''. We add axioms for operations <u>e</u>, <u>dm</u> where $\underline{e}(A,B) \simeq B^A$, $\underline{dm}f \simeq \mathfrak{D}(f)$.

IV is as before for join; to this is added an axiom
IV' for product, $\underline{pr}(A,f) \simeq \Pi_{x\eta A} fx$ under the same hypothesis.

V. Inductive generation is modified to an axiom for an operation $\underline{i}^*(A,S)$, replacing '$(y,x)\eta R$' throughout by '$\exists z.(y,x,z)\eta S$'. Again the logic of $T_0^{(-)}$ is taken to be intuitionistic, unless otherwise noted.

It is seen that $T_0^{(-)}$ has practically the same mathematical consequences as those indicated in 3.3 for T_0. Continuing the idea here we could consider a theory $T_0^{(--)}$ in which also the use of disjunction in defining properties is restricted. The only loss then are the general $\cup$ and $\bigcup$ operations; the disjoint union always serves for the remaining mathematical uses.

The classes $\mathcal{F}_0$, $\mathcal{F}_1$ of formulas of $\mathcal{L}$ are defined as follows.

(i) each $\mathcal{F}_i$ contains all atomic formulas and is closed under the operations of $\wedge$, $\vee$ and universal quantification;

(ii) If ϕ is in $\mathcal{F}_0$ and ψ is in $\mathcal{F}_1$ then $(\psi \to \phi)$ is in $\mathcal{F}_0$ and $(\phi \to \psi)$ is in $\mathcal{F}_1$;

(iii) If ϕ is in $\mathcal{F}_0$ then $\exists x\phi$ is in $\mathcal{F}_0$.

Thus all formulas without $\exists$ are in both $\mathcal{F}_0$ and $\mathcal{F}_1$.

THEOREM 4.3.1

$T_0^{(-)}$ + (AC) *is a conservative extension of* $T_0^{(-)}$ *for formulas in* $\mathcal{F}_0$; *in fact, if* $T_0^{(-)} + (AC) \vdash \exists x.\phi(x)$ *where* ϕ *is in* $\mathcal{F}_0$ *then* $T_0^{(-)} \vdash \phi(t)$ *for some application term* t.

Again the proof is sketched. We associate with each formula ϕ a formula ρ_ϕ with one new free variable f which we write $f\rho\phi$ and read "f realizes ϕ".[4]

(i) for ϕ atomic, $f\rho\phi$ is $(f=f) \wedge \phi$;

[4] Cf. Troelstra [73], Ch. III for similar variants of Kleene's definitions of realizability.

(ii) $f\rho(\phi \wedge \psi)$ is $(\underline{p}_1 f)\rho\phi \wedge (\underline{p}_2 f)\rho\psi$;

(iii) $f\rho(\phi \vee \psi)$ is $(\underline{p}_1 f)\rho\phi \vee (\underline{p}_2 f)\rho\psi$;

(iv) $f\rho(\phi \rightarrow \psi)$ is $\forall g[g\rho\phi \rightarrow fg\rho\psi]$;

(v) $(f\rho\forall x\phi)$ is $\forall x(fx\rho\phi)$

(vi) $(f\rho\exists x\phi)$ is $\exists x[f = (\underline{p}_1 f,x) \wedge (\underline{p}_1 f)\rho\phi]$.

Also with each ϕ is associated in $T_0^{(-)}$ a non-empty class $Typ(\phi)$ which includes all rules f which may realize ϕ. In particular, we take $Typ(\psi \rightarrow \phi) = Typ(\phi)^{Typ(\psi)}$.

The following may be shown:

(vii) If $T_0^{(-)} + (AC) \vdash \phi$ then $T_0^{(-)} \vdash (t\rho\phi)$ for some application term t.

(viii) If $\phi \in \mathscr{F}_0$ then $T_0^{(-)} \vdash \exists f(f\rho\phi) \rightarrow \phi$ and if $\psi \in \mathscr{F}_1$ then $T_0^{(-)} \vdash \psi \rightarrow \forall f \eta Typ(\psi)(f\rho\psi)$.

The theorem follows directly from (vii), (viii).

COROLLARY 4.3.2

$T_0^{(-)} + (AC) + (CT)$ *is consistent.*

More generally, $T_0^{(-)} + (AC)$ is consistent with any ψ such that $\exists f(f\rho\psi)$ is true in the model of 4.1.1.

A similar theorem can be established for $T_0^{(--)}$, by appropriately modifying the definition of $\mathscr{F}_0$, $\mathscr{F}_1$. One can also obtain analogous results for T_0 in place of $T_0^{(-)}$, but only for certain extensions (AC_A) of T_0 — roughly speaking for those classes A whose existence is established in $T_0^{(-)}$. But this seems to require a somewhat more delicate treatment of realizability starting with $f\rho(x\eta A)$ written as $(f,x)\eta A^*$ (A^* a variable associated with A).

It is easily seen that $T_0^{(-)} + (AC) + (CT)$ is inconsistent with classical logic.

QUESTION[5]

Is $T_0 + (AC_V) \pm (CT)$ *consistent with classical logic?*

Using the primitive recursive relation $<$ on N, the schema for the *least element principle* is the following:

(LE) $$\exists x \eta N . \phi \rightarrow \exists x \eta N [\phi \wedge \forall y (y < x \rightarrow \neg \phi(y/x)].$$

There is a corresponding rule (LER), to infer the conclusion of this implication from the hypothesis. If that were a derived rule of $T_0^{(-)} + (AC)$ then whenever $T_0^{(-)} + AC \vdash \exists x \eta N . \phi$, the conclusion ψ would be proved to be realizable in $T_0^{(-)}$. But then $\exists f (f \rho \psi)$ would be true in the model of 4.1.1. It would follow that if ψ is a number-theoretical statement then ψ is *recursively realizable.* Hence by the result of Kleene (Kleene [52], p.511) we obtain:

COROLLARY 4.3.3

(LER) *is not a derived rule of* $T_0^{(-)} + (AC)$ *even for hypotheses provable in* $T_0^{(-)}$.

5. *Relations with constructive and recursive mathematics*

The discussion in this section will be sketchy and programmatic.

---

[5]Raised by R. Statman.

5.1. *Constructivity* is understood here in the sense of *intuitionism*[6]. Bishop [67], [70] takes a more restrictive position but within which he redevelops substantial portions of mathematics (cf. also Bishop, Chang [72]). The essential difference is that he rejects use of Brouwer's notion of *choice sequence*, using alternative means for the treatment of analysis and topology. The dispensability of choice sequences was theoretically justified in some systems of intuitionistic analysis by Kreisel, Troelstra [70].

$\mathcal{L}$ is informally interpreted in intuitionistic terms as follows: $fx \simeq y$ holds if f is a *construction* (or *constructive function*) which gives the value y when applied to x. The notion of classification is interpreted as that of *species* (or *type*) and $x\eta A$ by: x belongs to the species A. Bishop's notion of *set* may be identified more particularly with pairs (A,E) for which A, E are classifications and E is an equivalence relation on A. (Inversely, classifications may be explained in Bishop's terms as sets equipped with the relation of literal identity; for alternative explanations cf §7.3 below.)

There is no notion in $\mathcal{L}$ which expresses that of choice sequence. Nor is there a means of expressing in $\mathcal{L}$ the notion of *constructive proof*. The latter is essential for the intuitionistic reduction of logic to mathematics (cf. Kreisel [65], §2).

(1) *Claim.* T_0 *is constructively correct.*

It seems to me that this should be accepted under all the explanations

[6] cf. e.g. Heyting [72], Kreisel [65], §2, Tait [68], and Troelstra [69] for various explanations of this position.

of the constructivist position mentioned here; cf. particularly the line of argument in (Tait [68]).

(2) *Claim. All of Bishop's work* (Bishop [67], Bishop, Chang [72]) *can be formalized in* T_0.

A related claim for a portion of Bishop [67], using a somewhat weaker theory of finite types in place of T_0, has been made by Goodman and Myhill [72]. However, they did not see how to deal with Bishop's general concept of set. As explained above, this is handled directly in T_0. Actually for (2) one should need only that part of V (along with I-IV) required to obtain N and inductively generated N-branching trees (used for countable ordinals and Borel sets).

5.2. *Relations with recursive mathematics*

There have been a number of investigations of *recursive analogues* of classical notions[7]. These yield results concerning statements $\phi^{(rec)}$ formulated in recursive terms analogous to some classical statement $\phi^{(cl)}$. The results for which $\phi^{(rec)}$ is true are often called *positive* while those for which it is false are called *negative*. For example, the theorem on the existence of the maximum of a continuous function on a closed interval has a positive recursive analogue; the statement that the maximum is taken on at some point has a negative analogue.

The interest of such a program obviously depends to a good extent on the choice of $\phi^{(rec)}$ given $\phi^{(cl)}$. It may be asked whether reasonable requirements for this choice can be formulated in

[7]For set theory cf. e.g. Dekker, Myhill [60], Crossley [69]; for algebra, Mal'cev [71], Rabin [62], Ershov [68]; for analysis, Specker [59], the Markov school (Sanin [68] and Tseytin, Zaslavsky, Shanin [66]); and for topology, Lacombe [59].

precise terms. The following is an example of such for a class of statements that covers many of the actual examples.

(1) *Suppose* $\phi^{(cl)}$ *is provably equivalent in set theory* (say ZFC) *to* $\theta^{(M)}$ *where* θ *is a sentence of* $\mathcal{L}_{FT}$; *then* $\theta^{(HEO,\equiv)}$ *is a candidate as the choice for* $\phi^{(rec)}$. (For example, where $\phi^{(cl)}$ concerns real numbers, θ will deal instead with Cauchy sequences of rationals under an equivalence relation.) When a choice is made according to (1) we can hope to learn much more from a positive result, in the light of 4.1.2 and 4.2.2.

(2) *Conjecture. For each known positive result of recursive mathematics of the form* $\theta^{(HEO,\equiv)}$ *where* θ *is a sentence of* $\mathcal{L}_{FT}$ *we have* $T_0 \vdash \theta^{(\overline{N},E)}$. We may regard $(\overline{N},E)$ in this case as a *constructive analogue* (or *substitute*) $\phi^{(cv)}$ of $\phi^{(cl)}$ which in fact is a *generalization* of both $\phi^{(rec)}$ and $\phi^{(cl)}$. Indeed, by 4.2.1(ii), read classically $\phi^{(cv)}$ *is equivalent* to $\phi^{(cl)}$. These relationships illustrate the following.

(3) *General expectation.* (i) *Each classical theorem* $\phi^{(cl)}$ *for which a recursive analogue* $\phi^{(rec)}$ *has been considered has a constructively meaningful form* $\phi^{(cv)}$. (ii) *When* $\phi^{(rec)}$ *is true,* $\phi^{(cv)}$ *is constructively provable.*

We may add, for the particular language and axioms considered:

(3) (iii) *When* $\phi^{(rec)}$ *is false then* $\phi^{(cv)}$ *is independent of* T_0.

Obviously we can also get independence results for any T such that $\phi^{(cv)}$ is interpreted as $\phi^{(rec)}$ in a suitable model.

REMARKS

(a) Requirements of the kind (1) above are only a first step to finding appropriate recursive and constructive analogues of classical statements. For even if such a choice is made, we may have θ_1, θ_2 with $\phi^{(cl)}$ equivalent in set theory to both $\theta_1^{(M)}$ and $\theta_2^{(M)}$, yet θ_1 is true and θ_2 is false in $(HEO,\equiv)$. For example, the classical theorem may have the form $\exists f\ \forall x \in A\ \phi(x,f(x))$. If A is definable in the form $\exists y.(x,y) \in B$, it may not be possible to find f as an effective function of x alone. On the other hand, for the classically equivalent statement $\exists g\ \forall(x,y) \in B \phi(x,g(x,y))$ we may be able to find g as an effective function. This is a well-recognized technique for finding positive recursive or constructive substitutes of classical theorems. For a smooth-running positive development one usually makes a choice of notions (e.g. Cauchy sequences considered only as paired with a rate-of-convergence function) which automatically involve this technique wherever needed.

(b) It is possible that the theory $T_0^{(-)} + (AC)$ could lend itself to the purpose of (3) above in the following way. First find a statement ψ of $\mathcal{L}$ which is equivalent in set theory to $\phi^{(cl)}$ and such that $T_0^{(-)} + (AC) \vdash \psi$. Then take $\phi^{(cv)}$ to be $\exists f(f\rho\psi)$. Note that $\phi^{(cv)}$ is also classically equivalent to $\phi^{(cl)}$ but now $T_0^{(-)} \vdash \phi^{(cv)}$. Finally, let $\phi^{(rec)}$ be the interpretation of $\phi^{(cv)}$ in the recursion-theoretic model of 4.1.1. Even with this approach one would still have to go through some of the work of the preceding remark, since when applying (AC_A) we can only use classifications proved to exist in $T_0^{(-)} + AC$; these in general do not include existentially definable A.

(c) Since T_0 is not extensional, when dealing with generalizations of classical theorems it is necessary throughout to replace sets by pairs (A,E) where E is an equivalence relation on A. Similarly, instead of algebraic structures $\mathfrak{A} = (A\ ;\ R_1,\ldots,f_1,\ldots,a_1,\ldots)$ one will consider more generally pairs $(\mathfrak{A},E)$ of such for which E is a congruence relation on $\mathfrak{A}$. The operation $\mathfrak{A} \to \mathfrak{A}/E$ cannot be performed, but when E is carried along, this is not necessary.

Call a classification X *decidable relative to* B if $X \subseteq B$ and X has a characteristic function g relative to B, i.e.

$$\forall x \eta B[(g_x \simeq \underline{0} \vee g_x \simeq \underline{1}) \wedge (g_x \simeq \underline{0} \leftrightarrow x\eta X)].$$

Call A *denumerable* if there exists $h : N \underset{\text{onto}}{\to} A$. When A is denumerable and E is a decidable congruence relation (relative to A^2) we can choose representatives of the E-equivalence classes and form a structure $\mathfrak{A}/E$. If each relation of $\mathfrak{A}$ is also decidable (relative to the appropriate A^n) then in the model of 4.1.1, $\mathfrak{A}/E$ is isomorphic to a *recursively enumerated structure* in the sense of Mal'cev [71] Ch.18. However, for a program of constructive generalization of algebra via formalization in T_0 it should not be necessary to demand of all the structures $(\mathfrak{A},E)$ considered that they be denumerable or decidable. Such additional information is only to be assumed where necessary and verified where possible.

One place where decidability restrictions may play an essential role in algebra is in the ideal theory of rings. For example, a non-trivial ideal X in the integers can only be shown to be principal if it has a least positive element z. But as observed in 4.3.3, this cannot be constructively derived in general. However, if X is decidable with characteristic function g and x is any given positive element in X then we can find z as $(\mu y \leq x)gy \simeq \underline{0}$.[8]

6. T_1 *and related theories*

6.1. Language and axioms. T_1 uses the same language $\mathcal{L}$ as T_0 except that (for simplicity) we adjoin one new constant symbol $\underline{e}_N$. There is only one new axiom:

VI *(Numerical quantification)*

$$(f : N \to N) \to (\underline{e}_N f \simeq \underline{0} \vee \underline{e}_N f \simeq \underline{1}) \wedge (\underline{e}_N f \simeq \underline{0} \leftrightarrow \exists x \eta N.\ fx \simeq \underline{0})$$

6.2. Some consequences. Using the partial minimum operator μ of §3.3(6) we define the *unbounded minimum operator* μ_0 in T_1 by

$$\mu_0 f \simeq \begin{cases} \underline{0} & \text{if } \underline{e}_N f \simeq \underline{1} \\ \mu f & \text{if } \underline{e}_N f \simeq \underline{0}\ . \end{cases}$$

Thus $\mu_0 f\downarrow$ for all $f : N \to N$.

The recursion-theoretic jump operator $J : N^N \to N^N$ is defined as $J(f) \simeq \lambda x.\underline{e}_N(\lambda y.\ t(f,x,y))$ for a certain primitive recursive t. Then the definition of the *hyperarithmetic hierarchy* $\langle H_a \rangle_{a \eta 0_1}$ can be given in T_1, iterating J along 0_1. From this, one defines the predicate $Hyp(f)$ expressing that an operation f in N^N is hyperarithmetic, i.e. recursive in some H_a.

There are several ways to introduce the notion of a function *partial recursive* in $\underline{e}_N$, and to give an associated enumeration

[8] This suggests a response to Bishop's question in Bishop [70], pp. 55-56.

$\{z\}^{\underline{e}_N}$ $(z=0,1,2,\ldots)$. One way is by means of Kleene's notion of *partial recursive functional of finite type* in Kleene [59]. This leads to an inductive definition of $\{z\}^{\underline{e}_N}(x_1,\ldots,x_n) \simeq y$ (for $x_1,\ldots,x_n,y$ in N) which falls under Axiom V and can be carried out conveniently in T_0. Kleene shows that

$$\forall f \eta N^N[\mathrm{Hyp}(f) \leftrightarrow \exists z \eta N \ \forall x \eta N (fx \simeq \{z\}^{\underline{e}_N}(x))].$$

Thus the following statement is analogous to Church's Thesis:

$$\text{(HT)} \qquad \forall f \eta N^N \ \mathrm{Hyp}(f).$$

6.3. Metamathematical results

THEOREM 6.3.1

There is a model of T_1 *in which the range of the variables is the set* ω *of natural numbers and* (HT) *is true.*

By 4.1 Remark (a), this may be proved in exactly the same way as 4.1.1. Note that as in Remark (c) there, this proof can also be formalized in classical 2nd order analysis.

There is an obvious generalization of HEO to any enumerative system. In particular, the enumeration indicated in 6.2 of the functions partial recursive in $\underline{e}_N$, induces a type structure

$$(\mathrm{HEO}_\sigma^{(\underline{e}_N)}, \equiv_\sigma^{(\underline{e}_N)})$$

or as we shall write it $(\mathrm{HEO},\equiv)^{(\underline{e}_N)}$.

COROLLARY 6.3.2

If θ *is a sentence of* $\mathcal{L}_{FT}$ *and* $T_1 \vdash \theta^{(\overline{N},E)}$ *with classical logic then* $\theta^{(HEO,\equiv)^{(e_N)}}$ *is true.*

Without further work we also obtain the results of 4.2.

THEOREM 6.3.3

Theorem 4.2.1 remains correct when "T_0" *is replaced by* "T_1".

COROLLARY 6.3.4

If θ *is a sentence of* $\mathcal{L}_{FT}$ *and* $T_1 \vdash \theta^{(\overline{N},E)}$ *then* $ZF \vdash \theta^{(M)}$.

Turning now to 4.3, define $T_1^{(-)}$ to be $T_1^{(-)}$ + (Axiom VI for e_N). The classes of formulas $\mathcal{F}_0$, $\mathcal{F}_1$ are defined in the same way as before.

THEOREM 6.3.5

Theorem 4.3.1 remains correct when "$T_0^{(-)}$" *is replaced by* "$T_1^{(-)}$".

The proof is as before.

6.4. Relations with predicativity. The informal conception of predicativity taken here is that one deals just with the *definitions and proofs implicit in assuming that the set of natural numbers is given* (as a kind of "completed infinite totality"). Precise proposals for explaining this have been given in terms of *autonomous progressions* *of ramified theories* R_α ; cf. Feferman [68] for a survey of work on these[9]. Viewed from the outside, the least non-autonomous ordinal is a certain (recursive) Γ_0. The *general concept of ordinal* (or *well-ordering*) is itself *not* predicative. One may speak of *particular ordinals* being predicative when it has been recognized by these means that corresponding principles of transfinite induction and recursion are justified.

The language $\mathcal{L}$ does not match directly with formal languages considered up to now in the study of predicativity. Nevertheless, it makes sense to interpret V as ω and the operations and classifications as ranging over predicative definitions of partial operations and subsets of ω, resp. Clearly, not all of Axiom V for inductive generation is justified under this interpretation. Let V_N be the special case used in 3.3(6) to derive N, and let $T_1^{(N)}$ be T_1 with V_N in place of V.

CONJECTURE

$T_1^{(N)}$ *is (proof-theoretically) reducible to predicative analysis* $\bigcup_{\alpha<\Gamma_0} R_\alpha$.

It may even be that $T_1^{(N)}$ is of the same strength as predicative analysis. (The latter is known to be weaker than the intuitionistic 1st order theory of 0_1, hence also weaker than T_0.)

The actual development of analysis by predicative means may be referred to the hierarchies N_σ or $(\overline{N}_\sigma, E_\sigma)$ in $T_1^{(N)}$. All of classical analysis and much of the modern theory of measure and

[9] Cf. also Feferman [64], Kreisel [70]. A more perspicuous formalization without progressions is given by Feferman in a paper to appear in the Lorenzen Festschrift.

integration can be accounted for predicatively, though the l.u.b. principle is available only for sequences rather than for sets in general[10].

6.5. Borelian mathematics and T_1. Obviously it is necessary to use Axiom V to deal with the parts of mathematics where ordinals enter unrestrictedly. In analysis this shows up in the sequence of derived sets of a closed set, in the theory of Borel sets, etc. Borel and his school (Baire, Lebesgue, etc.) talked of restricting mathematics to that which was explicitly definable (Borel [14]). However, they never made clear what means of definition or proof were to be admitted. Some idea of this can be drawn from their practice, which is seen to be accounted for in T_1. It would be of interest to see whether the *Borelian conception of mathematics* can be explained in a precise way; T_1 would seem to be a strong candidate for this.

The axiom VI with intuitionistic logic implies what Bishop calls the *limited principle of omniscience*; for $f : N \to N$,

(LPO) $$\forall x(fx \simeq 0) \vee \exists x,y(fx \simeq y + 1).$$

He says (Bishop [67], p.9) that each of his results ϕ is a constructive substitute for a classical theorem ψ such that ϕ together with (LPO) implies ψ. Relying on this and 5.1(2) we have:

CLAIM

All of the classical mathematics replaced by Bishop's work can be formalized in T_1.

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[10] cf. Lorenzen [65] for classical analysis. I have given a predicative development of measure theory in unpublished notes. Transcendental methods in algebraic number theory have also been treated predicatively (Larson [69]).

In most cases this is simpler to verify directly than to pass through this work.

6.6. *Relations with hyperarithmetic mathematics.* The idea here is the same as for recursive mathematics, and the discussion of §5.2 can be paralleled completely, but the subject itself has not been pursued to anywhere near the same extent. For the most part the positive results have already been realized as predicative theorems, hence as generalizations of both hyperarithmetic and classical results. By 6.3.2, negative results such as that of Kreisel [59] for the Cantor-Bendixson theorem may be used to give independence of some classical theorems from T_1.

There is one recent positive development in hyperarithmetic mathematics that ought to be re-examined with an eye to generalization by means of formalization in T_1, namely *hyperarithmetic model theory*. This was initiated by Cleave [68] particularly in a study of hyperarithmetic analogues of ultrafilters, ultrapowers, etc. Cutland [72] has continued this for saturated models and forms of categoricity. Denumerable models should play a special role since for these the satisfaction relation is decidable.

7. *Concluding questions and remarks*

7.1. *Systematic and ad hoc explicit mathematics.* What we have called here *systematic* explicit mathematics are attempts to redevelop substantial portions of mathematics by means of restricted methods of definition and/or proof. By contrast, *ad hoc* work examines particular existential results of classical mathematics with the aim to obtain more explicit or sharper information. No (deliberate) restriction is made on methods of definition or proof; rather one employs refined considerations or special new methods. The most striking (and frequently cited)[11] example is Baker's work on some classes of diophantine equations, giving explicit bounds for the solutions which had previously only been known to be finite in number. Taking a systematic approach will not automatically lead one to such improvements.

NOTE

Explicit definability results for theories with logic restricted to be intuitionistic give them an appearance of systematic explicit mathematics. Here the emphasis instead has been on the choice of basic notions to more accurately reflect actual practice. Intuitionist logic plays a role only in some metatheorems.

7.2. Proof-theoretical work on subsystems.

For certain subtheories T of T_0, T_1 it has been possible to characterize the operations from N to N which are proved in T to exist. These characterizations are given in terms of certain hierarchies of functions up to some familiar ordinals. The techniques are proof-theoretical, by Gödel's functional interpretation (Gödel [58]) followed by normalization of terms. This has been done for systems of finite type over N in Tait [65] and, with $\underset{\sim}{e}_N$, in Feferman [70]. Howard [72] treats a system of finite type over N, 0_1 which is interpretable in T_0, and Zucker (Troelstra [72]) gives some extensions of this to iterated inductive definitions; it should

[11]Particularly by Kreisel; cf. Kreisel [74] for a more adequate discussion.

be possible to use similar methods when the axiom VI for $\underline{e}_N$ is added. It may be of interest to see whether these techniques can be applied directly to related and stronger subsystems of T_0 and T_1 with "variable types". Some such has been done by Girard [71] and Martin-Löf [Ms] .

7.3. *Bounded classifications and sets*

The following kind of extension S_0 of T_0 might provide a more flexible comparison with set theory. The language $\mathcal{L}_{Bd}$ of S_0 has one new basic predicate symbol Bd(x) which we read as: x *is a bounded classification*; the idea is that x is contained in some classification built up by sums and products from N. The axioms of S_0 agree with those of T_0 through I-IV; V is modified so as to allow any formula ψ of $\mathcal{L}_{Bd}$. We have in addition the following axioms for Bd:

(i) $Bd(x) \rightarrow Cl(x)$

(ii) $Bd(N)$

(iii) $Bd(A) \wedge \forall x \eta A.\ Bd(fx) \rightarrow Bd(\Sigma_{x\eta A} fx) \wedge Bd(\Pi_{x\eta A} fx)$

(iv) $Bd(A) \wedge X \subseteq A \rightarrow Bd(X)$

(v) A scheme for proof by induction on Bd.

Since $A \times B \subseteq \Sigma_{x\eta A} B$ and $B^A \subseteq \Pi_{x\eta A} B$, we also have closure of Bd under these operations. Note that if $\sigma \eta Typ_N$ then $Bd(N_\sigma)$, $Bd(\overline{N}_\sigma)$ and $Bd(E_\sigma)$; more generally these hold for $\sigma \eta Typ_A$ where each X in A is bounded.

S_0 + (CT) has a model just like that for T_0 in 4.1.1. Simply take Bd to be the smallest set satisfying (ii)-(iv). The same method works in any model, where i(a,r) is set-theoretically defined by:

$$z \eta i(a,r) \Leftrightarrow z \in \cap X\{\forall x \eta a[\forall y(y,x)\ r \Rightarrow y \in X) \Rightarrow x \in X]\}$$

QUESTION

Is S_0 *a conservative extension of* T_0 *?*

We can associate with each model of set theory a model of S_0 in which the bounded classifications are just those coextensive with the sets:

THEOREM 7.3.1

Let $\mathfrak{U} = (A,\in)$ *be any model of* ZF. *There is a model* $\mathfrak{U}^*$ *of* S_0 *in which* $V = A$ *and* $Bd(a) \Leftrightarrow Cl(a) \wedge \exists b \in A\ \forall x \in A[x \eta a \Leftrightarrow x \in b]$.

The method of proof is the same as for 4.2.1.

For a nice result like 4.2.2 one should perhaps deal instead with axioms for the predicate $Set(A,E)$ which holds when $Bd(A)$ and E is an equivalence relation on A.

NOTE

Bishop's notion of set in Bishop [67] may be more appropriately interpreted by such a predicate.

7.4. Impredicatively defined operations

In analogy with the introduction of $\underline{e}_N$ we might further consider the theory T_2 obtained from T_1 by adding a constant $\underline{e}_{N^N}$ with the axiom:

$$(f : N^N \to N) \to (e_{N^N} f \simeq \underline{0} \vee e_{N^N} f \simeq \underline{1}) \wedge [e_{N^N} f \simeq 0 \Leftrightarrow \exists g \eta N^N . fg \simeq \underline{0}].$$

Mathematically, this permits application of the l.u.b. principle to decidable sets of real numbers. Questions here would be whether we could get a model like that in 6.3.1 for T_1, and to what extent we could get interesting proof-theoretical information about subsystems of T_2 including the new axiom.

7.5. *Impredicatively defined classifications*

Here we would like to know to what extent T_0 can be strengthened by classification existence axioms so that the resulting theory T^* is also intuitively correct, or at least for which we can get a model of $T^* + (CT)$ in which N is standard. Particularly to be considered are instances of the *comprehension axiom scheme*

$$(CA_\phi) \qquad \exists C\ \forall x[x\eta C \leftrightarrow \phi(x)]$$

where ϕ is not an elementary formula. The fact that the unrestricted application of this principle leads to contradiction shows that the concept of classification is not completely clear; cf. §7.7 below. Some experimentation to see how far the use of (CA) can be pushed may be helpful to obtain clarification in these circumstances. The specific cases considered in this section and the next were suggested by past experience.

Call ϕ *2nd order* if it is a formula of $\mathcal{L}^{(2)}$ satisfying the conditions to be an elementary formula (2.6) except that quantifiers with classifications variables $\forall X(\ldots)$, $\exists X(\ldots)$ are permitted. Write $\phi^{(\subseteq A)}$ for the result of replacing each such quantifier by $\forall X \subseteq A(\ldots)$, $\exists X \subseteq A(\ldots)$.

It turns out possible to get a model of $T_0 + (CT) +$

$(CA_\phi(\subseteq N))\ \phi$ 2nd order in which N is standard. This can be done more generally with N replaced by any bounded classification. Let $(CA^{(2)})$ be the schema (CA_ϕ) for all 2nd order ϕ.

QUESTION

Does $T_0 + (CT) + (CA^{(2)})$ *have a model, particularly one in which* N *is standard?*[12]

Note that the inductive generation axiom V is derivable from $(CA^{(2)})$.

7.6. Impredicatively defined classifications (cont.)

Another collection of instances of the scheme (CA) which should be considered is suggested by *self-applicable concepts in algebra* such as the "category of all categories". In T_0, (single-sorted) structures of signature $\nu = ((n_1,\ldots,n_k),\ (m_1,\ldots,m_\ell),\ (m_1,\ldots,m_\ell),p)$ are defined to be $(1 + k + \ell + p)$-tuples:

(i) $a = (A,R_1,\ldots,R_k,\ f_1,\ldots,f_\ell,\ c_1,\ldots,c_p)$

where

(ii) each $R_i \subseteq A^{n_i}$, $f_i : A^{m_i} \to A$, and $c_i \eta A$.

Write $Str_\nu(a)$ for the formula in $\mathcal{L}$ which expresses that there exist $A,R_1,\ldots,c_p$ satisfying (i), (ii). Given any sentence θ in the 1st order language $\mathcal{L}_\nu^{(1)}$ of structures of type ν, write $Sat_\theta(a)$ for the formula which expresses that the structure a *satisfies* (or is a *model* of) θ ; we also write $a \models \theta$ for this. Finally, write $a_1 \cong a_2$ to express that a_1, a_2 are isomorphic structures of the same signature.

The first question would be whether we can consistently assume for each ν : $\exists B\ \forall x[x\eta B \leftrightarrow Str_\nu(x)]$. This is not possible

[12](Added in proof) The answer is positive; cf. Addenda (A2) below.

with T_0 for then we could derive the existence of a C such that $\forall x[x\eta C \leftrightarrow C\ell(x)]$. A form of Russell's paradox follows by taking $J = \Sigma_{x\eta C}x$ and $D = \hat{x}.(x,x)\not\eta J$. But this does not exclude the possibility of having classes of *representatives* (with respect to $\cong$) of all structures of a given type, which is all that is important algebraically.

QUESTION

Is T_0 *consistent with the following sentences*

$$\exists C[\forall x(x\eta C \rightarrow Str_\nu(x) \wedge x \models \theta) \wedge \forall y(Str_\nu(y) \wedge y \models \theta \rightarrow \exists x\eta C(y \cong x))]$$

for each signature ν *and sentence* θ *of* $\mathcal{L}_\nu^{(1)}$*?*

Perhaps more promising is the use of $T_0 + (CA^{(2)})$ of the preceding section. We can already speak in that theory (to be more precise, in a slight extension) of the category of all functors between any given "large" categories (e.g. groups, classes, etc.): for, functors are just operations satisfying special conditions explained by quantifying over structures.

REMARK

The systems developed in (Feferman [Ms]) could deal with all these kinds of self-applicable concepts, but at a cost of other deficiencies. A principal difference is that operations were explained there in terms of classes rather than treated independently as here. The present language should permit greater flexibility.

7.7. *Partial and total classifications.*

Returning to the question of clarifying the concept Cl, one way of putting the difficulty is that not every well-formed formula $\phi(x,\ldots)$ of $\mathcal{L}$ need be recognized as determining a property which is *meaningful for all* x. For example, it might be said that the property of being a Cauchy sequence is only meaningful for sequences. This suggests considering a notion of *partial property* or *classification*, one whose *domain of significance* may be only a part A of V. We would only be able to say in this case:

$$\exists C\ \forall x \eta A\ [x \eta C \leftrightarrow \phi(x)].$$

This appealing idea goes back to Russell; one form of it has been pursued by Gilmore [70]. Another point that might appeal in the present context is that it appears to put operations and classifications on a similar footing.

If T_0 is to be embedded in a theory of partial classifications it seems we should have a new operation δ such that

$$Cl(x) \to \delta x\downarrow \wedge Cl(\delta x).$$

Here $Cl(x)$ is read as: x is a partial classification, and δx is read as: the domain of significance of x. We would call x a *total* classification if $V \subseteq \delta x$; it is such that we have had in mind up to this section. In a theory of this kind, the partial comprehension scheme would take the form for arbitrary ϕ :

$$\exists C\ \forall x \eta \delta C\ [x \eta C \leftrightarrow \phi(x)].$$

If nothing more is said about the members of C, this theory is trivially consistent. The problem with this idea is that we have shifted the initial question to: which ϕ have $V \subseteq \delta C$? One would hope to get simple evident (sufficient) conditions for this, to recapture at least axioms III-V, if not more[13].

7.8. *Perspective*

The study of systematic explicit mathematics may be more of logical and/or philosophical interest than of mathematical interest. In any case it is relevant to significant portions of actual mathematics. The problems raised in 7.5 - 7.7 are very intriguing from the logical point of view, but they have little mathematical relevance, as far as one can see now.

ADDENDA

(A1) To Footnote 2.

Following circulation of this paper I learned of a theory CST of functions and sets independently developed by Myhill, which has several aspects in common with T_0. His stated purpose is to provide a constructive framework for constructive mathematics as exemplified in Bishop [67]. CST differs from T_0 in the basic respect that extensionality for both functions and sets is taken among the axioms. For this reason, CST is not evidently constructive. It is possible though that Myhill's metamathematical work on the theory will show it reducible to constructive principles. (This is being prepared for publication.)

I also learned from Myhill of the paper by Cocchiarella [to appear] which introduces some axioms for predicates and corresponding sets (here: classifications) that may be said to anticipate the idea for

[13](Added in proof) Subsequent to the above I found some theories of partial operations and classifications which accomplish a good deal of this aim as well as that of 7.6. The work will appear in a paper for the Schütte Festschrift.

axiom schema III of T_0 (3.3 below).

(A2) To §7.5.

This gives an affirmative answer to the question raised in §7.5 whether there is a model $\mathfrak{N}$ for T_0 + (CT) with full 2nd order comprehension $CA^{(2)}$ (and in which N is standard). The proof is by a non-constructive modification of the proof of 4.1.1.
Let $\mathfrak{M} = (\omega, \mathcal{P}\omega, \simeq, \in)$; the interpretations of 0, k, s, d , p, p_1, p_2 are chosen as before. Associate with each 2nd order formula $\phi(x_1,\ldots,x_n, X_1,\ldots,X_m,Y)$ (where the variables listed include all free variables of ϕ) a Skolem function $Y = F_\phi(x_1,\ldots,x_n,X_1,\ldots,X_m)$ in $\mathfrak{M}$. Now define the subsets $C\ell_\alpha$ of ω and for $a \in C\ell_\alpha$ the set $e(a) = \{x : x\eta a\}$ by induction on α ($e(a)$ is the *extension* of a) as follows. $C\ell_0 = 0$ and for limit λ, $C\ell_\lambda = \cup_{\alpha<\lambda} C\ell_\alpha$. $C\ell_{\alpha+1}$ is $C\ell_\alpha$ together with

(i) all $(1,\phi,(x_1,\ldots,x_n,a_1,\ldots,a_m))$ where ϕ is 2nd order as above and each $a_1,\ldots,a_m \in C\ell_\alpha$, as well as

(ii) all $(2,a,f)$ such that $a \in C\ell_\alpha$ and $\forall x\eta a[\{f\}(x) \in C\ell_\alpha]$.

In case (i), take $z\eta(1,\phi,(x_1,\ldots,x_n,a_1,\ldots,a_m)) \Leftrightarrow z \in F_\phi(x_1,\ldots,x_n, e(a_1),\ldots,e(a_m))$ and in (ii), take $z\eta(2,a,f) \Leftrightarrow \exists x\eta a\ \exists y\eta\{f\}(x)[z = (x,y)]$. Let $C\ell = \cup_\alpha C\ell_\alpha$ and $N = (\omega, C\ell, \simeq, \eta)$.

LEMMA

For any 2nd order $\psi(x_1,\ldots,x_n,X_1,\ldots,X_m)$ *(considered as a formula of* $\mathcal{L}$*) and any* $x_1,\ldots,x_n \in \omega$, $a_1,\ldots,a_m \in C\ell$ *we have* $\mathfrak{N} \models \psi(x_1,\ldots,x_n,a_1,\ldots,a_m) \Leftrightarrow \mathfrak{M} \models \psi(x_1,\ldots,x_n,e(a_1),\ldots,e(a_m))$.

Thus $\mathfrak{N}$ is a model of $(CA^{(2)})$ + (CT) as well as axioms I, II, IV of T_0. The verification of $(CA^{(2)})$ takes care of III

(Elementary Comprehension) and V (Inductive generation).

In the same way we can strengthen Theorem 6.3.1.

* * * * * *

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