AND SO ON... REASONING WITH INFINITE DIAGRAMS

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What Proofs Must Do

- A proof must convince us of the truth of the statement being proved.
- "Truth" is taken in its *prima facie* sense, i.e. we are supposed to understand the meanings of the notions in that statement.
- To be convinced of a proof, one must follow the argument and check the steps using also background knowledge.

What Proofs Must Do (contd.)

- So to follow a proof we must also understand the meanings of the notions used in the proof and from background knowledge.
- Even given that, it is possible to go through the steps of a proof and not "really understand" the proof itself.

Really Understanding Proofs

- When we're led to say, "Oh, I see!"
- It's a special kind of insight into how and why the proof works.
- That kind of understanding of proofs is necessary in order to be a full-fledged consumer and producer of mathematics.

Diagrams in Proofs

- Ubiquitous in geometry from the Greeks to the present, as well as in early analysis.
- Doubts cast on their validity because the diagrams used might not be “typical”.

Diagrams in Proofs (cont'd)

- 19th c. rigorization of mathematics supposedly led to the elimination in principle of diagrams from proofs.
- But the practice of reliance on diagrams is still integral to the presentation of mathematical proofs of all sorts.
- That's because such use is often part of what is needed for real understanding.

What Are Diagrams?

- Two-dimensional representations of (possibly parts of possibly infinite) mathematical configurations.
- Lines, curves, arrows, labels, marks, shaded areas.
- Broken lines, dotted lines, dots.
- It's questionable whether one can define this concept in general.

Typicality in Diagrams

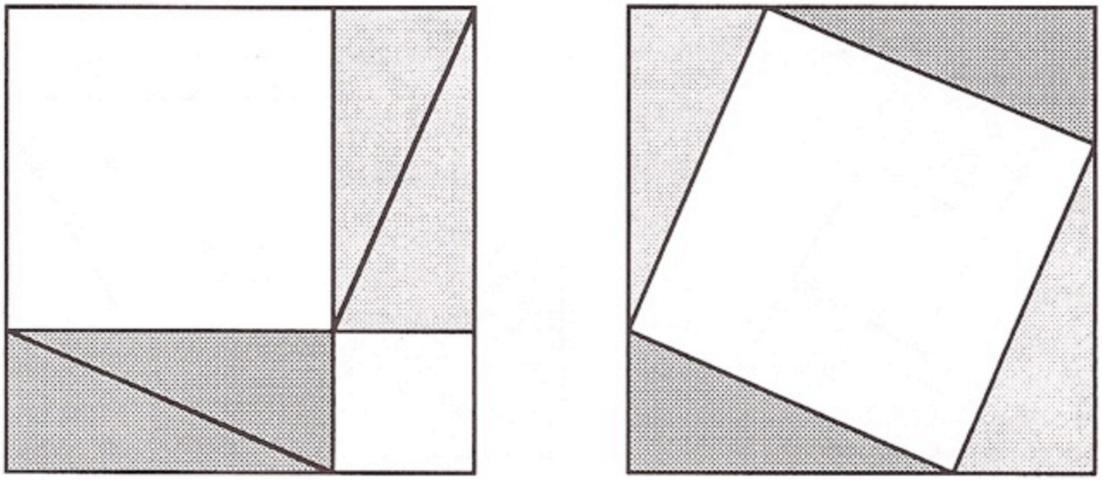
- Interesting mathematical theorems state a fact about infinitely many objects of a certain kind, e.g. triangles.
- But the diagram used in a proof represents only one such object.
- It is an issue whether the representation taken is typical.

Infinite Diagrams and Typicality

- In modern mathematics often deal with a diagram representing a typical part of a single infinite configuration, the balance indicated by dots.
- The use of such infinite diagrams is essential to understanding certain proofs.
- The statement of some theorems can't even be understood without reference to such a diagram.

The Dynamic View of Diagrams

- We should not think of diagrams, finite or infinite, as static completed figures.
- Rather, think of them as constructed and reasoned about in stages.
- Or retrace static representations in a dynamic way.
- Example: Pythagoras' Theorem.

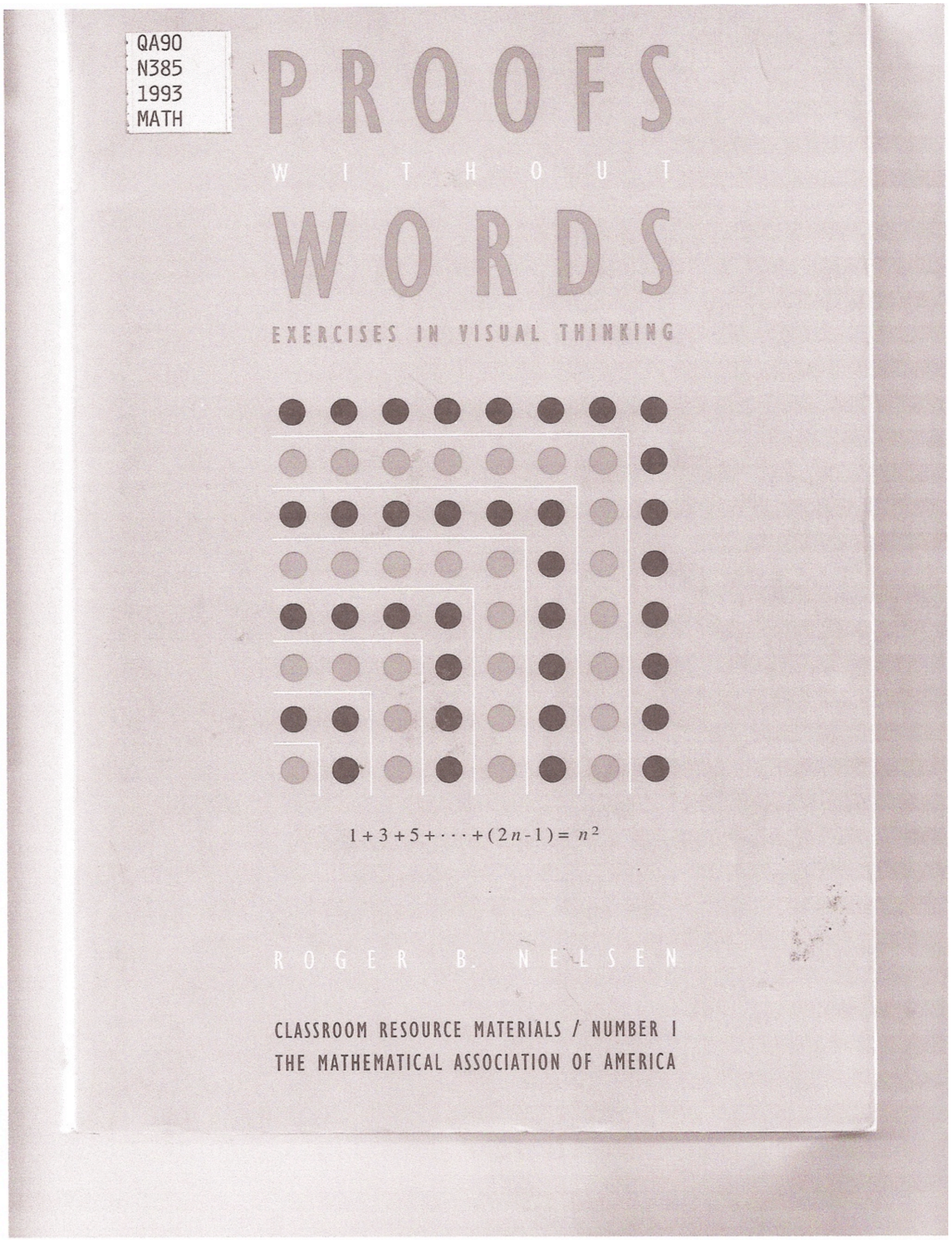


—adapted from the *Chou pei suan ching*
(author unknown, circa B.C. 200?)

Proofs Without Words

Roger B. Nelsen (ed.)

- The title is misleading: we need words to say what the diagrams are proofs of and words to guide us dynamically through the proofs.
- Nelsen: "generally, PWWs are pictures or diagrams that help the observers see why a particular statement may be true, and also to see how one might begin to go about proving it true."



PROOFS WITHOUT WORDS

EXERCISES IN VISUAL THINKING

$$1+3+5+\cdots+(2n-1)=n^2$$

ROGER B. NELSEN

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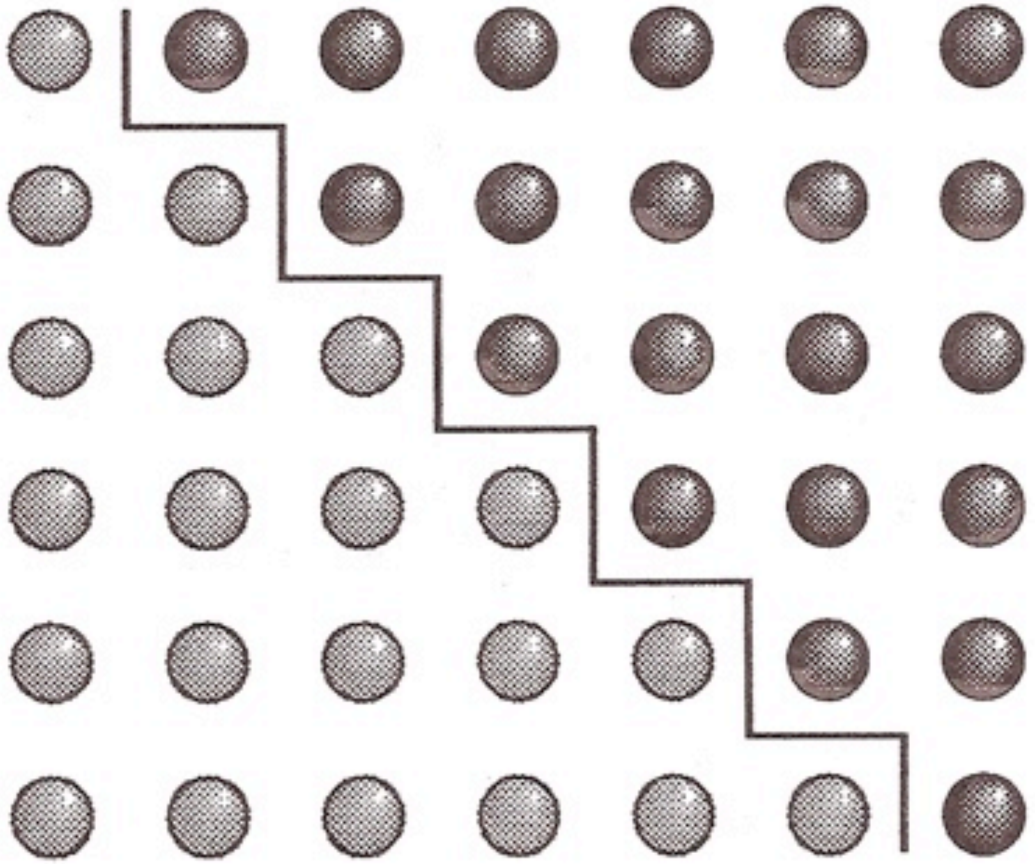
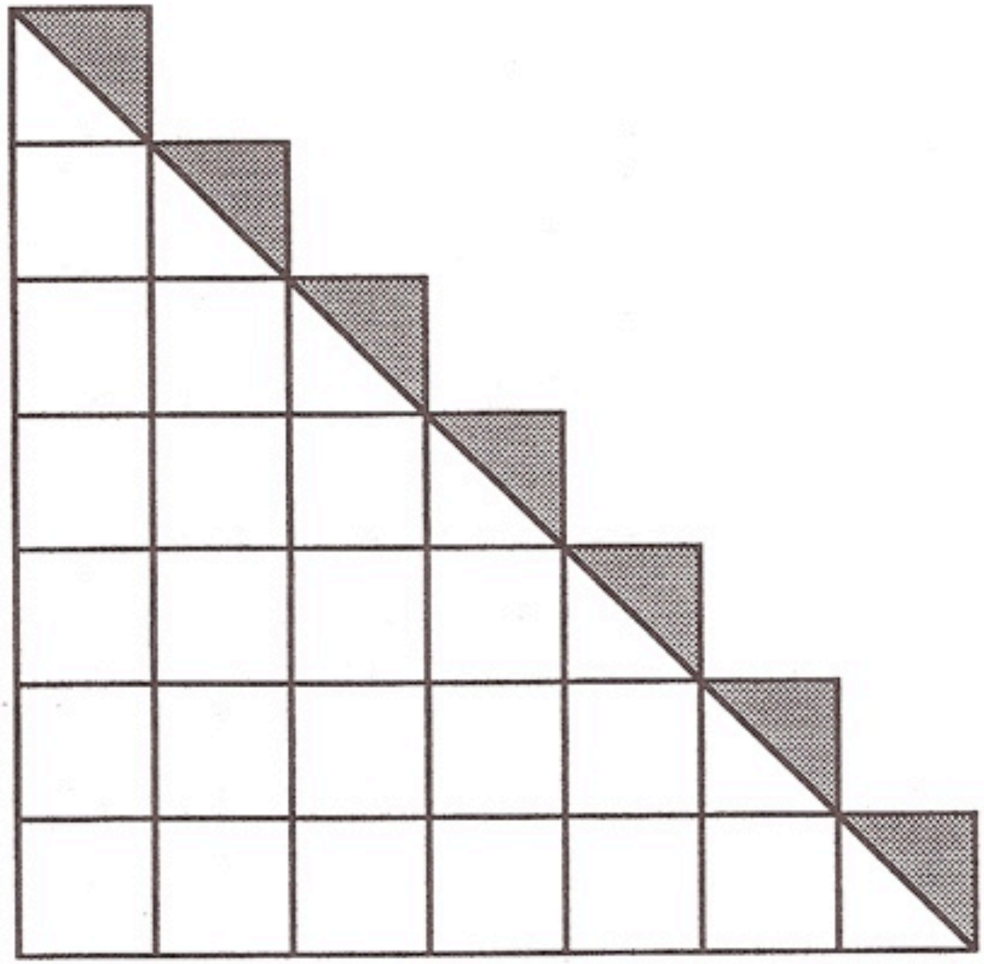
Some Arithmetical PWWs

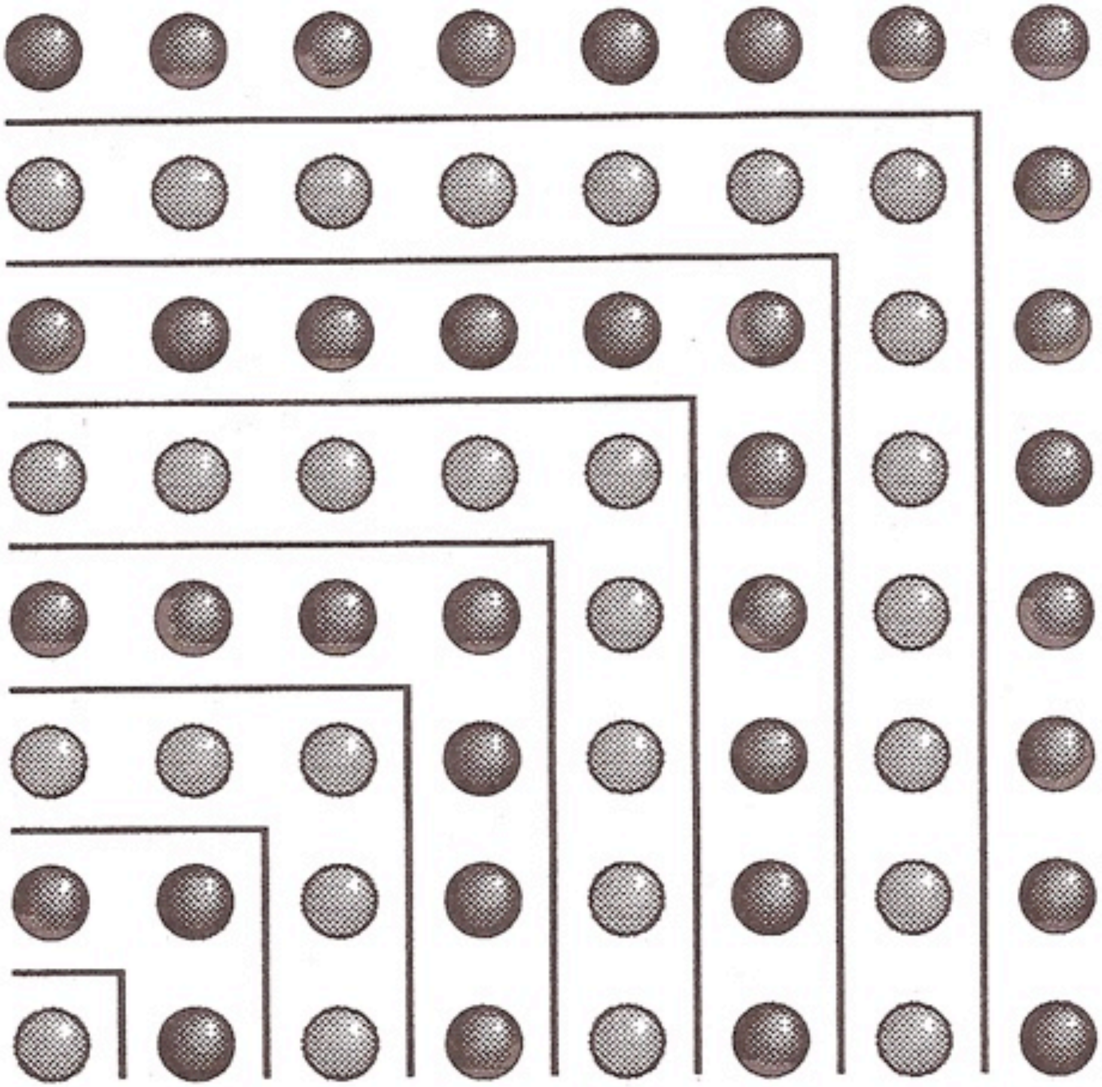
- Proofs of some identities f(1)+f(2)+...+f(n) = g(n).
- Typical diagram is given for some specific n, usually smaller than 10.
- They constitute completely convincing evidence for the truth of the identity.
- But they don't at all suggest the usual proof by induction.



$$1 + 2 + \cdots + n = \frac{1}{2}n(n + 1)$$

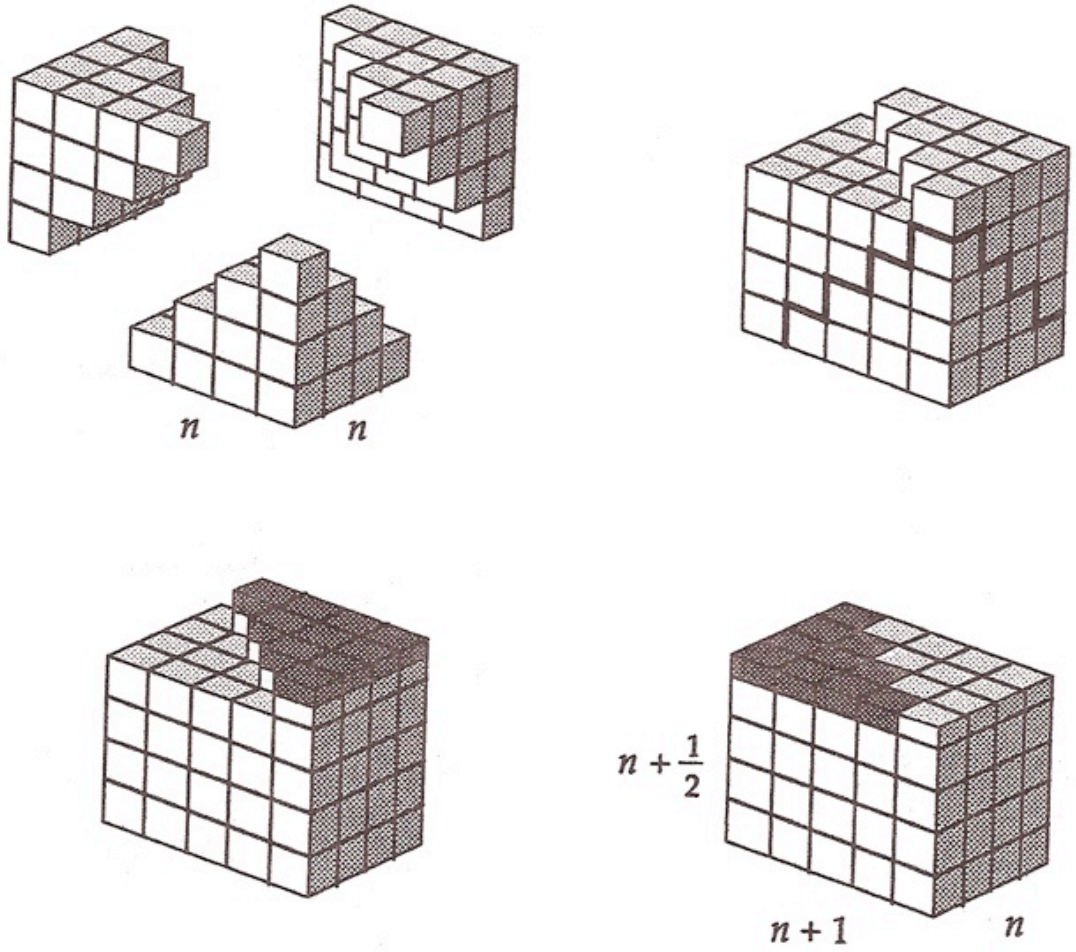


$$1 + 2 + \cdots + n = \frac{n^2}{2} + \frac{n}{2}$$



$$1 + 3 + 5 + \cdots + (2n-1) = n^2$$

$$1^2 + 2^2 + \cdots + n^2 = \frac{1}{3}n(n+1)(n+\frac{1}{2})$$

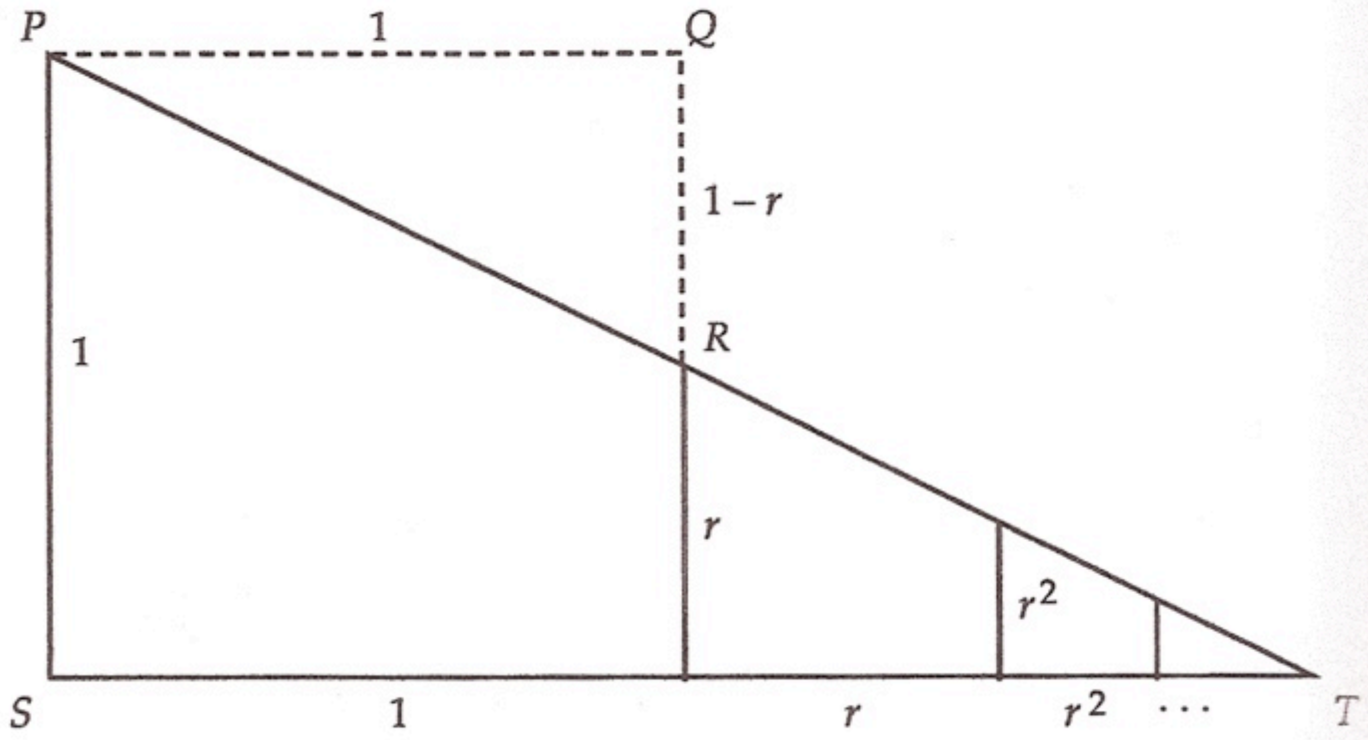


“Hilbert’s Thesis”

- The thesis that every proof can be formalized, i.e. turned into a formal proof in a formal system.
- Defenders and critics; cf., e.g., the Azzouni-Rav exchange in *Philosophia Mathematica*.
- The possible significance of metamathematical results for mathematical practice depends on the thesis.

The Main Challenge to Hilbert's Thesis

- Understanding of both meanings and proofs is essential to higher mathematical activity, and that is in no way reflected in the formal model.
- The cases of essential use of reasoning with diagrams is part of that challenge.



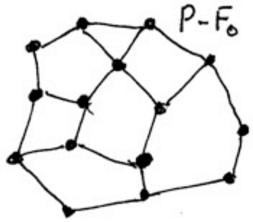
$$\Delta PQR \approx \Delta TSP$$

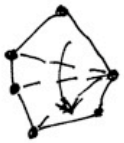
$$\therefore 1 + r + r^2 + \ldots = \frac{1}{1-r}.$$

Euler Descartes Theorem

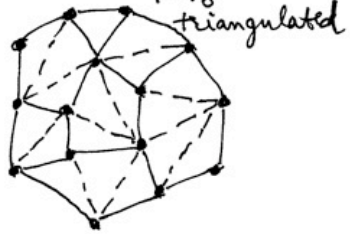
Theorem For a convex polyhedron P, $V-E+F=2$.

Proof. (1) Remove one face F_0 and then spread $P-F_0$ topologically out on the plane. Show $V-E+F=1$ for the resulting figure

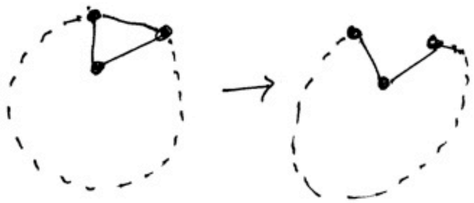




(2) Triangulate each face



(3) Remove triangles with at least one edge an outer edge, one at a time.



$V-E+F = V'-E'+F'$

$V-E+F = V'-E'+F'$

(4) For a single triangle, $V-E+F=1$ 

everal pieces having a total length of one foot from the original one foot ong piece, what's left? Whatever is left, if anything at all, will have a otal measure, or length, of zero. But what about the cardinality of the emaining point set? Would it be zero? Could it be anything but zero? f the pieces are removed in a special way, the remaining set, known as he *Cantor set*, or *Cantor dust*, turns out to be quite remarkable.

Begin with a line segment of unit length and remove the middle hird. That is, from [0, 1] remove all points strictly between $\frac{1}{3}$ and $\frac{2}{3}$. What remains is $[0, 1]-(\frac{1}{3},\frac{2}{3})=[0,\frac{1}{3}]\cup[\frac{2}{3}, 1]$. Now remove the niddle third from each of the two remaining pieces. Continue the rocess through infinitely many steps by always removing the middle hird of the remaining pieces. Figure 4.20 shows the first few steps of his process.

It is difficult to illustrate the Cantor set beyond this point; however, t appears as if there would be little if any left of our original segment fter infinitely many such steps. Supporting this is the fact that the total ength of all intervals removed is one, the length of our original segment. We show this by evaluating the infinite geometric series

$$\frac{1}{3}+\frac{2}{9}+\frac{4}{27}+\frac{8}{81}+\ldots=\frac{1}{3}\sum_{i=0}^{\infty}\left(\frac{2}{3}\right)^{i}=\frac{1}{3}\left(\frac{1}{1-\frac{2}{3}}\right)=1.$$

It follows that the measure, or total length of the remaining point et, is zero. In fact, if we randomly choose a point from the original

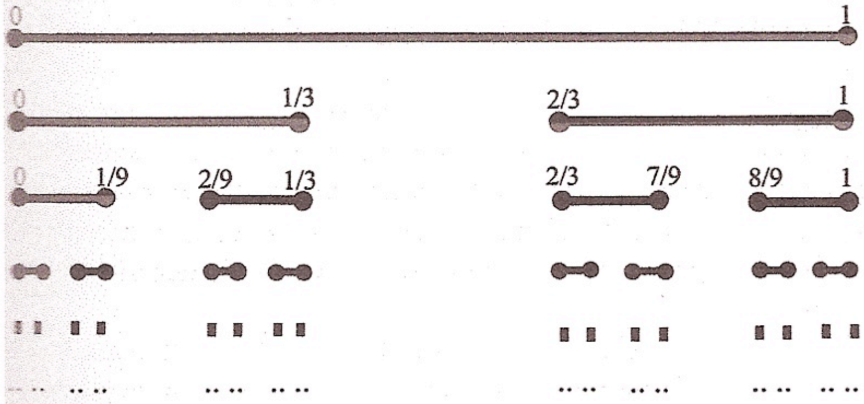
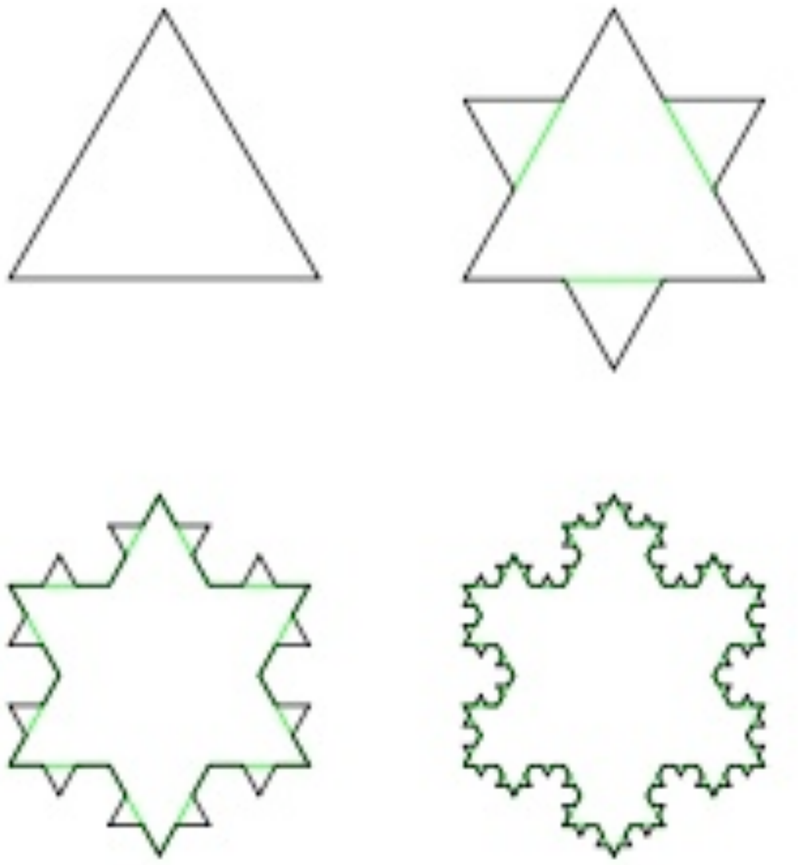


Figure 4.20. Formation of the Cantor set.

The Koch Snowflake

A Bounded Continuous Infinitely Long Curve



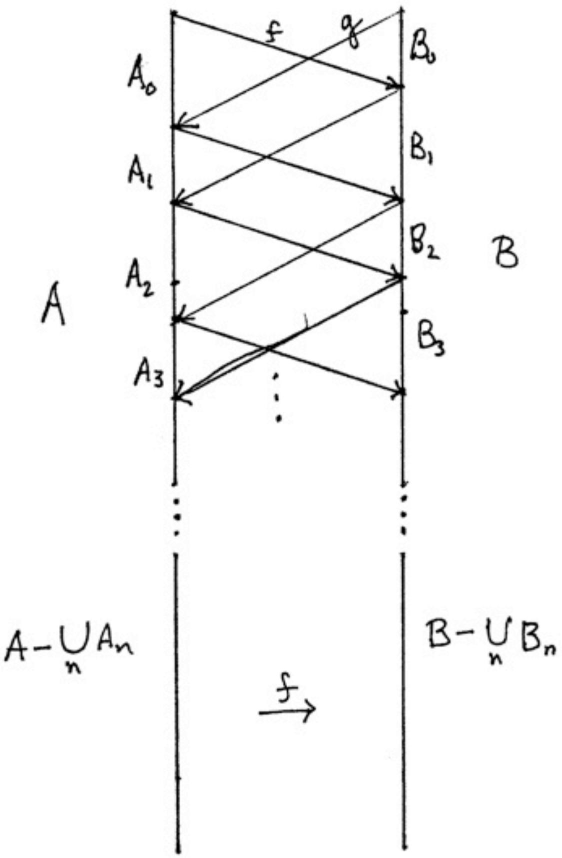
Cantor-Bernstein Theorem

Defn. $A \precsim B \iff \exists f : A \xrightarrow[\text{into}]{1-1} B$

$A \equiv B \iff \exists f : A \xrightarrow[\text{onto}]{1-1} B$

Theorem $A \precsim B \;\&\; B \precsim A \Rightarrow A \equiv B$

Proof. Suppose $f : A \xrightarrow[\text{into}]{1-1} B$ and $g : B \xrightarrow[\text{into}]{1-1} A$



Let $A_0 = A - g(B)$

$B_0 = B - f(A)$

$A_1 = g(B_0)$

$B_1 = f(A_0)$

$A_2 = g(B_1)$

$B_2 = f(A_1)$

etc.

Then $A_{2n} \equiv B_{2n+1}$ by f

$A_{2n+1} \equiv B_{2n}$ by g^{-1}

so $\bigcup_n A_n \equiv \bigcup_n B_n$

and $A - \bigcup_n A_n \equiv B - \bigcup_n B_n$ by f

An Amalgamation Theorem in Model Theory

Notions Structures $A, A', B, B', \ldots$ 1st order languages $L, L', \ldots$

$A \equiv A'$, $A \preceq A'$, $A \overset{\cdot}{\preceq} A'$ [means $A \cong A_1 \preceq A'$], $A \xrightarrow{\preceq} A'$, $A \to A'$

A in L_1, B in L_2, $L = L_1 \cap L_2$, $L \neq \phi$, $A|L$, $B|L$

Assume 1, 2:

1. Tarski-Vaught Theorem If $A_0 \preceq A_1 \preceq \cdots \preceq A_n \preceq \cdots$ and $A = \bigcup_n A_n$ then each $A_n \preceq A$

2. Weak Amalgamation Theorem Assume A in L_1, B in L_2, $L = L_1 \cap L_2$, $L \neq \phi$ and $A|L \equiv B|L$. Then $(\exists A')[A \preceq A'$ and $B|L \overset{\cdot}{\preceq} A'|L]$.

Picture
$$\begin{tikzcd}
A \arrow[r, "\preceq"] & A' \\
B \arrow[ur, "\overset{\cdot}{\preceq}(L)"'] &
\end{tikzcd}$$

3. Corollary Under same hypotheses there exist A', B'

$$\begin{tikzcd}
A \arrow[r, "\preceq"] & A' \arrow[r] \arrow[dr] & {} \\
B \arrow[ur] \arrow[rr, "\preceq"'] & & B'
\end{tikzcd}$$

$\nearrow \overset{\cdot}{\preceq}(L)$

$\searrow \overset{\cdot}{\preceq}(L)$

4. Strong Amalgamation Theorem Under the same hypotheses, $(\exists C)[C$ in $L_1 \cup L_2$ and $A \overset{\cdot}{\preceq} C$ and $B \overset{\cdot}{\preceq} C]$

Proof Let $A_0 = A$, $B_0 = B$. Form

$$\begin{tikzcd}
A_0 \arrow[r, "\preceq"] & A_1 \arrow[r, "\preceq"] \arrow[dr] & A_2 \arrow[r, "\preceq"] \arrow[dr] & \cdots \\
B_0 \arrow[ur] \arrow[r, "\preceq"'] & B_1 \arrow[ur] \arrow[r, "\preceq"'] & B_2 \arrow[r, "\preceq"'] & \cdots
\end{tikzcd}$$

$A_\omega = \bigcup_n A_n$ each $A_n \preceq A_\omega$ (L_1)

$B_\omega = \bigcup_n B_n$ each $B_n \preceq B_\omega$ (L_2)

$A_\omega|L \cong B_\omega|L$. Expand $A_\omega|L$ to C in $L_1 \cup L_2$ following B_ω.

5. Corollary Robinson's Joint Consistency Theorem, Craig's Interpolation Thm.

Complexes, Homology, and Ext

In this chapter we plan to define Ext and establish a few of its basic properties. As the reader will note, the process takes a considerable amount of space and lots of machinery, some of which will be used in later chapters.

DEFINITION. A *complex* (sometimes called a *graded differential complex*) is a sequence of R modules

$$C = \{C_n\}_{n=-\infty}^{\infty}$$

together with a collection of R homomorphisms

$$\{d_n : C_n \to C_{n-1}\}_{n=-\infty}^{\infty}$$

or

$$\{d_n : C_n \to C_{n+1}\}_{n=-\infty}^{\infty}$$

called *differentials* such that

$$d_{n-1}d_n = 0 \quad \text{or} \quad d_{n+1}d_n = 0.$$

For simplicity in the following discussion, we shall treat only the case that the differential goes down ($d_n : C_n \to C_{n-1}$). All the analogous properties can be shown in the other case by merely renumbering. In application, both cases will occur.

If C is a complex of R modules with differentials $d_n : C_n \to C_{n-1}$, then $\operatorname{Ker} d_n \supseteq \operatorname{Im} d_{n+1}$. Let $H_n(C) = \operatorname{Ker} d_n / \operatorname{Im} d_{n+1}$. The groups, $H_n(C)$, are called the *homology groups* of the complex. If $H_n(C) = (0)$ for every n, then the complex is an exact sequence, and conversely.

If A and C are two complexes of R modules with differentials $d_n^A : A_n \to A_{n-1}$ and $d_n^C : C_n \to C_{n-1}$, respectively, then a *complex map* $f: A \to C$ is a collection of R homomorphisms $f_n : A_n \to C_n$ such that the following diagram is commutative:

$$\begin{tikzcd}
\cdots \arrow[r] & A_n \arrow[r, "d_n^A"] \arrow[d, "f_n"] & A_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & C_n \arrow[r, "d_n^C"'] & C_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

That is, for each n, $f_{n-1}d_n^A = d_n^C f_n$.

In the following text, we are going to drop the subscripts and superscripts on differentials and complex maps except in those cases where confusion would otherwise result. Usually it will be clear from the context which subscripts and superscripts are called for.

Proposition. If A and C are complexes of R modules, then a complex map $f\colon A \to C$ induces (for each n) an R homomorphism $f_*\colon H_n(A) \to H_n(C)$.

Proof. Consider $f_n\colon A_n \to C_n$. Notice that the relation $f_n d_{n+1} = d_{n+1} f_{n+1}$ implies that $f_n(\operatorname{Im} d_{n+1}) \subseteq \operatorname{Im} d_{n+1}$ in C_n. Thus, f_n induces $f_n'\colon A_n/\operatorname{Im} d_{n+1} \to C_n/\operatorname{Im} d_{n+1}$. Similarly, the relation $d_n f_n = f_{n-1} d_n$ shows that $f_n(\operatorname{Ker} d_n) \subseteq \operatorname{Ker} d_n$ in C_n. That is, we have the induced map

$$f_n'|_{H_n(A)}\colon H_n(A) \to H_n(C).$$

This mapping is f_*.

DEFINITION. The sequence

$$0 \to A \xrightarrow{j} B \xrightarrow{\pi} C \to 0$$

where A, B, C, and 0 are complexes of R modules (0 is the complex of zero R modules). The sequence is called an *exact sequence of complexes* if $0 \to A$, j, π, and $C \to 0$ are complex maps and

$$0 \to A_n \xrightarrow{j_n} B_n \xrightarrow{\pi_n} C_n \to 0$$

is exact for each n.

The Exact Sequence of Homology Theorem. If

$$0 \to A \xrightarrow{j} B \xrightarrow{\pi} C \to 0$$

is an exact sequence of complexes, it induces the following exact sequence of homology:

$$\begin{aligned} \cdots \to H_{n+1}(C) \xrightarrow{\theta} H_n(A) \xrightarrow{j_*} H_n(B) \xrightarrow{\pi_*} \\ H_n(C) \xrightarrow{\theta} H_{n-1}(A) \xrightarrow{j_*} H_{n-1}(B) \xrightarrow{\pi_*} H_{n-1}(C) \xrightarrow{\theta} \cdots \end{aligned}$$

Remarks. The R homomorphisms θ are called *connecting maps*. Note that we have omitted subscripts on the connecting maps. Many proofs in this chapter involve a certain amount of "diagram chasing," and for this proof the following diagram is appropriate:

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$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots & A_{n+2} \arrow[r] \arrow[d] & A_{n+1} \arrow[r, "d"] \arrow[d] & A_n \arrow[r, "d"] \arrow[d, "j"] & A_{n-1} \arrow[r, "d"] \arrow[d] & A_{n-2} \arrow[d, "j"'] & \cdots \\
\cdots & B_{n+2} \arrow[r] \arrow[d] & B_{n+1} \arrow[r, "d"] \arrow[d] & B_n \arrow[r, "d"] \arrow[d, "\pi"'] & B_{n-1} \arrow[r] \arrow[d] & B_{n-2} & \cdots \\
\cdots & C_{n+2} \arrow[r, "d"] \arrow[d] & C_{n+1} \arrow[r, "d"] \arrow[d] & C_n \arrow[r, "d"] \arrow[d] \arrow[uur, dashed] & C_{n-1} \arrow[r] \arrow[d] & C_{n-2} \arrow[d] & \cdots \\
 & 0 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

Recall that this diagram is commutative.

Proof. First we wish to define the connecting homomorphism $\theta\colon H_n(C) \to H_{n-1}(A)$. Note on the preceding diagram the dashed arrow from C_n to A_{n-1}. This indicates the path to follow for the construction of θ. There will be several choices involved in the construction, but after we have completed the definition, we shall show that it did not depend on these choices.

Let $x \in H_n(C)$ and choose $c \in C_n$ such that c is in the coset x. By the definition of the homology groups, we see that $d(c) = 0$. Since $B_n \xrightarrow{\pi} C_n \to 0$ is exact, there exists $b \in B_n$ such that $\pi(b) = c$. Now form $d(b) \in B_{n-1}$ and observe that, from the commutativity of the diagram, $\pi d(b) = d\pi(b) = d(c) = 0$. Since $A_{n-1} \xrightarrow{j} B_{n-1} \xrightarrow{\pi} C_{n-1}$ is exact, there exists $a \in A_{n-1}$ such that $j(a) = d(b)$. There is no arbitrary choice involved in the selection of a, since j is a monomorphism. The definition of $\theta(x)$ now emerges: Let $\theta(x)$ be the coset in $H_{n-1}(A)$ containing the element a. We note first that a is in such a coset, since $jd(a) = dj(a) = dd(b) = 0$ and the fact that j is a monomorphism imply that $d(a) = 0$.

The reader will note that in the construction of $\theta(x)$, we made two arbitrary choices, the selection of the elements b and c. In the following discussion, we shall show that $\theta(x)$ is independent of these selections.

(a) $\theta(x)$ is independent of the choice of b. Suppose that c has been selected and that b, b' have the property that $\pi(b) = \pi(b') = c$. Then, from the exactness of $A_n \xrightarrow{j} B_n \xrightarrow{\pi} C_n$, we see that $b = b' + j(a_0)$ for $a_0 \in A_n$. Applying d and a little commutativity, we see that $d(b) = d(b') + jd(a_0)$. Now suppose that we try to construct $\theta(x)$, using b' instead of b; then we obtain an (unique) element $a' \in A_{n-1}$ such that $j(a') = d(b')$. However, the above equation implies that $a = a' + d(a_0)$ and that a and a' are in the same coset mod Im d. That is, once c has been chosen, $\theta(x)$ does not depend on the choice of b.

(b) $\theta(x)$ is independent of the choice of c. Suppose that c and c' are two elements in the coset x; then $c = c' + d(c_0)$ for $c_0 \in C_{n+1}$. Since the map π is an epimorphism, there exists $b_0 \in B_{n+1}$ such that $\pi(b_0) = c_0$. The above equation can be rewritten $c = c' + \pi d(b_0)$, using the commutativity of the big diagram.

THE END