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Systems of explicit mathematics with non-constructive μ-operator. Part I

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Abstract

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This paper is mainly concerned with the proof-theoretic analysis of systems of explicit mathematics with a non-constructive minimum operator. We start off from a basic theory *BON* of operators and numbers and add some principles of set and formula induction on the natural numbers as well as axioms for μ. The principal results then state: (i) $BON(\mu)$ plus set induction is proof-theoretically equivalent to Peano arithmetic PA; (ii) $BON(\mu)$ plus formula induction is proof-theoretically equivalent to the system $(\Pi^0_\infty\text{-}CA)_{<\varepsilon_0}$ of second-order arithmetic.

1. Introduction

Systems of explicit mathematics were introduced in Feferman [4]; these provide axiomatic theories of operations and classes for the abstract development and proof-theoretic analysis of a variety of constructive and semi-constructive approaches to mathematics. In particular, two such systems T_0 and T_1 were introduced there, related roughly to constructive and predicative mathematics, respectively. T_1 is obtained from T_0 by adding a single axiom for the non-constructive but predicatively acceptable quantification operator e_N over the

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natural numbers. However, since T_1 (like T_0) contains an axiom IG for a general impredicative inductive generation operator, it actually goes far beyond the limits of predicativity as measured by the Feferman–Schütte ordinal Γ_0.

Much precise proof-theoretic information was subsequently obtained about T_0 and various of its subsystems; cf. Feferman [7], the two chapters of Feferman and Sieg in [2], Jäger and Pohlers [16] and Jäger [14]. Corresponding work on subsystems of T_1 has been slower to be achieved. The first was for a theory $VT(\mu)$ of variable types with non-constructive μ-operator (interdefinable with e_N) in Feferman [5], which may be considered to be a subtheory of T_1 without the J (join) and IG axioms. A proof was sketched there of the proof-theoretic equivalence of $VT(\mu)$ with $(\Pi^0_\infty\text{-}CA)_{<\varepsilon_0}$ (corresponding to ramified or predicative analysis up to level ε_0), and of the equivalence of a subsystem $Res\text{-}VT(\mu)$ with Peano arithmetic PA, where in $Res\text{-}VT(\mu)$, induction is restricted to (abstractly) decidable sets. Improved versions of these systems with corresponding results due to the present authors were stated in Feferman [9], but without proofs.

The purpose of this paper is to present full proofs of these results, in two parts. In this first part we deal only with theories of operations and numbers which may contain the μ-operator. Then, in Part II, we shall consider the effect of adding class axioms. Essential use will be made in this part of proof-theoretic results by Jäger [15] on certain formal theories of ordinals over PA.

2. The basic theory *BON* of operations and numbers

A useful fragment of T_0 with axioms for (partial) operations and (natural) numbers was isolated by Beeson [1] under the name *elementary theory of operations and numbers* (*EON*). In order to examine the effect of various induction principles we shall have to work here over a still weaker fragment, *BON*, which we call the *basic theory of operations and numbers.*

The language L_p of the basic theory of partial operations and numbers is a first-order language with the individual variables $a, b, c, v, w, x, y, z, f, g, h, \ldots$ (possibly with subscripts). In addition there are individual constants and relation symbols, to be specified. The individual constants include the symbols 0, $\mathbf{k}$, $\mathbf{s}$, $\mathbf{p}$, $\mathbf{p}_0$, $\mathbf{p}_1$, $\mathbf{s}_N$, $\mathbf{p}_N$, $\mathbf{d}_N$, $\mathbf{r}_N$ and μ, the meaning of which will be explained later. The basic relation symbols are $\downarrow$, $=$ and N. The principal term formation operation is term application which we write as $(s \cdot t)$ or often just as (st) or st. In this simplified form we adopt the convention of association to the left so that $s_1 s_2 \cdots s_n$ stands for $(\cdots(s_1 \cdot s_2)\cdots s_n)$. We also use the notation $s[t_1, \ldots, t_n]$ for $st_1 \cdots t_n$.

The *individual terms* $(r, s, t, r_1, s_1, t_1, \ldots)$ of L_p are generated as follows:

1. Each individual variable is an individual term.
2. Each individual constant is an individual term.
3. If s and t are individual terms, then so also is $(s \cdot t)$.

The *atomic formulas* of L_p are those of the form $t\downarrow$, $(s = t)$ and $N(t)$; if R is an additional n-ary relation symbol in an expansion of the language L_p, then $R(t_1, \ldots, t_n)$ is also considered as an atomic formula. In the following we will make use of the logic of partial terms. Then $t\downarrow$ is read 't is defined' or 't has a value'.

The *formulas* $(\varphi, \chi, \psi, \varphi_1, \chi_1, \psi_1, \ldots)$ of L_p are generated as follows[1]:

1. Each atomic formula is a formula.
2. If φ and ψ are formulas, then so also are $\neg\varphi$ and $(\varphi \vee \psi)$.
3. If φ is a formula, then so also is $(\exists x)\, \varphi$.

The underlying logic of *BON* is the classical first-order predicate calculus. Thus the remaining logical operations are defined by

$$(\varphi \wedge \psi) := \neg(\neg\varphi \vee \neg\psi), \qquad (\varphi \rightarrow \psi) := (\neg\varphi \vee \psi),$$

$$(\varphi \leftrightarrow \psi) := (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi) \quad \text{and} \quad (\forall x)\, \varphi := \neg(\exists x) \neg\varphi.$$

The *partial equality relation* $\simeq$ is introduced by

$$(s \simeq t) := ((s\downarrow \vee t\downarrow) \rightarrow (s = t))$$

and $(s \neq t)$ is written for $(s\downarrow \wedge t\downarrow \wedge \neg(s = t))$. Further we put $t' := \mathbf{s}_N t$ and $1 := 0'$. As additional abbreviations in connection with the relation symbol N for the natural numbers we will use:

$$t \in N := N(t),$$

$$(\exists x \in N)\, \varphi := (\exists x)(x \in N \wedge \varphi),$$

$$(\forall x \in N)\, \varphi := (\forall x)(x \in N \rightarrow \varphi),$$

$$(t : N \rightarrow N) := (\forall x \in N)(tx \in N),$$

$$(t : N^{m+1} \rightarrow N) := (\forall x \in N)(tx : N^m \rightarrow N),$$

$$(t : (N \rightarrow N) \rightarrow N) := (\forall x)((x : N \rightarrow N) \rightarrow tx \in N).$$

The logic of *BON* is the (classical) *logic of partial terms* due to Beeson [1]. It corresponds to the E^+-logic with equality and strictness of Troelstra and van Dalen [20], where $E(t)$ is written instead of $t\downarrow$. The non-logical axioms of *BON* can be divided into the following five groups:

I. *Partial combinatory algebra*

(1) $\mathbf{k}xy = x$,

(2) $\mathbf{s}xy\downarrow \wedge \mathbf{s}xyz \simeq xz(yz)$,

(3) $\mathbf{k} \neq \mathbf{s}$.

II. *Pairing and projection*

(4) $\mathbf{p}xy\downarrow \wedge \mathbf{p}_0(\mathbf{p}xy) = x \wedge \mathbf{p}_1(\mathbf{p}xy) = y$,

(5) $\mathbf{p}xy \neq 0$.

[1] In Part II of this paper, these will be called *first-order formulas*.

III. *Natural numbers*

(6) $0 \in N \wedge (\forall x \in N)(x' \in N)$,

(7) $(\forall x \in N)(x' \neq 0 \wedge \mathbf{p}_N(x') = x)$,

(8) $(\forall x \in N)(x \neq 0 \rightarrow \mathbf{p}_N x \in N \wedge (\mathbf{p}_N x)' = x)$.

IV. *Definition by cases on N*

(9) $v \in N \wedge w \in N \wedge v = w \rightarrow \mathbf{d}_N vwxy = x$,

(10) $v \in N \wedge w \in N \wedge v \neq w \rightarrow \mathbf{d}_N vwxy = y$.

V. *Primitive recursion on N*

(11) $(f : N \rightarrow N) \wedge (g : N^3 \rightarrow N) \rightarrow (\mathbf{r}_N fg : N^2 \rightarrow N)$,

(12) $(f : N \rightarrow N) \wedge (g : N^3 \rightarrow N) \wedge x \in N \wedge y \in N \wedge h = \mathbf{r}_N fg$
$\rightarrow hx0 = fx \wedge hx(y') = gxy(hxy)$.

k and **s** are the partial versions of the well-known combinators of Curry's combinatory logic. **p** provides an injective pairing of the universe with the inverse functions $\mathbf{p}_0$ and $\mathbf{p}_1$. $\mathbf{s}_N$ represents the successor function on the natural numbers and $\mathbf{p}_N$ the predecessor function. $\mathbf{d}_N$ gives definition by integer cases; the original versions of T_0 and T_1 used $\mathbf{d}_V$, definition by cases on the universe. However, $\mathbf{d}_N$ suffices for most applications. $\mathbf{r}_N$ acts as a recursion operator which guarantees closure under primitive recursion. It is an immediate consequence of the work in [4, 1] that the following two theorems can be proved in *BON*, using only the partial combinatory axioms (1)–(3).

Theorem 1 (λ abstraction). *For each variable x and individual term t of L_p we can construct an individual term $\lambda x.t$ of L_p whose free variables are those of t, excluding x, so that*

$$BON \vdash \lambda x.t\downarrow \wedge (\lambda x.t)x \simeq t.$$

Theorem 2 (Recursion theorem). *There exists an individual term $\mathbf{r}_{\text{rec}}$ of L_p so that*

$$BON \vdash \mathbf{r}_{\text{rec}}x\downarrow \wedge (y = \mathbf{r}_{\text{rec}}x \rightarrow (\forall z)(yz \simeq xyz)).$$

3. Set and formula induction

In the following we extend the basic theory *BON* by complete induction on the natural numbers. We introduce two principles of increasing strength: an axiom of set induction and a schema of formula induction (full induction).

With each individual a we associate as its extension the collection of x such that $ax \simeq 0$; ax may be defined for other x, but not necessarily all x. In this way, a is regarded as a *semi-decidable set*, or simply a *semiset*. By a *decidable set* is meant an a such that for all x, $ax = 0 \vee ax = 1$, and by a *decidable subset of* N is meant an a such that for all $x \in N$, $ax = 0 \vee ax = 1$. In accordance with these ideas we introduce the following definitions:

$$b \epsilon a := (ab = 0),$$

$$a \in P(N) := (\forall x \in N)(ax = 0 \vee ax = 1).$$

Observe, however, that the symbols 'ϵ' and '$P(N)$'—as well as the earlier introduced $\in$—do not belong to the language L_p; they are introduced as abbreviations only to increase readability. The main principles of complete induction on the natural numbers are the following.

Set induction on N $(\mathit{Set\text{-}IND}_N)$

$$a \in P(N) \wedge 0 \epsilon a \wedge (\forall x \in N)(x \epsilon a \rightarrow x' \epsilon a) \rightarrow (\forall x \in N)(x \epsilon a),$$

Formula induction on N $(\mathit{Fmla\text{-}IND}_N)$

$$\varphi(0) \wedge (\forall x \in N)(\varphi(x) \rightarrow \varphi(x')) \rightarrow (\forall x \in N)\, \varphi(x)$$

for all formulas φ of L_p. Obviously $(\mathit{Set\text{-}IND}_N)$ can be regarded as special cases of $(\mathit{Fmla\text{-}IND}_N)$ where $\varphi(x)$ is the formula $(x \epsilon a)$. Adding these induction principles to the theory *BON* yields the following new theories

$$BON + (\mathit{Set\text{-}IND}_N) \quad \text{and} \quad BON + (\mathit{Fmla\text{-}IND}_N).$$

If $(\mathit{Fmla\text{-}IND}_N)$ is assumed, we can derive axioms (11) and (12) by Theorem 2, using a suitable definition of $\mathbf{r}_N$ in terms of $\mathbf{r}_{\mathrm{rec}}$; however, $(\mathit{Set\text{-}IND}_N)$ is not sufficient for this. The theory *EON* is *BON* minus axioms (11) and (12) plus $(\mathit{Fmla\text{-}IND}_N)$ so that *EON* is equivalent to $BON + (\mathit{Fmla\text{-}IND}_N)$. It is known from Beeson [1] that *EON* is proof-theoretically equivalent to Peano arithmetic *PA*; it also follows with techniques known from other work in the literature (e.g. Feferman [11]) that $BON + (\mathit{Set\text{-}IND}_N)$ is proof-theoretically equivalent to primitive recursive arithmetic *PRA*. For the sake of completeness, both those results will be given again below.

There are also interesting forms of so-called *semiset induction* on N, i.e., induction on the natural numbers for objects which are not assumed to be total on N. Semiset induction follows from formula induction and comprises set induction, hence is in strength between $(\mathit{Set\text{-}IND}_N)$ and $(\mathit{Fmla\text{-}IND}_N)$. However, in this paper we will not study this intermediate form of induction.

4. The non-constructive minimum operator

For the development of classical mathematics within the framework of operations and numbers one often needs stronger operation existence axioms.

This section presents one method of achieving this goal: the *non-constructive unbounded minimum operator* μ. This is a functional on $(N \to N)$ which assigns to each f with $(f : N \to N)$ an $x \in N$ with $fx = 0$, if there is any such x, and 0 otherwise. It thus satisfies the following axioms.

Axioms of the unbounded minimum operator

(μ.1) $(\mu : (N \to N) \to N)$,

(μ.2) $(f : N \to N) \wedge (\exists x \in N)(fx = 0) \to f(\mu f) = 0.$

These are sufficient for our purposes. Note that we then have:

$$f \in P(N) \to [(\exists x \in N)(x \,\epsilon\, f) \leftrightarrow \mu f \,\epsilon\, f].$$

We shall write $BON(\mu)$ for $BON + (\mu.1, \mu.2)$. The main results of this paper establish the proof-theoretic strength of this system with ($Set\text{-}IND_N$), respectively ($Fmla\text{-}IND_N$) respectively as:

$$BON(\mu) + (Set\text{-}IND_N) \equiv PA,$$

$$BON(\mu) + (Fmla\text{-}IND_N) \equiv (\Pi^0_\infty\text{-}CA)_{<\varepsilon_0},$$

where $\equiv$ means proof-theoretic equivalence as it is usually defined, for example in Feferman [10]. These results are established in Sections 6–8 below.

5. The proof-theoretic strength of *BON* with set and with formula induction

In this section we determine the proof-theoretic strength of *BON* with set and formula induction on N (but without μ) and show, in particular, that the theory $BON + (Set\text{-}IND_N)$ is proof-theoretically equivalent to *PRA*, and that $BON +$ ($Fmla\text{-}IND_N$) is proof-theoretically equivalent to *PA*.

Let L_2 be the usual second-order language of arithmetic with number variables $v, w, x, y, z, f, g, \ldots$, set variables $X, Y, Z, \ldots$ (both possibly with subscripts), the constant 0, as well as function and relation symbols for all primitive recursive functions and relations. The number terms $(r, s, t, r_1, s_1, t_1, \ldots)$ of L_2 are as usual. An L_2 formula is called *arithmetic* if it contains no bound set variables, though it may contain free set variables; the class of all arithmetic L_2 formulas is denoted by Π^0_∞. L_2 *sentences* are L_2 formulas without free variables.

The first-order sublanguage of L_2 which is built up without referring to set variables will be denoted by L_1 in the following. Hence every L_1 formula is arithmetic. A Σ^0_1 formula is an L_1 formula of the form $(\exists x)\,\varphi$ with φ quantifier-free.

In the following we use standard notation of first- and second-order arithmetic: $\langle \cdots \rangle$ is a standard primitive recursive function for forming n-tuples $\langle t_1, \ldots, t_n \rangle$; *Seq* is the primitive recursive set of sequence numbers; $\mathrm{lh}(t)$ denotes the length of (the sequence coded by) t; $(t)_i$ is the ith component of (the sequence coded by) t if $i < \mathrm{lh}(t)$, i.e., $t = \langle (t)_0, \ldots, (t)_{\mathrm{lh}(t)-1} \rangle$ if t is a sequence number; $s \in (X)_t$ stands for $\langle s, t \rangle \in X$.

Peano arithmetic PA is formulated in L_1 and given by the axioms for 0, successor and the defining axioms for all primitive recursive functions and relations together with all instances of complete induction on the natural numbers

$$(L_1\text{-}IND_N) \qquad \varphi(0) \wedge (\forall x)(\varphi(x) \rightarrow \varphi(x')) \rightarrow (\forall x)\varphi(x)$$

where $\varphi(x)$ is any L_1 formula. Primitive recursive arithmetic PRA is the subsystem of PA which is obtained by restricting the scheme of complete induction $(L_1\text{-}IND_N)$ to the quantifier-free formulas of L_1. In general, if $\mathscr{C}$ is a class of L_2 formulas, then we write $(\mathscr{C}\text{-}IND_N)$ for the restriction of $(L_1\text{-}IND_N)$ to $\mathscr{C}$. As known from the work of Parsons [17], PRA is equivalent to the subsystem of PA based on $(\Sigma_1^0\text{-}IND_N)$, and also to the quantifier-free system with a rule of induction.

5.1 Lower bounds

The lower bounds for the proof-theoretic strength of BON plus $(Set\text{-}IND_N)$, resp. $(Fmla\text{-}IND_N)$ and the same with the non-constructive μ-operator will be established by translating suitable systems of first- and second-order arithmetic into these theories. The basic idea is that the number variables of L_2 are interpreted as ranging over N and the set variables as ranging over $P(N)$. Accordingly, an atomic formula of the form $(x \in Y)$ is translated into $yx = 0$ where x and y are the variables of L_p which are associated to the variables x and Y of L_2, respectively.

Using the recursion operator $\mathbf{r}_N$, each primitive recursive function on the natural numbers can be represented in BON by an individual term of L_p and its recursion equation can be proved there. Then every L_2 formula $\varphi(\boldsymbol{X}, \boldsymbol{y})$ is translated into a formula $\varphi^N(\boldsymbol{x}, \boldsymbol{y})$ of the language L_p in a natural way. This translation is such that

$$((\exists z)\, \varphi(\boldsymbol{X}, \boldsymbol{y}, z))^N = (\exists z \in N)\, \varphi^N(\boldsymbol{x}, \boldsymbol{y}, z),$$
$$((\exists Z)\, \varphi(\boldsymbol{X}, Z, \boldsymbol{y}))^N = (\exists z \in P(N))\, \varphi^N(\boldsymbol{x}, z, \boldsymbol{y})$$

and similarly for the universal quantifiers. To keep the notation as simple as possible we use the same expressions for the individual terms of L_1 and their translation into L_p, and identify L_2 formulas with their translations into L_p, when there is no confusion. Moreover, as is straightforward to check, every quantifier-free formula of L_2 can be represented by an individual term of L_p, in the following sense:

Lemma 3. *For every quantifier-free formula $\varphi(\boldsymbol{X}, \boldsymbol{y})$ of L_2 with at most $\boldsymbol{X}, \boldsymbol{y}$ free there exists an individual term t of L_p so that*

1. $BON \vdash (\forall \boldsymbol{x} \in P(N))(\forall \boldsymbol{y} \in N)(t[\boldsymbol{x}, \boldsymbol{y}] = 0 \vee t[\boldsymbol{x}, \boldsymbol{y}] = 1)$,
2. $BON \vdash (\forall \boldsymbol{x} \in P(N))(\forall \boldsymbol{y} \in N)(\varphi^N(\boldsymbol{x}, \boldsymbol{y}) \leftrightarrow t[\boldsymbol{x}, \boldsymbol{y}] = 0)$.

As a consequence of this lemma we obtain that (the translation of) complete induction for quantifier-free formulas in *PRA* follows from (*Set-IND*$_N$) in the theory $BON + (\textit{Set-IND}_N)$. Therefore *PRA* is contained in $BON + (\textit{Set-IND}_N)$ and *PA* in $BON + (\textit{Fmla-IND}_N)$.

Theorem 4. *We have for every L_1 sentence φ:*
1. $PRA \vdash \varphi \Rightarrow BON + (\textit{Set-IND}_N) \vdash \varphi^N$,
2. $PA \vdash \varphi \Rightarrow BON + (\textit{Fmla-IND}_N) \vdash \varphi^N$.

5.2. *Upper bounds*

Upper bounds for $BON + (\textit{Set-IND}_n)$, $BON + (\textit{Fmla-IND}_N)$ and the corresponding versions with the unbounded μ-operator are obtained by interpreting them into appropriate systems of first-order arithmetic. The main step in each case is to find a suitable formula $App(x, y, z)$ which translates the L_p formula $xy \simeq z$.

Any such formula leads to a translation of L_p as follows: Assume that L is a first-order language which contains L_1; in addition assume that $App(x, y, z)$ is an L formula and I a mapping which assigns a numeral $I(t)$ to each constant t of L_p. Then let $*$ be the pair (App, I) and define an interpretation of L_p into L depending on $*$, by the following conditions 1–7.

The $*$ translation of an individual term t of L_p is an L_1 formula $\mathcal{T}_t^*(x)$ which is inductively defined as follows (where x does not occur in t).

1. If t is an individual variable, then $\mathcal{T}_t^*(x)$ is $(t = x)$.
2. If t is an individual constant, then $\mathcal{T}_t^*(x)$ is $(I(t) = x)$.
3. If t is the individual term (rs), then

$$\mathcal{T}_t^*(x) := (\exists z_1)(\exists z_2)(\mathcal{T}_r^*(z_1) \wedge \mathcal{T}_s^*(z_2) \wedge App(z_1, z_2, x)).$$

The $*$ translation φ^* of an L_p formula φ is then inductively defined as follows.

4. First, for atomic formulas of L_p, we put

$$(t\downarrow)^* := (\exists x)\, \mathcal{T}_t^*(x),$$

$$(s = t)^* := (\exists x)(\mathcal{T}_s^*(x) \wedge \mathcal{T}_t^*(x)),$$

$$N(t)^* := (\exists x)\, \mathcal{T}_t^*(x),$$

$$R(t_1, \ldots, t_n)^* := (\exists x_1) \cdots (\exists x_n)\Big(\bigwedge_{i=1}^{n} \mathcal{T}_{t_i}^*(x_i) \wedge R(x_1, \ldots, x_n)\Big)$$

if R is an n-ary relation symbol of L_p.

5. If φ is the formula $\neg\psi$, then φ^* is $\neg(\psi^*)$.
6. If φ is the formula $(\psi \vee \chi)$, then φ^* is $(\psi^* \vee \chi^*)$.
7. If φ is the formula $(\exists x)\, \psi$, then φ^* is $(\exists x)(\psi^*)$

The treatment of $BON + (Set\text{-}IND_N)$ and $BON + (Fmla\text{-}IND_N)$ is now straightforward: We interpret L_p into L_1 and handle the application operation of L_p in the sense of ordinary recursion theory by taking

$$App(x, y, z) := \{x\}(y) \simeq z,$$

where $\{n\}$ for $n = 0, 1, 2, \ldots$ is a standard enumeration of the partial recursive functions and $\simeq$ is the recursion-theoretic partial equality. In this case $\mathcal{T}_t^*(x)$ is (equivalent to) a Σ_1^0 formula for all L_p terms t. It is now an easy exercise in formalized recursion theory to show that there exist translations $I(t)$ of the L_p constants t so that

$$PRA + (\Sigma_1^0\text{-}IND_N) \vdash \varphi^*$$

for each axiom φ of BON. Since $(Set\text{-}IND_N)$ in the language L_p translates into $(\Sigma_1^0\text{-}IND_N)$ in the language L_1, we obtain the following theorem. Together with Parsons' result mentioned earlier and Theorem 4 it establishes the proof-theoretic equivalences stated in the corollary below.

Theorem 5. *We have for every L_p formula φ:*
1. $BON + (Set\text{-}IND_N) \vdash \varphi \Rightarrow PRA + (\Sigma_1^0\text{-}IND_N) \vdash \varphi^*$,
2. $BON + (Fmla\text{-}IND_N) \vdash \varphi \Rightarrow PA \vdash \varphi^*$.

Corollary 6. *We have:*
1. $BON + (Set\text{-}IND_N) \equiv PRA$,
2. $BON + (Fmla\text{-}IND_N) \equiv PA$.

6. Lower bounds for the proof-theoretic strength of $BON(\mu)$ with set and with formula induction

The rest of this paper is devoted to the proof-theoretic analysis of $BON(\mu)$ with set and with formula induction on the natural numbers. We begin these investigations by determining the lower bounds for both theories in this section.

6.1. Lower bounds for $BON(\mu) + (Set\text{-}IND_N)$

The lower bound for the theory $BON(\mu) + (Set\text{-}IND_N)$ can be established directly by applying the unbounded minimum operator μ in order to eliminate the (number) quantifiers of arithmetic L_2 formulas. Using Lemma 3 and induction on the length of arithmetic formulas one then easily verifies the following.

Lemma 7. *For every arithmetic formula $\varphi(X, y)$ of L_2 with at most X, y free there exists an individual term t of L_p so that*
1. $BON(\mu) \vdash (\forall \boldsymbol{x} \in P(N))(\forall \boldsymbol{y} \in N)(t[\boldsymbol{x}, \boldsymbol{y}] = 0 \lor t[\boldsymbol{x}, \boldsymbol{y}] = 1)$,
2. $BON(\mu) \vdash (\forall \boldsymbol{x} \in P(N))(\forall \boldsymbol{y} \in N)(\varphi^N(\boldsymbol{x}, \boldsymbol{y}) \leftrightarrow t[\boldsymbol{x}, \boldsymbol{y}] = 0)$.

This lemma implies that (the translation of) complete induction for arbitrary L_1 formulas can be derived from $(\mathit{Set\text{-}IND}_N)$ in the theory $BON(\mu)+(\mathit{Set\text{-}IND}_N)$. Hence PA may be regarded as a subtheory of $BON(\mu)+(\mathit{Set\text{-}IND}_N)$.

Theorem 8. *We have for every L_1 sentence φ:*

$$PA \vdash \varphi \Rightarrow BON(\mu)+(\mathit{Set\text{-}IND}_N) \vdash \varphi^N.$$

6.2. Lower bounds for $BON(\mu)+(\mathit{Fmla\text{-}IND}_N)$

This part shows that the second-order theory $(\Pi^0_\infty\text{-}CA)_{<\varepsilon_0}$ can be embedded into $BON(\mu)+(\mathit{Fmla\text{-}IND}_N)$; this takes a bit more work, though of a relatively familiar kind. Let $<$ be a standard primitive recursive well-ordering of order type ε_0. The idea is to define an operation h such that for each $\alpha<\varepsilon_0$ and for each n of order type α, provably in $BON(\mu)+(\mathit{Fmla\text{-}IND}_N)$ we have that hn represents H_α in the hyperarithmetic (iterated jump) hierarchy. Moreover, we can relativize this to any initial set.

Let us first recall the theory $(\Pi^0_\infty\text{-}CA)_{<\varepsilon_0}$ of the arithmetic comprehension axiom iterated through each ordinal less than ε_0; more on systems of this kind can be found, for example, in Feferman [3] and Friedman [12]. By *arithmetic comprehension* one means the axiom scheme

$$(\Pi^0_\infty\text{-}CA) \qquad (\exists X)(\forall x)(x\in X \leftrightarrow \varphi(x))$$

for all arithmetic L_2 formulas φ. This is well known to be equivalent to the scheme $(\Pi^0_1\text{-}CA)$ which restricts comprehension to formulas φ in Π^0_1 form. If $<$ is a primitive recursive well-ordering and $R_<$ the corresponding relation symbol, then we write $(x<y)$ for $R_<(x,y)$, $(\exists x<y)\,\varphi(x)$ for $(\exists x)(x<y \wedge \varphi(x))$ and $(\forall x<y)\,\varphi(x)$ for $(\forall x)(x<y\rightarrow\varphi(x))$. The principle of *transfinite induction* for an L_2 formula $\varphi(x)$ along $<$ is expressed by the formula $TI(<,\varphi)$ defined by

$$TI(<,\varphi) := (\forall x)((\forall y<x)\,\varphi(y)\rightarrow\varphi(x))\rightarrow(\forall x)\,\varphi(x).$$

In the following we assume that $<$ is a primitive recursive standard well-ordering of order type ε_0 with least element 0 and field $\mathbb{N}$. If n is a natural number, then $<_n$ denotes the restriction of $<$ to the numbers $m<n$. For details about such primitive recursive standard well-orderings we refer to Girard [13], Schütte [18] or Takeuti [19].

Given an arithmetic L_2 formula $\chi(X,y)$ with at most X, y free, an arbitrary set X of natural numbers and a natural number n, we define the *χ-jump hierarchy* along $<_n$ starting with X, by the following transfinite recursion

$$(Y)_0 := X,$$

$$(Y)_i := \{\langle m,j\rangle : j<i \wedge \chi((Y)_j, m)\}$$

for all $0<i<n$ and denote the arithmetic formula which formalizes this definition up to any given n by $\mathscr{H}_\chi(X,Y,n)$. If α is an ordinal less than ε_0, then we write

$(\Pi^0_\infty - CA)_\alpha$ for the second-order theory which consists of the axioms of PA plus the additional axioms $TI(<_n, \varphi)$ for all L_2 formulas φ and $(\forall X)(\exists Y)\ \mathcal{H}_\chi(X, Y, n)$ for all arithmetic formulas $\chi(X, y)$ with at most X, y free where the order type of $<_n$ is α. The union of all theories $(\Pi^0_\infty\text{-}CA)_\beta$ with $\beta < \alpha$ is called $(\Pi^0_\infty\text{-}CA)_{<\alpha}$.

The following theorem shows that there exists an L_p term h which represents the χ-jump hierarchy uniformly in the initial set parameter. The detailed proof of this theorem is given in the Appendix.

Theorem 9. *Let $\chi(X, y)$ be an arithmetic L_2 formula with at most X, y free and assume that n is an arbitrary natural number. Then there exists an L_p term h so that $BON(\mu) + (Fmla\text{-}IND_N)$ proves:*
1. $x \in P(N) \rightarrow hx \in P(N)$,
2. $x \in P(N) \rightarrow \mathcal{H}^N_\chi(x, hx, n)$.

It is an obvious consequence of this theorem that the translations of the L_2 formulas $(\forall X)(\exists Y)\ \mathcal{H}_\chi(X, Y, n)$ are provable in $BON(\mu) + (Fmla\text{-}IND_N)$ for all arithmetic L_2 formulas $\chi(X, y)$ with at most X, y free and all natural numbers n. From standard proof theory it is also known[2] that PA proves $TI(<_n, \varphi)$ for all L_1 formulas φ. Hence it is also clear that the translations of the formulas $TI(<_n, \varphi)$ are provable in the latter theory for all L_2 formulas φ. Therefore $BON(\mu) + (Fmla\text{-}IND_N)$ contains $(\Pi^0_\infty\text{-}CA)_{<\varepsilon_0}$.

Theorem 10. *We have for every L_2 sentence φ:*

$$(\Pi^0_\infty\text{-}CA)_{<\varepsilon_0} \vdash \varphi \;\Rightarrow\; BON(\mu) + (Fmla\text{-}IND_N) \vdash \varphi^N.$$

7. Theories of ordinals over PA

The upper bounds for BON with set and with formula induction on the natural numbers were determined in Section 5 by making use of the recursion-theoretic model of BON. In contrast to that approach, more delicate considerations are needed to establish the upper bounds for the proof-theoretic strength of the corresponding theories with the unbounded minimum operator. In order to achieve this aim we introduce the fixed point theories with ordinals PA^r_Ω and PA^w_Ω whose proof-theoretic analysis has been carried through in Jäger [15].

Let P be a new n-ary relation symbol, i.e., a relation symbol which does not belong to the language L_1. Then $L_1(P)$ is the extension of L_1 by P. An $L_1(P)$ formula is called *P-positive* if each occurrence of P in this formula is positive. We call P-positive formulas which contain at most $\boldsymbol{x}$ free *inductive operator forms,* and let $A(P, \boldsymbol{x})$ range over such forms.

[2] Since $<$ is a well-ordering of order type ε_0, the order type of each segment $<_n$ is less than ε_0.

Now we extend L_1 to a new first-order language L_Ω by adding a new sort of *ordinal variables* $\alpha, \beta, \gamma, \ldots$ (possibly with subscripts), a new binary relation symbol $<$ for the less relation on the ordinals[3] and an $(n+1)$-ary relation symbol P_A for each inductive operator form $A(P, \boldsymbol{x})$ for which P is n-ary.

The *number terms* of L_Ω are the number terms of L_1; the *ordinal terms* of L_Ω are the ordinal variables. The formulas $(\varphi, \psi, \chi, \theta, \varphi_1, \psi_1, \chi_1, \theta_1, \ldots)$ of L_Ω are inductively generated as follows:

1. If R is an n-ary relation symbol of L_1, then $R(s_1, \ldots, s_n)$ is an (atomic) formula of L_Ω.
2. $(\alpha < \beta)$, $(\alpha = \beta)$ and $P_A(\alpha, \boldsymbol{s})$ are (atomic) formulas of L_Ω. We write $P_A^\alpha(\boldsymbol{s})$ for $P_A(\alpha, \boldsymbol{s})$.
3. If φ and ψ are formulas of L_Ω, then $\neg\varphi$ and $\varphi \vee \psi$ are formulas of L_Ω.
4. If φ is a formula of L_Ω, then $(\exists x)\, \varphi$ and $(\forall x)\, \varphi$ are formulas of L_Ω.
5. If φ is a formula of L_Ω, then $(\exists \alpha)\, \varphi$ and $(\forall \alpha)\, \varphi$ are formulas of L_Ω.
6. If φ is a formula of L_Ω, then $(\exists \alpha < \beta)\, \varphi$ and $(\forall \alpha < \beta)\, \varphi$ are formulas of L_Ω.

Parentheses can be omitted if there is no danger of confusion. If $\varphi(P)$ is an $L_1(P)$ formula and $\psi(\boldsymbol{x})$ an L_Ω formula (where P is n-ary and $\boldsymbol{x} = x_1, \ldots, x_n$), then $\varphi(\psi)$ denotes the result of substituting $\psi(\boldsymbol{s})$ for every occurrence of $P(\boldsymbol{s})$ in $\varphi(P)$. For every L_Ω formula φ we write φ^α to denote the L_Ω formula which is obtained by replacing all unbounded quantifiers $(Q\beta)$ in φ by $(Q\beta < \alpha)$. Additional abbreviations are:

$$P_A^{<\alpha}(\boldsymbol{s}) := (\exists \beta < \alpha)\, P_A^\beta(\boldsymbol{s}),$$

$$P_A(\boldsymbol{s}) := (\exists \alpha)\, P_A^\alpha(\boldsymbol{s}).$$

An L_Ω formula is called a Δ_0^Ω formula if all its ordinal quantifiers are bounded. It is called a Σ^Ω formula if all positive universal ordinal quantifiers and all negative existential ordinal quantifiers are bounded; correspondingly it is called a Π^Ω formula if all negative universal ordinal quantifiers and all positive existential ordinal quantifiers are bounded. See Jäger [15] for the precise definitions.

Now we introduce three L_Ω theories which differ in the strength of their induction principles. The weakest of those, PA_Ω^r, is given by the following axioms.

Number-theoretic axioms. These comprise the axioms of Peano arithmetic PA with the exception of complete induction on the natural numbers.

Inductive operator axioms. For all inductive operator forms $A(P, \boldsymbol{x})$:

$$P_A^\alpha(\boldsymbol{s}) \leftrightarrow A(P_A^{<\alpha}, \boldsymbol{s}).$$

[3] It will always be clear from the context whether $<$ denotes the less relation on the nonnegative integers or on the ordinals.

Σ^Ω reflection axioms. For every Σ^Ω formula φ:

$(\Sigma^\Omega\text{-}Ref)$ $\quad \varphi \rightarrow (\exists\alpha)\,\varphi^\alpha$.

Linearity of the relation $<$ on the ordinals:

(LO) $\quad \alpha \nless \alpha \wedge (\alpha < \beta \wedge \beta < \gamma \rightarrow \alpha < \gamma) \wedge (\alpha < \beta \vee \alpha = \beta \vee \beta < \alpha)$.

Δ_0^Ω induction on the natural numbers. For all Δ_0^Ω formulas $\varphi(x)$:

$(\Delta_0^\Omega\text{-}IND_N)$ $\quad \varphi(0) \wedge (\forall x)(\varphi(x) \rightarrow \varphi(x')) \rightarrow (\forall x)\,\varphi(x)$.

Δ_0^Ω induction on the ordinals. For all Δ_0^Ω formulas $\varphi(\alpha)$:

$(\Delta_0^\Omega\text{-}IND_\Omega)$ $\quad (\forall\alpha)((\forall\beta < \alpha)\,\varphi(\beta) \rightarrow \varphi(\alpha)) \rightarrow (\forall\alpha)\,\varphi(\alpha)$.

PA_Ω^w is the extension of PA_Ω^r by the following *scheme of complete induction on the natural numbers*:

$(L_\Omega\text{-}IND_N)$ $\quad \varphi(0) \wedge (\forall x)(\varphi(x) \rightarrow \varphi(x')) \rightarrow (\forall x)\,\varphi(x)$

for all L_Ω formulas $\varphi(x)$. PA_Ω is the extension of PA_Ω^w by the following *scheme of induction on the ordinals*

$(L_\Omega\text{-}IND_\Omega)$ $\quad (\forall\alpha)((\forall\beta < \alpha)\,\varphi(\beta) \rightarrow \varphi(\alpha)) \rightarrow (\forall\alpha)\,\varphi(\alpha)$

for all L_Ω formulas $\varphi(\alpha)$.

It follows from the P-positivity of the inductive operator forms $A(P, \boldsymbol{x})$ and the inductive operator axioms that the formulas $P_A^\alpha(x)$ are monotonic in their ordinal arguments.

Lemma 11. *We have for all ordinal variables α, β and all number terms s:*

$$PA_\Omega^r \vdash \alpha < \beta \rightarrow (P_A^\alpha(s) \rightarrow P_A^\beta(s)).$$

Corresponding to the well-known result that every total recursively enumerable function is recursive, we have that every total Σ^Ω function is Δ_0^Ω. More precisely, in PA_Ω^r every total functional relation on the numbers which is defined by a Σ^Ω formula can already be defined by a Δ_0^Ω formula, in the following sense.

Lemma 12. *We have for all Σ^Ω formulas $\varphi(\boldsymbol{x}, y)$:*

$$PA_\Omega^r \vdash (\forall\boldsymbol{x})(\exists!\,y)\,\varphi(\boldsymbol{x}, y) \rightarrow (\exists\alpha)(\forall\boldsymbol{x})(\forall y)(\varphi(\boldsymbol{x}, y) \leftrightarrow \varphi^\alpha(\boldsymbol{x}, y)).$$

Proof. We work in PA_Ω^r and assume that $(\forall\boldsymbol{x})(\exists!\,y)\,\varphi(\boldsymbol{x}, y)$. Hence by Σ^Ω reflection there exists an ordinal α so that $(\forall\boldsymbol{x})(\exists y)\,\varphi^\alpha(\boldsymbol{x}, y)$. Σ^Ω persistency is easily provable in PA_Ω^r, and so we also have

$$(\forall\boldsymbol{x})(\forall y)(\varphi^\alpha(\boldsymbol{x}, y) \rightarrow \varphi(\boldsymbol{x}, y)).$$

Hence we have for all $\boldsymbol{s}$, t that $\varphi(\boldsymbol{s}, t)$ if and only if $\varphi^\alpha(\boldsymbol{s}, t)$. $\square$

From the inductive operator and Σ^{Ω} reflection axioms we can easily deduce that the Σ^{Ω} formula $P_A(\boldsymbol{x})$ describes a fixed point of the inductive operator form $A(P, \boldsymbol{x})$. If $(L_{\Omega}\text{-}IND_{\Omega})$ is available as well, then this fixed point can be proved to be the least L_{Ω} definable fixed point of $A(P, \boldsymbol{x})$. These constitute the following statement.

Theorem 13. *We have for all inductive operator forms $A(P, \boldsymbol{x})$ of $L_1(P)$ and all formulas $\varphi(\boldsymbol{x})$ of L_{Ω}:*

1. $PA^r_{\Omega} \vdash (\forall \boldsymbol{x})(P_A(\boldsymbol{x}) \leftrightarrow A(P_A, \boldsymbol{x}))$,
2. $PA_{\Omega} \vdash (\forall \boldsymbol{x})(A(\varphi, \boldsymbol{x}) \rightarrow \varphi(\boldsymbol{x})) \rightarrow (\forall \boldsymbol{x})(P_A(\boldsymbol{x}) \rightarrow \varphi(\boldsymbol{x}))$.

This theorem suggests that there is a close relationship between the theory PA_{Ω} and the well-known theory ID_1 (cf. e.g. [2, 3]) as well as between PA^w_{Ω} and the fixed point theory $\widehat{ID}_1$ of Feferman [8]. Both theories, ID_1 and $\widehat{ID}_1$, are formulated in the language $L_1(FP)$ which extends L_1 by adding fixed point constants $\mathcal{P}_A$ for all inductive operator forms $A(P, \boldsymbol{x})$, and there is a natural translation of $L_1(FP)$ into L_{Ω}: One only has to interpret the atomic formulas $\mathcal{P}_A(\boldsymbol{x})$ of $L_1(FP)$ by the Σ^{Ω} formulas $P_A(\boldsymbol{x})$ of L_{Ω}. Hence complete induction on the natural numbers for L_1 formulas is a consequence of $(\Delta^{\Omega}_0\text{-}IND_N)$, whereas $(L_{\Omega}\text{-}IND_N)$ is needed to prove the translations of complete induction on the natural numbers for $L_1(FP)$ formulas.

Obviously PA^r_{Ω} contains PA. Although the (translations of the) fixed point axioms of $\widehat{ID}_1$ are provable in PA^r_{Ω} according to the previous theorem, we need $(L_{\Omega}\text{-}IND_N)$ for dealing with the scheme of complete induction which is available in $\widehat{ID}_1$ for all $L_1(FP)$ formulas. Hence $\widehat{ID}_1$ can be directly interpreted in PA^w_{Ω} but not in PA^r_{Ω}. Finally we also obtain from Theorem 13 that PA_{Ω} contains ID_1. In addition to these remarks, the following results of Jäger [15] describe the exact proof-theoretic strength of PA_{Ω}, PA^w_{Ω} and PA^r_{Ω}.

Theorem 14. *We have*:

1. *PA_{Ω} is a conservative extension of ID_1 with respect to all L_1 formulas.*
2. *PA^w_{Ω} is a conservative extension of $\widehat{ID}_1$ with respect to all L_1 formulas.*
3. *PA^r_{Ω} is a conservative extension of PA with respect to all L_1 formulas.*

8. Upper bounds for the proof-theoretic strength of $BON(\mu)$ with set and with formula induction

For the embedding of $BON(\mu) + (Set\text{-}IND_N)$ and $BON(\mu) + (Fmla\text{-}IND_N)$ into PA^r_{Ω} and PA^w_{Ω}, respectively, we interpret the application relation $xy \simeq z$ by means of a fixed point of a suitable inductive operator form to be introduced below. Special difficulties are caused by the recursion operator $\mathbf{r}_N$, and we turn to this problem first.

Let $\varphi(f, x, y)$ be an L_Ω formula with at most f, x, y free, and n a natural number greater 0. Then we define L_Ω formulas $Ap^n_\varphi(f, x_1, \ldots, x_n, y)$ by recursion on n and, from those, L_Ω formulas $Fun^n_\varphi(f)$ and $Un^n_\varphi(f)$:

$$Ap^1_\varphi(f, x_1, y) := \varphi(f, x_1, y),$$

$$Ap^{n+1}_\varphi(f, x_1, \ldots, x_{n+1}, y) := (\exists z)(Ap^n_\varphi(f, x_1, \ldots, x_n, z) \wedge \varphi(z, x_{n+1}, y)),$$

$$Fun^n_\varphi(f) := (\forall x_1, \ldots, x_n)(\exists!\, y)\, Ap^n_\varphi(f, x_1, \ldots, x_n, y),$$

$$Un^n_\varphi(f) := (\forall \mathbf{x})(\forall y, z)(Ap^n_\varphi(f, \mathbf{x}, y) \wedge Ap^n_\varphi(f, \mathbf{x}, z) \rightarrow y = z).$$

Hence, if $\varphi(f, x, y)$ is used as an interpretation of the application relation $fx \simeq y$ in L_p, then $Ap^n_\varphi(f, x_1, \ldots, x_n, y)$ represents the L_p formula $fx_1, \ldots, x_n \simeq y$. In this context $Fun^n_\varphi(f)$ expresses that f is (a code of) an n-ary total function in the sense of φ; $Un^n_\varphi(f)$ says that f is (a code of) an n-ary partial function in the sense of φ. When it is clear by the number of the variables shown, we shall drop the superscript 'n' in the above notations.

Remark 15. *If $\varphi(f, x, y)$ is a Σ^Ω formula, then $Ap_\varphi(f, x_1, \ldots, x_n, y)$ is a Σ^Ω formula for each $n \geq 1$.*

If f is (a code of) a 1-ary function in the sense of φ and g (a code of) a 3-ary function in the sense of φ, then the formula $Rec_\varphi(f, g, x, y, z)$ below can be used to describe the graph of the function which is defined from f and g by primitive recursion in the sense of φ:

$$Rec_\varphi(f, g, x, y, z) := \begin{cases} (\exists v)(Seq(v) \wedge \mathrm{lh}(v) = y + 1 \wedge \varphi(f, x, (v)_0) \wedge \\ (\forall w < y)\, Ap_\varphi(g, x, w, (v)_w, (v)_{w+1}) \wedge z = (v)_y). \end{cases}$$

Remark 16. *Let P be a 3-ary relation symbol. Then $Rec_P(f, g, x, y, z)$ is a P-positive formula of the language $L_1(P)$.*

$Rec_\varphi(f, g, x, y, z)$ is the standard formula for primitive recursion from f and g where we use φ to interpret application. It will be important to know later that it has the properties in the following lemma. The first part of this is concerned with the uniqueness of the formula $Rec_\varphi(f, g, x, y, z)$ in its fifth argument and the second with its functionality. Sufficient conditions for uniqueness and functionality are given.

Lemma 17. 1. *If $\varphi(x, y, z)$ is an L_Ω formula with at most x, y, z free, then PA^r_Ω proves*

$$Un^1_\varphi(f) \wedge Un^3_\varphi(g) \wedge Rec_\varphi(f, g, x, y, z_1) \wedge Rec_\varphi(f, g, x, y, z_2) \rightarrow z_1 = z_2.$$

2. *If $\varphi(x, y, z)$ is a Σ^Ω formula with at most x, y, z free, then PA^r_Ω proves*

$$Fun^1_\varphi(f) \wedge Fun^3_\varphi(g) \rightarrow (\forall x)(\forall y)(\exists!\, z) Rec_\varphi(f, g, x, y, z).$$

Proof. The first assertion follows from the uniqueness of f and g and the definition of Rec_φ by an easy inductive argument. For the proof of the second we work in PA^r_Ω, assume that $Fun^1_\varphi(f)$ and $Fun^3_\varphi(g)$ and choose an arbitrary x_0. Since $\varphi(x, y, z)$ is a Σ^Ω formula, $Ap_\varphi(x, y_1, y_2, y_3, z)$ is a Σ^Ω formulas as well. Hence by Lemma 12 there exist Δ^Ω_0 formulas $\psi(x, y, z)$ and $\chi(x, y_1, y_2, y_3, z)$ so that

$$\varphi(f, v, w) \leftrightarrow \psi(f, v, w), \tag{1}$$

$$Ap_\varphi(g, v_1, v_2, v_2, w) \leftrightarrow \chi(g, v_1, v_2, v_3, w) \tag{2}$$

for all v, v_1, v_2, v_3, w. Observe that ψ and χ may have an additional ordinal parameter. It follows that $Rec_\varphi(f, g, x_0, y, z)$ is equivalent for all y, z to the Δ^Ω_0 formula $\theta(y, z)$,

$$\theta(y, z) := \begin{cases} (\exists v)(Seq(v) \wedge \mathrm{lh}(v) = y + 1 \wedge \psi(f, x_0, (v)_0) \wedge \\ (\forall w < y)\, \chi(g, x_0, w, (v)_w, (v)_{w+1}) \wedge z = (v)_y). \end{cases} \tag{3}$$

Using Δ^Ω_0 induction on the natural numbers, which is available in PA^r_Ω, we obtain

$$(\forall y)(\exists z)\, \theta(y, z), \tag{4}$$

$$(\forall y, z_1, z_2)(\theta(y, z_1) \wedge \theta(y, z_2) \rightarrow z_1 = z_2) \tag{5}$$

as usual. This completes the proof of our assertion. □

Next we turn to the representation of the application operation of L_p in L_Ω. This will be achieved by means of a fixed point of an inductive operator form $A(P, x, y, z)$. Specific such constructions are carried through, for example in Feferman [7, p. 200] and Beeson [1, p. 144]. First we choose pairwise different numerals $\hat{\mathbf{k}}$, $\hat{\mathbf{s}}$, $\hat{\mathbf{p}}$, $\hat{\mathbf{p}}_0$, $\hat{\mathbf{p}}_1$, $\hat{\mathbf{s}}_N$, $\hat{\mathbf{p}}_N$, $\hat{\mathbf{d}}_N$, $\hat{\mathbf{r}}_N$ and $\hat{\mu}$ (the values of) which do not belong to the set $\{0\} \cup \{x \in \mathbb{N}: Seq(x)\}$; they will later act as codes of the corresponding constants of L_p. Besides that we define for all natural numbers n:

$$Seq_n(t) := Seq(t) \wedge \mathrm{lh}(t) = n$$

and assume that our coding of sequences is such that $\neg(Seq_m(t) \wedge Seq_n(t))$ if $m \neq n$. We are going to code the L_p terms $\mathbf{k}x$, $\mathbf{s}x$, $\mathbf{s}xy$, $\mathbf{p}x, \ldots$ by the sequence numbers $\langle \hat{\mathbf{k}}, x\rangle$, $\langle \hat{\mathbf{s}}, x\rangle$, $\langle \hat{\mathbf{s}}, x, y\rangle$, $\langle \hat{\mathbf{p}}, x\rangle, \ldots$ of the corresponding form; for example, to satisfy $\mathbf{k}xy = x$ we interpret $\mathbf{k}x$ as $\langle \hat{\mathbf{k}}, x\rangle$ and then $\langle \hat{\mathbf{k}}, x\rangle y$ is taken to be x.

In detail, let P be a 3-ary relation symbol which does not belong to the language L_1 and define $A(P, x, y, z)$ to be the disjunction of the following formulas (1)–(22):

(1) $x = \hat{\mathbf{k}} \wedge z = \langle \hat{\mathbf{k}}, y\rangle$,
(2) $Seq_2(x) \wedge (x)_0 = \hat{\mathbf{k}} \wedge (x)_1 = z$,
(3) $x = \hat{\mathbf{s}} \wedge z = \langle \hat{\mathbf{s}}, y\rangle$,
(4) $Seq_2(x) \wedge (x)_0 = \hat{\mathbf{s}} \wedge z = \langle \hat{\mathbf{s}}, (x)_1, y\rangle$,

(5) $Seq_3(x) \wedge (x)_0 = \hat{\mathbf{s}} \wedge (\exists v, w)(P((x)_1, y, v) \wedge P((x)_2, y, w) \wedge P(v, w, z))$,
(6) $x = \hat{\mathbf{p}} \wedge z = \langle \hat{\mathbf{p}}, y \rangle$,
(7) $Seq_2(x) \wedge (x)_0 = \hat{\mathbf{p}} \wedge z = \langle (x)_1, y \rangle$,
(8) $x = \hat{\mathbf{p}}_0 \wedge (\exists v)(y = \langle z, v \rangle)$,
(9) $x = \hat{\mathbf{p}}_1 \wedge (\exists v)(y = \langle v, z \rangle)$,
(10) $x = \hat{\mathbf{s}}_N \wedge z = y + 1$,
(11) $x = \hat{\mathbf{p}}_N \wedge y = z + 1$,
(12) $x = \hat{\mathbf{d}}_N \wedge z = \langle \hat{\mathbf{d}}_N, y \rangle$,
(13) $Seq_2(x) \wedge (x)_0 = \hat{\mathbf{d}}_N \wedge z = \langle \hat{\mathbf{d}}_N, (x)_1, y \rangle$,
(14) $Seq_3(x) \wedge (x)_0 = \hat{\mathbf{d}}_N \wedge z = \langle \hat{\mathbf{d}}_N, (x)_1, (x)_2, y \rangle$,
(15) $Seq_4(x) \wedge (x)_0 = \hat{\mathbf{d}}_N \wedge (x)_1 = (x)_2 \wedge z = (x)_3$,
(16) $Seq_4(x) \wedge (x)_0 = \hat{\mathbf{d}}_N \wedge (x)_1 \neq (x)_2 \wedge z = y$,
(17) $x = \hat{\mathbf{r}}_N \wedge z = \langle \hat{\mathbf{r}}_N, y \rangle$,
(18) $Seq_2(x) \wedge (x)_0 = \hat{\mathbf{r}}_N \wedge z = \langle \hat{\mathbf{r}}_N, (x)_1, y \rangle$,
(19) $Seq_3(x) \wedge (x)_0 = \hat{\mathbf{r}}_N \wedge z = \langle \hat{\mathbf{r}}_N, (x)_1, (x)_2, y \rangle$,
(20) $Seq_4(x) \wedge (x)_0 = \hat{\mathbf{r}}_N \wedge Rec_P((x)_1, (x)_2, (x)_3, y, z))$,
(21) $x = \hat{\mu} \wedge (\forall v)(\exists w)(w \neq 0 \wedge P(y, v, w)) \wedge z = 0$,
(22) $x = \hat{\mu} \wedge P(y, z, 0) \wedge (\forall v)(v < z \rightarrow (\exists w)(w \neq 0 \wedge P(y, v, w)))$.

In view of Remark 16 we see immediately that $A(P, x, y, z)$ is a P-positive formula of $L_1(P)$, hence an inductive operator form. If we write $A_i(P, x, y, z)$ for the clause (i) of the definition of $A(P, x, y, z)$, then this operator form is deterministic in the following sense:

Remark 18. *We have for all L_Ω formulas $\varphi(x, y, z)$ with at most x, y, z free and all $1 \leq i < j \leq 22$:*

$$PA^r_\Omega \vdash (\forall v)\, Un^1_\varphi(v) \rightarrow \neg(A_i(\varphi, x, y, z) \wedge A_j(\varphi, x, y, z)).$$

Hence $A(\varphi, x, y, z)$ implies that exactly one of its definition clauses (1)–(22) is satisfied if we have $(\forall v)\, Un^1_\varphi(v)$, i.e., if each v is a partial function in the sense of φ. This assumption is necessary in order to distinguish between clause (21) and clause (22). In the following we will often make use of the previous remark without explicitly mentioning it. The next results are concerned with properties of the formulas $P^\alpha_A(x)$ and $P_A(x)$ which are induced by the operator form $A(P, x, y, z)$.

Lemma 19. *PA^r_Ω proves for variable α and all number terms r, s, t_1, t_2:*

1. $P^\alpha_A(r, s, t_1) \wedge P^\alpha_A(r, s, t_2) \rightarrow t_1 = t_2$,
2. $P_A(r, s, t_1) \wedge P_A(r, s, t_2) \rightarrow t_1 = t_2$.

Proof. Let $\psi(\alpha)$ be the Δ^Ω_0 formula $(\forall x)\, Un_{P^\alpha_A}(x)$. Then our first assertion follows from $PA^r_\Omega \vdash \psi(\alpha)$. To establish this we work in PA^r_Ω and prove $\psi(\alpha)$ by Δ^Ω_0 induction on the ordinals. For this purpose assume that

$$P^\alpha_A(x, y, v) \wedge P^\alpha_A(x, y, w) \tag{1}$$

for arbitrary x, y, v, w. We have to show that $v = w$. From the induction hypothesis and Lemma 11 we obtain that

$$(\forall x)\, Un_{P_A^{<\alpha}}(x), \tag{2}$$

and the operator axioms yield

$$A(P_A^{<\alpha}, x, y, v) \wedge A(P_A^{<\alpha}, x, y, w). \tag{3}$$

If there exist z_1, z_2, z_3 so that $x = \langle \hat{\mathbf{r}}_N, z_1, z_2, z_3 \rangle$, then $v = w$ follows from (3), (2) and the first part of Lemma 17. In all other cases we obtain $v = w$ either directly from (3) or from (3) and (2). This establishes the first assertion of the present lemma; in view of Lemma 11 the second assertion is an immediate consequence of the first. □

Now we come back to the interpretation of the theories $BON(\mu) + (Set\text{-}IND_N)$ and $BON(\mu) + (Fmla\text{-}IND_N)$ into PA_Ω^r and PA_Ω^w, respectively. In order to represent the application operation of L_p in L_Ω we define

$$App(x, y, z) := P_A(x, y, z)$$

and take a translation I of the constants of L_p to numerals so that $I(0) = 0$ and $I(t) = \hat{t}$ for all L_p constants different from 0. Using $* = (App, I)$ we then define the translations $\mathcal{T}_t^*(x)$ of the L_p terms t and φ^* of the L_p formulas φ as in Subsection 5.2. It follows that $\mathcal{T}_t^*(x)$ is a Σ^Ω formula for every L_p term t so that all atomic formulas of L_p are translated into Σ^Ω formulas of L_Ω.

It is an easy exercise to check that the $*$ translation of all axioms of the logic of partial terms are provable in PA_Ω^r. The following lemma gives the same for all the mathematical axioms of $BON(\mu)$.

Lemma 20. *We have for every axiom φ of $BON(\mu)$:*

$$PA_\Omega^r \vdash \varphi^*.$$

Proof. Obviously the definition of $A(P, x, y, z)$ has been tailored so that this lemma goes through. It can be checked by straightforward but tedious calculations that φ^* can be proved in PA_Ω^r for each axiom φ of $BON(\mu)$. In the case of the axioms about primitive recursion on N Lemma 17 gives the desired results. □

The discussion of the induction principles of our theories of operations and numbers is still missing. As before, we distinguish between $(Set\text{-}IND_N)$ and $(Fmla\text{-}IND_N)$ and takes care of the former version of induction in PA_Ω^r whereas PA_Ω^w provides the framework to handle the latter.

Lemma 21. *The $*$ translation of each instance of $(Set\text{-}IND_N)$ is provable in PA_Ω^r; i.e., PA_Ω^r proves*

$$[a \in P(N) \wedge 0 \,\epsilon\, a \wedge (\forall x \in N)(x \,\epsilon\, a \rightarrow x' \,\epsilon\, a) \rightarrow (\forall x \in N)(x \,\epsilon\, a)]^*.$$

Proof. We work in PA^r_Ω and assume that the $*$ translations of $a \in P(N)$, $0 \,\epsilon\, a$ and $(\forall x \in N)(x \,\epsilon\, a \rightarrow x' \,\epsilon\, a)$ are true. Then we have

$$(\forall x)(\exists! y)\, P_A(a, x, y), \tag{1}$$

$$P_A(a, 0, 0) \wedge (\forall x)(P_A(a, x, 0) \rightarrow P_A(a, x', 0)). \tag{2}$$

By Lemma 12 we obtain from (1) that there exists an ordinal α so that we have

$$P_A^{<\alpha}(a, 0, 0) \wedge (\forall x)(P_A^{<\alpha}(a, x, 0) \rightarrow P_A^{<\alpha}(a, x', 0)). \tag{3}$$

By Δ_0^Ω induction on the natural numbers this gives $(\forall x)(P_A^\alpha(a, x, 0))$, and we obtain $(\forall x \in N)(x \,\epsilon\, a)^*$. $\square$

The treatment of $(\mathit{Fmla}\text{-}\mathit{IND}_N)$ in PA^w_Ω is much simpler, since the $*$ translation of each instance of $(\mathit{Fmla}\text{-}\mathit{IND}_N)$ is an instance of $(L_\Omega\text{-}\mathit{IND}_N)$ and thus an axiom of PA^w_Ω.

Lemma 22. *The $*$ translation of each instance of $(\mathit{Fmla}\text{-}\mathit{IND}_N)$ is provable in PA^w_Ω; i.e., PA^w_Ω proves*

$$[\varphi(0) \wedge (\forall x \in N)(\varphi(x) \rightarrow \varphi(x')) \rightarrow (\forall x \in N)\, \varphi(x)]^*$$

for all formulas φ of L_p.

The reductions of $BON(\mu) + (\mathit{Set}\text{-}\mathit{IND}_N)$ to PA^r_Ω and $BON(\mu) + (\mathit{Fmla}\text{-}\mathit{IND}_N)$ to PA^w_Ω are thus now established by combining Lemmas 20, 21 and 22.

Theorem 23. *We have for every L_p formula φ:*
1. $BON(\mu) + (\mathit{Set}\text{-}\mathit{IND}_N) \vdash \varphi \;\Rightarrow\; PA^r_\Omega \vdash \varphi^*$,
2. $BON(\mu) + (\mathit{Fmla}\text{-}\mathit{IND}_N) \vdash \varphi \;\Rightarrow\; PA^w_\Omega \vdash \varphi^*$.

Now all results are available in order to present the proof-theoretic characterization of $BON(\mu)$ plus set and formula induction. Besides the previous theorem we only need Theorems 8, 10 and 14 and the result due to Aczel (cf. [8]) concerning the strength of $\widehat{ID}_1$.

Corollary 24. *We have*:
1. $BON(\mu) + (\mathit{Set}\text{-}\mathit{IND}_N) \equiv PA^r_\Omega \equiv PA$.
2. $BON(\mu) + (\mathit{Fmla}\text{-}\mathit{IND}_N) \equiv PA^w_\Omega \equiv \widehat{ID}_1 \equiv (\Pi^0_\infty\text{-}CA)_{<\varepsilon_0}$.

Appendix

In this appendix we give a proof of Theorem 9. For this we take up the notations of Subsection 6.2 again and assume that $<$ is a primitive recursive standard well-ordering of order type ε_0 with least element 0 and field $\mathbb{N}$, that n is an arbitrary natural number and that $\chi(X, y)$ is an arithmetic formula with at most X, y free.

Remember that the χ-jump hierarchy along $\prec_n$ starting with a set X of natural numbers, is defined by the following transfinite recursion:

$$(Y)_0 := X,$$

$$(Y)_i := \{\langle m, j\rangle : j \prec i \wedge \chi((Y)_j, m)\}$$

for all $0 \prec i \prec n$. It is our aim to show that, provably in the theory $BON(\mu) + (Fmla\text{-}IND_N)$, there exists an L_p term f with the following property: If $x \in P(N)$ codes the set of natural numbers X and if $y \in N$, then fxy codes the set Y_y.

To achieve this, we first observe that by Lemma 7 there exist L_p terms s and t so that $BON(\mu)$ proves

$$(\forall x, y \in N)[sxy = 0 \vee sxy = 1], \tag{1}$$

$$(\forall x, y \in N)[(x = \langle (x)_0, (x)_1\rangle \wedge (x)_1 \prec y) \leftrightarrow sxy = 0], \tag{2}$$

$$(\forall x \in P(N))(\forall y \in N)[txy = 0 \vee txy = 1], \tag{3}$$

$$(\forall x \in P(N))(\forall y \in N)[\chi^N(x, y) \leftrightarrow txy = 0]. \tag{4}$$

Making use of these terms s and t we thus look for a term f which satisfies the following equation for all $x \in P(N)$ and $y, z \in N$:

$$fxyz \simeq \begin{cases} xz, & \text{if } y = 0, \\ t(fx(z)_1)(z)_0, & \text{if } szy = 0, \\ 1, & \text{otherwise.} \end{cases}$$

This equation may be considered as a definition of the codes fxy with parameter x which is recursive in y. Such an L_p term f can be defined in $BON(\mu)$ by means of the recursion theorem:

$$f := \lambda x.\mathbf{r}_{rec}(\lambda g.\lambda y.\mathbf{d}_N y0x(\lambda z.\mathbf{d}_N(szy)0(\lambda v.(t(g(v)_1)(v)_0))(\lambda v.1)z)).$$

Then it is a matter of routine to check that the following properties of f can be proved in $BON(\mu)$:

$$x \in P(N) \rightarrow fx0 = x, \tag{5}$$

$$x \in P(N) \wedge y \in N \wedge z \in N \wedge szy = 0 \rightarrow fxyz \simeq t(fx(z)_1)(x)_0, \tag{6}$$

$$x \in P(N) \wedge y \in N \wedge z \in N \wedge 0 \prec y \wedge szy = 1 \rightarrow fxyz \simeq 1. \tag{7}$$

It remains to show that the objects fxy code sets of natural numbers, i.e., belong to $P(N)$, for all $x \in P(N)$ and y in the field of $\prec_n$. To this end let m be a successor of n in the well-ordering $\prec$. As the order type of $\prec_m$ is less than ε_0, we know from standard proof theory that

$$BON(\mu) + (Fmla\text{-}IND_N) \vdash TI(\prec_m, \varphi) \tag{8}$$

for all L_p formulas φ. By the properties of f mentioned above, we obtain therefore by straightforward induction along $\prec_m$ that $BON(\mu) + (Fmla\text{-}IND_n)$ proves:

$$x \in P(N) \wedge y \in N \wedge y \prec m \rightarrow fxy \in P(N). \tag{9}$$

From (2), (3) and (6) we can also conclude that $BON(\mu) + (Fmla\text{-}IND_N)$ proves for all $x \in P(N)$ and $y, z \in N$ with $0 < y < m$ that

$$z \in fxy \leftrightarrow (z = \langle (z)_0, (z)_1 \rangle \wedge (z)_1 < y \wedge \chi^N(fx(z)_1, (z)_0)). \tag{10}$$

If we now set $h := \lambda x.fxn$, then it follows immediately from what has been shown above that h is an L_p term which satisfies Theorem 9. This finishes our proof.

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