Conceptions of the Continuum

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Abstract: A number of conceptions of the continuum are examined from the perspective of conceptual structuralism, a view of the nature of mathematics according to which mathematics emerges from humanly constructed, intersubjectively established, basic structural conceptions. This puts into question the idea from current set theory that the continuum is somehow a uniquely determined concept.

Key words: the continuum, structuralism, conceptual structuralism, basic structural conceptions, Euclidean geometry, Hilbertian geometry, the real number system, set-theoretical conceptions, phenomenological conceptions, foundational conceptions, physical conceptions.

1. What is the continuum? On the face of it, there are several distinct forms of the continuum as a mathematical concept: in geometry, as a straight line, in analysis as the real number system (characterized in one of several ways), and in set theory as the power set of the natural numbers and, alternatively, as the set of all infinite sequences of zeros and ones. Since it is common to refer to *the* continuum, in what sense are these all instances of the same concept? When one speaks of the continuum in current set-theoretical terms it is implicitly understood that one is paying attention only to the cardinal number that these sets have in common. Besides ignoring the differences in structure involved, that requires, to begin with, recasting geometry in analytic terms. More centrally, it presumes acceptance of the overall set-theoretical framework. And (unless one is a formalist or pragmatist about axiomatic set theory) underlying that is a philosophically realist stance as to the nature of mathematics, exemplified by Gödel in "What is Cantor's continuum problem?" (1947). Other philosophies of mathematics have

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led to divergent conceptions of the continuum, such as that of Weyl (1918) in terms of predicative definability, and of Brouwer (1927) in terms of his intuitionistic theory of free choice sequences. My purpose here is to provide still another philosophical perspective that I call *conceptual structuralism* for which the above conceptions of the continuum from geometry, analysis and set theory are essentially different. Comparisons will also be made with some other conceptions of the continuum, including phenomenological, non-standard, predicativist, intuitionist, and physical interpretations, all of which fail to satisfy as basic structural conceptions.

This article is a companion to another one (2008), in which my main concern is to examine Cantor's Continuum Hypothesis (CH) from the point of view of conceptual structuralism. I argue there, contrary to Gödel and his successors, that CH is not a definite mathematical problem since the concepts of set and function used in its formulation are inherently vague; that is, no sharpening of these concepts is possible that does not violate what they are supposed to be about. That argument is *not* repeated here, but my general point of view *is*, as presented in the next section.

2. Conceptual structuralism. Though the idea that the primary objects of mathematics are structures rather than individuals such as points, lines, numbers, etc., became a commonplace of 20th century mathematics, systematic structuralist philosophies of mathematics—due among others to Hellman (1989), Resnik (1997), Shapiro (1997), Chihara (2004), Parsons (2008) and Isaacson (2008)—are more recent. Conceptual structuralism differs from all these. By way of background to the reexamination below of conceptions of the continuum, let me repeat ten theses that spell it out as stated in Feferman (2008).[2]

[2] A slightly different version of the following theses was first presented in a lecture that I gave for the Philosophy Department Colloquium at Columbia University on Dec. 9, 1977, under the title "Mathematics as objective subjectivity." The text of that lecture with some supplementary notes was later circulated but never published. The theses by themselves were put on the internet on Jan. 3, 1998 for the FOM (Foundations of Mathematics) list at http://www.cs.nyu.edu/pipermail/fom/1998-January/000682.html , where they gave rise to interesting follow-up discussion.

1. The basic objects of mathematical thought exist only as mental conceptions, though the source of these conceptions lies in everyday experience in manifold ways, in the processes of counting, ordering, matching, combining, separating, and locating in space and time.

2. Theoretical mathematics has its source in the recognition that these processes are independent of the materials or objects to which they are applied and that they are potentially endlessly repeatable.

3. The basic conceptions of mathematics are of certain kinds of relatively simple ideal-world pictures which are not of objects in isolation but of structures, i.e. coherently conceived groups of objects interconnected by a few simple relations and operations. They are communicated and understood prior to any axiomatics, indeed prior to any systematic logical development.

4. Some significant features of these structures are elicited directly from the world-pictures which describe them, while other features may be less certain. Mathematics needs little to get started and, once started, a little bit goes a long way.

5. Basic conceptions differ in their degree of clarity. One may speak of what is true in a given conception, but that notion of truth may be partial. Truth in full is applicable only to completely clear conceptions.

6. What is clear in a given conception is time dependent, both for the individual and historically.

7. Pure (theoretical) mathematics is a body of thought developed systematically by successive refinement and reflective expansion of basic structural conceptions.

8. The general ideas of order, succession, collection, relation, rule and operation are pre-mathematical; some implicit understanding of them is necessary to the understanding of mathematics.

9. The general idea of property is pre-logical; some implicit understanding of that and of the logical particles is also a prerequisite to the understanding of mathematics. The

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The comparison with various views of "social constructivism" deserves elaboration, not provided here; the essential difference lies in the central, explicit role that I give to structural conceptions.

reasoning of mathematics is in principle logical, but in practice relies to a considerable extent on various forms of intuition in order to arrive at understanding and conviction.

10. The objectivity of mathematics lies in its stability and coherence under repeated communication, critical scrutiny and expansion by many individuals often working independently of each other. Incoherent concepts, or ones which fail to withstand critical examination or lead to conflicting conclusions are eventually filtered out from mathematics. The objectivity of mathematics is a special case of intersubjective objectivity that is ubiquitous in social reality.

By way of reinforcement of this last point, many examples of intersubjective objectivity in the constructions of social reality are given in the book by Searle (1995).[3] As he says,

> [T]here are portions of the real world, objective facts in the world, that are only facts by human agreement. In a sense there are things that exist only because we believe them to exist. ... things like money, property, governments, and marriages. Yet many facts regarding these things are 'objective' facts in the sense that they are not a matter of [our] preferences, evaluations, or moral attitudes. (Searle 1995, p.1)

He might well have added board games to the list of things that exist only because we believe them to exist, and facts such as that in the game of chess, it is not possible to force a checkmate with a king and two knights against a lone king.[4] Unlike facts about one's government, citizenship, finances, property, marital relations, and so on that are vitally important to our daily welfare since they constrain one's actions and determine one's "rights", "responsibilities" and "obligations", facts about the structure and execution of athletic games and board games are not essential to our well-being even though they may engage us passionately. In this respect, mathematics is akin to games; the fact that there are infinitely many prime numbers is an example of a fact that is about

[3] Searle prefers the term 'collective intentionality' to 'intersubjectivity', which has a phenomenological history. Since my emphasis is on firmly shared conceptions, understanding and knowledge, I prefer 'intersubjective objectivity' at the risk of that association.

[4] http://en.wikipedia.org/wiki/Two_knights_endgame .

our conception of the integers, a conception that is as clear as what we mean, for example, by the game of chess or the game of go.

The nature of language leads us to talk of social institutions, games and mathematical conceptions (among other aspects of social reality) as if they were things of some sort or other with a temporal existence, much like that of living beings. So, for example, we speak of the game of chess in its present form as having evolved from earlier similar games in southern Europe in the 15th century; moreover, like mathematical conceptions it surely did not exist in any form in, say, 1,000,000 B.C. Taken at face value, this kind of talk leads to philosophical puzzles. What kinds of things are these? And if they did not exist at a certain time, how is it possible to ascribe timeless truths to them, such as the facts about prime numbers and end games in chess just cited? Must one say that it was *not* true in 1,000,000 B.C. that there are infinitely many primes, because the conception of the structure of integers and its derived concept of prime numbers did not exist then? Of course not. But it doesn't help to posit a kind of Popperian existence to a world of products of the human mind (another "thing") to give the facts about chess and prime numbers timeless validity, because that is just another form of platonism. It is simply a demonstrated fact, for example, that *in the game of chess*, it is not possible to force a checkmate against a lone king by means of a king and two knights. And though the existence of mathematical conceptions is time dependent both for the human race and for individual humans, it doesn't make sense to say that for those humans who do not grasp the concept of prime numbers, the infinity of primes is false; one can only say that that is a demonstrated truth about the standard conception of the prime numbers that must surely be accepted by whoever does grasp that concept and the usual proof.

Conceptual structuralism is illustrated in (Feferman 2008) by discussion of two constellations of structural notions, first of the positive integer sequence and second of the power set conception of the continuum. For comparison with what I say below about the latter and other conceptions of the continuum, let me review briefly what I have to say there about the former.

The most primitive mathematical conception is that of the positive integer sequence as represented by the tallies: |, ||, |||, ... From the structural point of view, our conception is that of a structure (N^+, 1, Sc, <), where N^+ is generated from the initial unit 1 by closure under the successor operation Sc, and for which $m < n$ if m precedes n in the generation procedure. Certain facts about this structure (if one formulates them explicitly at all), are evident: that < is a total ordering of N^+ for which 1 is the least element, and that $m < n$ implies $Sc(m) < Sc(n)$. Reflecting on a given structure may lead us to elaborate it by adjoining further relations and operations and to expand basic principles accordingly. For example, in the case of N^+, thinking of concatenation of tallies immediately leads us to the operation of addition, $m + n$, and that leads us to $m \times n$ as "m added to itself n times". The basic properties of the + and × operations such as commutativity, associativity, distributivity, and cancellation are initially recognized only implicitly. We may then go on to introduce more distinctively mathematical notions such as the relations $n|m$ and $k \equiv m \pmod{n}$, and the property "n is a prime number". In this language, a wealth of interesting mathematical statements can be formulated and investigated as to their truth or falsity, for example, that there are infinitely many twin prime numbers, that there are no odd perfect numbers, Goldbach's conjecture, and so on.

The conception of the structure (N^+, 1, Sc, <, +, ×) is so intuitively clear that (again implicitly, at least) there is no question in the minds of mathematicians as to the definite meaning of such statements and the assertion that they are true or false, independently of whether we can establish them one way or the other. In other words, in more modern terms, realism in truth values is accepted for statements about this structure, and the application of classical logic in reasoning about such statements is automatically legitimized. Despite the subjective source of the positive integer structure in the collective human understanding, it is in the domain of objective concepts and there is no reason to restrict oneself to intuitionistic logic on subjectivist grounds. Further reflection on the structure of positive integers with the aim to simplify calculations and algebraic operations and laws leads directly to its extension to the structure of natural numbers (N, 0, Sc, <, +, ×), and then the structures for the integers Z and the rational numbers Q. The latter are relatively refined conceptions, not basic ones, but we are no less clear in our dealings with them than for the basic conceptions of N^+.

At a further stage of reflection we may recognize the least number principle: if P(n) is any well-defined property of members of N and there is some n such that P(n) holds then there is a least such n. More advanced reflection leads to general principles of proof by induction and definition by recursion on N. Furthermore, the general scheme of induction,

$$P(0) \wedge \forall n[P(n) \rightarrow P(Sc(n))] \rightarrow \forall n\, P(n),$$

is taken to be open-ended in the sense that it is accepted for any definite property P of natural numbers that one meets in the process of doing mathematics, no matter what the subject matter and what the notions used in the formulation of P. The question—What is a definite property?—requires in each instance the mathematician's judgment. For example, the property,"n is an odd perfect number", is a definite property, while "n is a feasibly representable number" is not, nor is "n is the number of grains of sand in a heap".

3. Six conceptions of the continuum. These are: (i) the Euclidean continuum, (ii) Cantor's continuum, (iii) Dedekind's continuum, (iv) the Hilbertian continuum, (v) the set of all paths in the full binary tree, and (vi) the set of all subsets of the natural numbers. By (ii) and (iii) I mean the two familiar ways of "constructing" and characterizing the real number structure. By (iv), I mean the continuum as it appears in Hilbert's system of geometry, whose continuity property is directly informed by the Dedekind completeness condition. Let us look at these in more detail from the perspective of conceptual structuralism; neither historical accuracy nor comprehensiveness are guiding concerns.

3.1 The Euclidean continuum. In Euclidean geometry the continuum is exemplified by straight lines, indefinitely extended. However, these are not conceived of as existing in isolation but rather as parts of the structural framework of plane geometry (and by extension, spatial geometry). There, our informal intuitions are of the plane as being "perfectly flat", of lines as being "perfectly straight," and of points as being "perfectly fine." These intuitions are Janus faced. On the one hand, our language leads us to treat points and lines as objects for which the basic relation is that of a point lying on a line, viz. in such statements as, "For any two distinct points there exists a unique line on which they lie." On the other hand, our intuitions of points as being dimensionless—in Euclid's terminology, as having "no parts"—and of lines as having "no breadth", requires us to imagine entities which have no substance. An alternative to objectual parlance is to think of points as being "pure locations", but that makes talk of a specific point lying on a specific line sound odd. Despite the fact that our intuitions must do double duty, they are sufficiently clear to recognize immediately, the validity of each of Euclid's postulates other than, perhaps, the parallel postulate.

The main thing to be emphasized about the conception of the continuum as it appears in Euclidean geometry is that the general concept of set is *not* part of the basic picture, and that Dedekind style continuity considerations of the sort discussed below are at odds with that picture. It does not make sense, for example, to think of deleting a point from a line, or to remove the end point of a line segment. Given two line segments L and L′, we can form a right triangle with legs L_1 and L_1' congruent to the given segments, respectively; but these share a vertex as a common point, each an end point. Thought of as a set, L is transformed into L′ by a rigid motion, and the same for L_1 and L_1'. Thought of in that way, the vertex of the right triangle has displaced one of the end points, but which one? There are many similar thought experiments which dictate that lines, line segments and other figures in Euclidean geometry are not to be identified with their sets of points.[5] It is true that we have some talk in the Euclidean framework that can be construed in set-theoretical terms, as when we speak of a circle as the locus of points equidistant from a given point, or of conic sections as the intersection of a plane and a cone. But these are in all cases specific kinds of sets given by visualizable descriptions.

Dedekind style continuity considerations only emerged in the critical re-examination of the Euclidean development in the 19th century when it was recognized that several of its proofs assume the existence of certain points that are "supposed" to be there, but whose existence is not guaranteed by the Euclidean postulates. In fact, this occurs at the very

[5] Cf. Giaquinto (2008), p. 58 for similar considerations.

outset in proposition I.1, which asserts that an equilateral triangle can be constructed on any line segment AB as base. This is proved by constructing two circles C_1 and C_2, with centers A and B, respectively, and radius equal to AB. The vertex of the equilateral triangle to be constructed is taken as either point of intersection of C_1 and C_2. But there is nothing in the Euclidean postulates which guarantees that such points exist. From the modern point of view, the circles may be "gappy". But that way of thinking is entirely foreign to Euclidean geometry, as is the general continuity axiom formulated in Hilbertian geometry below. Some modern developments of Euclidean geometry add special continuity axioms ("line-line", "line-circle", "circle-circle") asserting the existence of points that "ought to exist", and indeed evidently exist in our basic picture, but don't follow from the Euclidean postulates. (For details, see Greenberg (2007) or Hartshorne (1997).) One can imagine that Euclid himself might have acknowledged the logical necessity of adding such specific continuity postulates in one form or another if that were pointed out to him, but one cannot say that consideration of these questions is part of the basic picture.[6]

3.2 Cantor's continuum. The following is an ancient idea, surely predating the Pythagoreans: given an arbitrarily fixed "unit line segment" U to which is ascribed the length 1, every line segment L has a number attached to it as length lh(L) relative to U. In abstract terms the physical process of measuring a line segment L is to lay off U within L as many times as possible, resulting, say, in n_0U and then to measure the difference $L - n_0U$ by means of fractions U/k of U, obtained by subdividing U into k equal parts for some integer $k \geq 2$, and then repeating the process. From this picture it is seen that lh(L) is either rational or the limit of an increasing sequence of rational numbers, $q_0 = n_0 = \mathrm{lh}(n_0U)$, $q_1 = n_0 + n_1/k = \mathrm{lh}(n_0U + n_1U/k)$, $q_2 = n_0 + n_1/k + n_2/k^2 = \mathrm{lh}(n_0U + n_1U/k + n_2U/k^2)$, … , where $0 \leq n_i < k$ for each $i > 0$. Any system of numbers including the

[6] There is a considerable literature on these matters that would take us too far afield to plumb properly. One view that is not part of the account given here of the basic structural conception in Euclidean geometry stresses the central role of constructions in Euclid's proofs; according to that, for example, the intersection of the two circles in I.1 exists because one can construct it. But that does not obviously take care of Pasch's axiom, that if a line goes through one side AB of a triangle ABC, it must intersect AC or BC (line-line continuity).

rational numbers that is to be complete for measurement in this way must be closed under the limit of increasing sequences of rational numbers that are bounded above; the two conditions together are met by closure under the limits of bounded non-decreasing sequences of rationals. Assuming the real number system R, Cauchy found a necessary and sufficient condition on arbitrary sequences (x_n) of real numbers in order for them to be convergent, namely that the $\lim_{n,m \to \infty} |x_n - x_m| = 0$ or, as we would put it since Weierstrass, that for any $k > 0$, there exists a p such that for all $n, m > p$, $|x_n - x_m| < 1/k$; these are called fundamental sequences. In particular, bounded non-decreasing sequences are fundamental. So the first idea for "construction" of the real number system R would be to take it to consist of all bounded non-decreasing sequences of rational numbers, with two such sequences (x_n) and (y_n) identified if and only if $\lim_{n\to\infty} |x_n - y_n| = 0$. But if real numbers are to be defined in such a way that we can also extend the arithmetical operations +, × and their inverses to them, it works out better to take R to consist of *all* fundamental sequences of rational numbers, identified in the same way; this construction, given in Cantor (1872) §1 and repeated in Cantor (1883) §9 is what I call Cantor's continuum. In that system, the ordering relation $x < y$ between real numbers given by fundamental sequences of rational numbers $x = (x_n)$ and $y = (y_n)$ respectively is defined to hold when there exists $k > 0$ and a p such that for all $n > p$, $(y_n - x_n) > 1/k$. The system thus constructed is an ordered field that satisfies the Cauchy criterion in general. Abstractly, R can then be characterized as the completion of the ordered field of rational numbers with respect to fundamental sequences. This is a relatively sophisticated structural concept which hybridizes basic geometric and arithmetic ideas with the general (pre-)set-theoretical idea of arbitrary sequence.

3.3 Dedekind's continuum. Here the basic intuition is quite different from that of the Cantor's continuum. Each line segment is to have a length equal to that of a segment AB where A and B are points in the continuum conceived of as a linearly ordered set going from "left" to "right", and points are conceived to be elements of that set; in addition, the choice of AB is to be arbitrary up to rigid translation. In particular, designating a point O chosen arbitrarily as the "origin", each line segment is to have the same length as that of some segment OP, where P is to the right of O. Hence if U is an arbitrarily designated unit line segment then there is to be a point I with lh(OI) = 1; by simple geometric constructions, every length commensurable with U is then represented as a rational number lh(OP) for some P. But then by the existence of incommensurable lengths such as that of the diagonal of the square whose side is equal to U, it is inadequate to take the continuum to consist just of those points P (in either direction from O) for which lh(OP) is rational. In other words, there are "gaps" in the linearly ordered rational number structure, while the continuum must have no gaps. As Dedekind puts it:

> I find the essence of continuity … in the following principle:
> 'If all points of the straight line fall into two classes such that every point of the first class lies to the left of every point of the second class, then there exists one and only one point which produces this division of all points into two classes, this severing of the straight line into two portions.' (Dedekind 1872, translation in Ewald 1996 p. 771)

The more precise formulation uses the notion of a *cut* (X_1, X_2) in a linearly ordered set $(M, <)$, which holds when both X_1 and X_2 are non-empty subsets of M whose union is M and is such that every member of X_1 precedes every member of X_2. Dedekind's continuity requirement on a line C is that given any cut (X_1, X_2) in C, either X_1 has a largest element or X_2 has a least element. Then the real numbers are explained to be a linearly ordered set $(R, <)$ which contains the rationals Q as a densely ordered subset and which satisfies this continuity requirement. One argues that any two such ordered sets $(R, <)$ and $(R^*, <^*)$ are isomorphic, by mapping each element p of R to the element p* in R* having the same left section in the rationals, i.e. $\{q \in Q : q < p\} = \{q \in Q: q <^* p^*\}$.[7] (One passes from the cut in R determined by p to its restriction to Q and then back to the natural extension of the latter to a cut in R* that determines p*). Finally, Dedekind's construction of such an $(R, <)$ is obtained by taking it to consist of the rational numbers together with the numbers corresponding to all those cuts (X_1, X_2) in Q for which X_1 has no largest element and X_2 has no least element, ordered in correspondence to the ordering

[7] Dedekind did not himself explicitly state the uniqueness up to isomorphism of this structural concept, though it seems that he took it implicitly for granted.

of cuts $(X_1, X_2) < (Y_1, Y_2)$ when X_1 is a proper subset of Y_1. Dedekind himself spoke of this construction of R at individual cuts in Q for which X_1 has no largest element and X_2 no least element as *the creation of an irrational number* (ibid., p. 773), though he did not identify the numbers themselves with those cuts.[8]

To sum up, from the point of view of conceptual structuralism and just as with Cantor's construction, Dedekind's construction and characterization of the continuum is not a basic conception but a hybrid of geometric, arithmetic and set-theoretical notions.

3.4 The Hilbertian continuum. By this is meant the consequences for the straight line of the Axioms of Continuity (Axiom Group V) V.1 and V.2 in Hilbert's *Grundlagen der Geometrie*, whose first edition in 1899 underwent a number of revisions in his lifetime. Reference here is to the Tenth Edition as translated into English in Hilbert (1988).[9]

V.1 (Axiom of measure or Archimedes' Axiom). *If AB and CD are any segments then there exists a number n such that n segments CD constructed contiguously from A, along the ray from A through B, will pass beyond the point B.*
V.2 (Axiom of line completeness). *An extension of a set of points on a line with its order and congruence relations that would preserve the relations existing among the original elements as well as the fundamental properties of line order and congruence that follows from Axioms I-III, and from V1 is impossible.*[10] (Hilbert 1988, p. 26)

[8] Incidentally, Cantor (1883) referred to Dedekind's construction as "eigenartige", or "idiosyncratic" as Ewald (1996) p. 897 translates it. In Cantor's further discussion in comparison with other constructions of the reals, including his own, he says that Dedekind's has the "undeniable advantage … that every number b corresponds to only a *single* cut. But this definition has the great disadvantage that the numbers of analysis *never* occur as 'cuts', but must be brought into this form with a great deal of artificiality and effort." (Ibid., p. 899)
[9] Seven editions of the *Grundlagen der Geometrie* appeared in Hilbert's lifetime. The editions 8-10 (the last in 1968) were edited by Paul Bernays and, as stated in his Prefaces, only minor additions and corrections were made there to the body of the text of the Seventh Edition; he did, however, contribute a number of supplements to the latter.
[10] What is meant by the properties in question is elaborated in Hilbert (1988) p. 26. Incidentally, the Axioms of Continuity were only introduced in the 2nd edition of the *Grundlagen.*

From the modern logical point of view, both of these axioms are non-elementary, since the first requires quantifying over the natural numbers and the second over arbitrary sets of points, taken both syntactically and semantically. The Archimedean axiom insures that if one imposes a system of measurement on any ordered line L relative to some unit length and origin O, the points with rational distance from O are dense in L. The completeness axiom is equivalent to the Dedekind continuity condition, that there are no gaps in the line.

As with the Euclidean line, the Hilbertian continuum is not conceived of independently but only within the framework of plane (and spatial) geometry as a whole. Thus, from the point of view of conceptual structuralism, the Hilbertian conception of the continuum is not a basic one but rather is a hybrid of geometrical and set-theoretical notions. The second-order logic required for Hilbert's group V axioms can be eliminated in favor of a first-order axiomatization of geometry, as was done in Tarski's system; cf. Tarski and Givant (1999). However, the price that is paid thereby is that the continuity axiom is replaced by an axiom scheme, and the system, while formally complete, is not categorical. In any case, consideration of such a system is based on primarily logical grounds and only secondarily on conceptual grounds.

3.5 The set of all paths in the full binary tree. The conception of the generation of the full binary tree T as obtained from an initial node by successive branching to the left or right is practically as basic and clear as that of the natural number sequence. Denoting 'left' by 0 and 'right' by 1, we recognize this as equivalent to the generation of all finite sequences of zeros and ones, beginning with the empty sequence. A path in the full binary tree is a subset of T which contains the initial node and contains with each node exactly one of its successors; a path is thus represented by an infinite sequence of zeros and ones, and every such sequence represents a unique path. One standard set-theoretical conception of the continuum is as the set of *all* paths in T, or equivalently as the set 2^N of all functions from N into the set {0, 1}. However, intuitively, it is natural to think of sequences (x_n) of natural numbers as being generated by some sort of rule which tells us how, at each n, x_n is to be evaluated given the values of x_i for all $i < n$, or more generally simply as a rule which provides the value of x_n for each n. On the other hand, the idea of a function from N into {0, 1} is based on a quite different intuition, namely as an arbitrary many-one correspondence considered extensionally, i.e. independently of how the correspondence may be effected, while rules are thought of intensionally, i.e. as specific procedures to carry out the requisite evaluations. Any such rule determines a function, but not conversely. On either view, the conception is of a two-sorted structure $(N, 2^N, Val)$ where $Val(n, x) = x_n$ for x in 2^N.

Reflection on the function explanation of 2^N leads one to accept the principle that for each definite correspondence (i.e., relation) P(n, m) between natural numbers such that $\forall n \exists! m\, P(n,m)$ there exists a function $F: N \to \{0, 1\}$ such that for every n, m,

$$F(n) = m \leftrightarrow P(n, m).$$

As with the natural numbers, this leaves open the question which P are definite. Clearly, the relation P(n, m) between arbitrary $n \in N$ and $m = 0, 1$ for which ($m = 1 \leftrightarrow$ n is prime) is definite, while that for which ($m = 1 \leftrightarrow$ n is the number of grains of sand in a heap) is not. But what about the relation, ($m = 1 \leftrightarrow$ the GCH holds at n)?

Appealing as the idea is of an arbitrary path through the binary tree, or an arbitrary sequence of 0s and 1s, the problem with this set-theoretical conception of the continuum is grasping the meaning of 'all' in the description of 2^N as consisting of *all* such sequences. If that meaning is taken to be well-determined, then any correspondence P expressed using quantification over 2^N is definite, i.e. 2^N is a definite totality. But on the face of it that requires a platonist ontology, according to which the totality in question somehow exists independently of human conceptions; in that respect it goes beyond conceptual structuralism.[11] This is not to deny that we can *imagine* the ideal world in

[11] The platonism in question pertains only to the existence of the particular set 2^N as a definite totality independent of human conceptions; similarly for the view of the set of all subsets of N as a definite totality, treated next. Following Bernays (1935) one can be a "moderate" platonist via the assumption of only these specific totalities, as opposed to being an unrestricted or "absolute" platonist as to the nature of all mathematical objects.

which 2^N is a definite totality just as we can imagine our physical universe being totally bounded in space and in time. Since the consequent issues are essentially the same as for the concept of the set of all subsets of N, I delay taking them up until the end of the next section.

3.6 The set of all subsets of the natural numbers. Given the idea of an arbitrary set X of elements of a given set A, considered independently of how membership in X may be defined, the purest conception of the continuum in the set-theoretical sense is that of the set of all subsets of N, S(N).[12] This is a two-sorted structure, (N, S(N), $\in$), where $\in$ is the relation of membership of natural numbers to sets of natural numbers. Two principles are evident for this conception, using letters 'X', 'Y' to range over S(N) and 'n' to range over N.

I. *Extensionality* $\forall X \forall Y[\ \forall n(n \in X \leftrightarrow n \in Y) \rightarrow X = Y\ ]$

II. *Comprehension* For any definite property P(n) of members of N,

$$\exists X[\ \forall n(\ n \in X \leftrightarrow\ P(n)\ ]\ .$$

Again what is problematic here for conceptual structuralism is the meaning of 'all' in the description of S(N) as the set of *all* subsets of N. In the usual set-theoretical view, S(N) is a definite totality, so that quantification over it is well-determined and may be used to express definite properties P. But again that requires on the face of it a platonist ontology and in that respect goes beyond conceptual structuralism (cf. footnote 11). The usual set-theoretical view of functions as many-one sets of ordered pairs puts 2^N in one-one correspondence with S(N), by associating with each function F from N to {0, 1) the set X of all n such that F(n) = 1; inversely, each member X of S(N) determines the function F given by F(n) = 1 $\leftrightarrow$ n $\in$ X. But the pre-set-theoretical conceptions of sequence and set distinguish these: sequences are given by rules of generation and sets by membership conditions, so that every sequence determines a set, but not conversely. In a sequence we step from each term to the next, but a set is just a scattering of natural numbers

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But Bernays does not try to mark out what such a moderate platonism ought to admit with respect to how far such operations on sets may be iterated, if at all.

[12] S(N) is commonly referred to as the power set of N, denoted $\mathcal{P}$(N), though that terminology is more appropriately applied to 2^N.

considered without regard to order. The correspondence does work for the extensional conception of sequences as functions, for which the order in which elements are generated is irrelevant, and that serves more generally to put S(A) in one-one correspondence with 2^A for any set A.

As with 2^N in the preceding section, we can certainly conceive of a world in which S(N) is a definite totality and quantification over it is well-determined; in that ideal world, one may take for the property P in the above Comprehension Principle any formula of full second-order logic over the language of arithmetic. Then a number of theorems can be drawn as consequences in the corresponding system PA^2, including purely arithmetical theorems. Since the truth definition for arithmetic can be defined in PA^2 and transfinite induction can be proved in it for very large recursive well-orderings, PA^2 goes in strength far beyond PA even when that is enlarged by the successive adjunction of consistency statements transfinitely iterated over such well-orderings. What confidence are we to have in the resulting purely arithmetical theorems? There is hardly any reason to doubt the consistency of PA^2 itself, even though by Gödel's theorem, we cannot prove it by means of any weaker means. Indeed, the ideal world picture of $(N, S(N), \in)$ that we have been countenancing would surely lead us to say more, since in it the natural numbers are taken in their standard comception. On this account, any arithmetical statement that we can prove in PA^2 ought simply to be accepted as true. But given that the assumption of S(N) as a definite totality is a purely hypothetical and philosophically problematic one, the best we can rightly say is that *in that picture*, everything proved of the natural numbers is true.

All of this and more comes into question when we move one type level up to the structure $(N, S(N), S(S(N)), \in_1, \in_2)$ in which Cantor's continuum hypothesis may be formulated. A more extensive discussion of the conception of *that* structure and the question of its definiteness in connection with the continuum problem is given in Feferman (2008).

4. Non-structural and other conceptions of the continuum. Several other conceptions of the continuum have been suggested for comparison with the preceding: (i) phenomenological, (ii) non-standard, (iii) foundational (intuitionistic and predicative), and, finally, (iv) physical. The interesting motivations for each of these aside, they do not stand as basic structural conceptions.

4.1 Phenomenological conceptions. I shall refer here to only one attempt, that by Weyl (1918)—whose predicative treatment of the mathematical continuum is taken up below—to elaborate an intuitive conception of the continuum.[13] This is carried out in the two final sections of his monograph; quotations are taken from the English translation, Weyl (1987). In the natural effort to somehow firmly root the mathematical continuum in experience, Weyl was strongly influenced by Husserl's phenomenology (1913), to which he refers (ibid., p.2), following which he writes that "[o]ur examination of the continuum problem contributes to critical epistemology's investigation into the relations between what is immediately (intuitively) given and the formal (mathematical) concepts through which we seek to construct the given in geometry and physics."

Weyl's point of departure is the following:

> In order to better understand the relation between an intuitively given continuum and the concept of number … let us stick to *time* as the most fundamental continuum. And in order to remain entirely within the domain of the immediately given, let us adhere to *phenomenal* time …, i.e. to that constant form of my experiences of consciousness by virtue of which they appear to me to flow by successively. … In order to have some hope of connecting phenomenal time with the world of mathematical concepts, let us grant the ideal possibility that a rigidly punctual "now" can be placed within this species of time and that time-points can be exhibited. Given any two distinct time-points, one is the *earlier*, the other the *later*.[14] (Ibid., p. 88, italics Weyl's)

[13] For a fuller account of Weyl's thoughts and efforts in the phenomenological direction, see Bell (2000).

[14] The intuitionist Brouwer also took time as the source of the concept of the continuum in his inaugural address (1912), where he talks of the "intuition of two-oneness, the basal intuition of mathematics" derived from "the falling apart of moments of life into qualitatively different parts, to be reunited only while separated by time, as the

In addition, Weyl posits the relation of equality between time intervals A_1B_1 and A_2B_2 where the A_i and B_i are points in time. But almost immediately he abandons this approach as *nonsense* (p. 90, italics his), because of the difficulty of ascribing precise meaning to the experience of the "Now" as a point in time. He then shifts (pp. 97ff) to a proto-mathematical theory taking *time spans* a, b,… with the relations of equality a = b and addition a + b =c as basic, from which the conception of number is to be obtained as ratios of time spans. In the end, though, Weyl is deeply dissatisfied with these efforts and concludes by writing:

> To the criticism that the intuition of the continuum in no way contains those logical principles on which we must rely for the exact definition of the concept "real number," we respond that the conceptual world of mathematics is so foreign to what the intuitive continuum presents to us that the demand for coincidence between the two must be dismissed as absurd. Nevertheless, those abstract schemata which supply us with mathematics must also underlie the exact science of domains of objects in which continua play a role. (Ibid., p. 108)

For a (relatively) recent stimulating comparison of Weyl's venture with some other intuitive conceptions as well as mathematical constructions, see Longo (1999).[15]

4.2 The continuum in infinitesimal analysis. Extensive reasoning with infinitesimal quantities was ubiquitous in the early development of mathematical analysis and its applications, but was subject to puzzles and outright contradictions. Eventually eliminated in the 19th century through the systematic use of the limit concept introduced by Cauchy and made precise by Weierstrass, infinitesimals fell by the wayside in proofs, though they retain their value as a heuristic to this day. Robinson (1966) discovered a

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fundamental phenomenon of the human intellect" [translation from Benacerraf and Putnam (1983) p. 80]. This leads Brouwer to treat the continuum not linearly, as in Weyl's attempt here, but rather in terms of sequences in the binary branching tree; cf. sec. 4.3 below.

[15] See also the essays by Bouveresse, Petitot and Thom in Salanskis and Sinaceur (1993).

way to provide a rigorous foundation for certain uses of infinitesimals via a model-theoretic construction of a proper extension *R of the real number system, in which the infinitesimal quantities are those elements whose absolute values lie between 0 and all the positive elements of R. Compared to Cantor's and Dedekind's construction of R, Robinson's construction of such *R is relatively sophisticated, but in its use of model theory, relatively simple for that subject. In addition, from the set-theoretical point of view, it is not categorically determined—there are many *R, any one of which serves the purposes of non-standard analysis. There is no question that one can develop good working intuitions about the use of such *R as in other parts of mathematics; witness its many applications subsequent to Robinson's work.[16] But that does not qualify it as a basic structural conception; the use of the appellation 'non-standard' is an implicit recognition of its status.

A quite different approach to infinitesimal analysis using a "nonpunctiform", "smooth" modification R_{sm} of the real number system was initiated by Lawvere in lectures in 1967 (cf. Lawvere 1979) and later pursued systematically by Kock (1981) among others; Bell (2008) provides an elegant and intuitively motivated exposition. The infinitesimals in this version form a non-trivial set of quantities ε in R_{sm} whose square is 0; the logic is necessarily intuitionistic, since it can be proved for every such ε that it is not the case that $(\varepsilon = 0)$ or $\neg(\varepsilon = 0)$. Every point on a curve C has an infinitesimal neighborhood along which C is straight; if f is a function on R_{sm} then for each x, the slope of that straight line to its curve at f(x) is the value of the derivative of f at x, and f itself is smooth, i.e. infinitely differentiable. There is an elegant axiomatization of the "smooth world" containing R_{sm} but a proof of its consistency is quite sophisticated—via topos theory—beginning with the category of smooth manifolds (itself defined in terms of our "standard" R); in that respect it certainly does not qualify as a basic structural conception.[17] As with non-standard analysis, smooth infinitesimal analysis has direct applications to differential geometry and physics in the cited sources.

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[16] Cf., for example Cutland (1988). For a recent exposition, see Robert (2003).
[17] Cf. the appendix to Bell (2008) for references to the work producing a model for the smooth world.

4.3 Two foundational conceptions: intuitionism and predicativism. These are not structural conceptions to begin with but rather developments of the continuum within two foundational programs. Both take the intuitive conception of the natural number structure and the principle of proof by induction as basic. The difference is that the predicativists (at least those stemming from Poincaré and Weyl) view the natural numbers as a definite totality, for which quantification over N is thus definite and classical logic is admissible for all arithmetical statements, while the intuitionists view N as a potential totality and reject the law of excluded middle as applied to such statements. For the predicativists, sets which can be explicitly enumerated, such as the rational numbers, are also definite totalities. Both the intuitionists and the predicativists reject the assumption of any completed infinite totalities of uncountable cardinality, and in particular the set-theoretical conceptions of 2^N and S(N) as definite totalities.

The intuitionists and the predicativists both follow Cantor in the construction of the real numbers, namely as fundamental sequences of rational numbers; equality of real numbers is also defined the way Cantor did it. In order to carry this out, in each framework one must presume the general notion of sequence of rational numbers, though not the totality of such sequences; thus there is no totality of real numbers, only the concept of what it means to be a real number. Moreover, sequences must be regarded intensionally, not as functions determined by their values. For the predicativists, one has the usual properties of sequences, among them closure under relative arithmetical definability. However, for the development of the theory of real numbers, the intuitionists generally follow Brouwer (1927) in interpreting them as (free) choice sequences, of which one has only a finite amount of information at any given time. Thus, for the intuitionists, any total operation on choice sequences to the natural numbers is continuous, and that leads one to Brouwer's conclusion that every function of real numbers on a closed interval is continuous. The consequent development of analysis based on the theory of choice sequences is thus *prima facie* in conflict with classical analysis, though if one understands the meaning of the terms in the way intended by Brouwer and his followers, there is no direct conflict, only a radically different development.[18],[19]

By contrast, the predicative development leads to a part of classical analysis and is nowhere in conflict with it. The initial redevelopment of analysis on strictly predicative grounds was due to Hermann Weyl in his monograph *Das Kontinuum* (1918), in which he showed that all of the classical analysis of pointwise continuous functions can be carried out predicatively.[20] As explained in Feferman (1988), Weyl's work can be formalized in a 2nd order system, usually denoted ACA_0, which is a conservative extension of the first order system PA of Peano Arithmetic. The second order variables range over sets of natural numbers, for which the comprehension principle simply asserts closure under relative arithmetical definability, while the induction principle says that every set which contains 0 and is closed under the successor operation contains all natural numbers. Sequences of natural numbers and thence of rational numbers are then treated as sets of pairs. A system W (so designated in honor of Weyl) was introduced in Feferman (1988) to allow more flexible use of various notions of finite type, including the type R of real numbers and the type $R \rightarrow R$ of functions of real numbers. In W, the abstract notion of rule is taken as basic, and sequences are rules which are total on N. It was shown by Feferman and Jäger (1996) that W, too, is conservative over PA.

Though predicativity in principle goes far beyond PA in strength, it turns out in practice that the part of mathematics that can be justified on predicative grounds is *robustly predicative* in the sense that if a result can be obtained on those grounds at all, it can already be established in a system conservative over PA such as ACA_0 or W. This has been verified through a number of case studies: for ACA_0 they have been carried out in the Reverse Mathematics program introduced by Friedman and extended and exposited in detail by Simpson (1999), while for W they have been carried out in my unpublished

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[18] For an introduction to the theory of choice sequences and the intuitionistic development of analysis on their basis, see Troelstra and van Dalen (1988), Ch. 12.
[19] For the differing understanding of the terms involved, see Hellman (1989a).
[20] A few years later, Weyl fell under Brouwer's spell and made his own contributions to the intuitionistic approach. But years later he became quite pessimistic about its prospects; cf. Part II of Mancosu (1998) for a full description of Weyl's intuitionistic excursion with relevant articles, and Feferman (2005) p. 601.

notes as described in Feferman (1988, 1993). Roughly speaking, among other things the robustly predicative part of mathematics includes the Lebesgue theory of measurable functions and integration and various parts of standard functional analysis, such as the spectral theory of bounded and unbounded operators on a separable Hilbert space.[21] These case studies led me to the following working hypothesis in Feferman (1988): *all of scientifically applicable mathematics can be developed in the system W*. Some possible counterexamples (in the use, for example, of non-separable spaces or non-measurable sets) are discussed in the Postscript to the reprinting of that article in Feferman (1998), pp. 281-283, but so far all such seem to be very much at the margin. The significance of this hypothesis for the physical conception of the continuum is taken up next.

4.4 The continuum in natural science. The indispensability of mathematics to the practice of natural science is indisputable, but the reasons for that have been the subject of endless philosophical and semi-philosophical discussion (cf., e.g. Steiner (1998, 2005), Mickens (1990), Colyvan (2001) and Livio (2009)).[22] One view, as encapsulated in the famous quotes of Galileo, "Nature's great book is written in the language of mathematics," and Jeans, "God is a mathematician," is that mathematics is integrally involved in nature itself. In particular, since the real number structure and its many superstructures are at the core of the mathematics that has proved to be indispensable, one may read this view as asserting that one version or other of the continuum is part of the natural order. In *Science Without Numbers* (1980), Field's ambitious attempt to show that a purely nominalistic account can be given of Newtonian gravitational theory, the assumption is that Hilbertian geometry in its second-order formulation (via quantification over both space-time points and arbitrary space-time regions) holds of the space-time structure and thus that *numbers* need not be assumed for that part of natural science.[23]

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[21] Subsequent to that work, it has been shown by Ye (2000) that much of that work can already be carried out in a constructive subsystem of W, conservative over the much weaker system PRA of primitive recursive arithmetic.

[22] I shall not concern myself here with the particular side of this discussion dealing with the Quine-Putnam indispensability argument; for that, see the Steiner and Colyvan references as well as Resnik (2005), Maddy (1992) and Feferman (1993).

[23] Field hoped to extend that in some way to relativistic gravitational theory, but as far as I know, that has not been achieved. Moreover, as explained in the Malament (1982)

But, as we have seen, the Dedekind conception of the continuum is taken as basic for the characteristic properties of straight lines in Hilbert's geometry, and the arithmetical continuum can be constructed from that as usual. In addition, this assumes that the Hilbert or Dedekind continuum is somehow embodied in the physical world.[24]

Given the patent fact that the subject matter of mathematics is of an essentially different kind from the subject matter of natural science, and that mathematical knowledge is in some sense *a priori* as opposed to the *a posteriori* character of scientific knowledge, the great difficulty—in the words of Wigner (1960)—lies in explaining the unreasonable effectiveness of mathematics in the natural sciences. But as Penelope Maddy observed, the problem goes still deeper, namely to account for the unreasonable effectiveness of highly idealized physical models which are "literally false", for example in "the analysis of water waves by assuming the water to be infinitely deep or the treatment of matter as continuous in fluid dynamics or the treatment of energy as a continuously varying quantity" (Maddy 1992 p.281). One could add to such problematic examples the assumption of point masses in Newtonian mechanics, and the derivation of planetary motions around the sun ignoring all but one planet.[25]

Whichever view of the relation of mathematics to nature one takes, there is no independent physical conception of the continuum on offer in all this, since all the mathematics is filtered through the real number system (or Hilbertian geometry as a surrogate). Moreover, I don't see that any argument can be made from the enormously successful applications of mathematics in natural science to the conclusion that one or another of the mathematical conceptions of the continuum surveyed above is uniquely

review of *Science Without Numbers,* quantum mechanics poses serious obstacles to Field's program. In addition, Geoffrey Hellman has pointed out to me that the program runs into severe difficulties when applied more generally to state-space theories, including Newtonian mechanics of many particles and statistical mechanics.

[24] Incidentally, Dedekind (1872) himself warned against that assumption: "If space has a real existence at all it is not necessary for it to be continuous." (Ewald (1996) p. 772)

[25] Another aspect of "how the laws of physics lie," namely in the use of *ceteris paribus* assumptions, has been dealt with by Cartwright (1983).

singled out as the "real one". In any case, the work on the reach of predicative mathematics cited at the end of the preceding section shows that the properties of the continuum needed for its applications in natural science do not require it to have a definite reality in the platonistic sense.

5. Conclusion. Of all the conceptions of the continuum considered here, only those of sec. 3 stand as structural ones, and of those, only 2^N and S(N) stand as *basic* structural conceptions. For, the continuum in Euclidean and Hilbertian geometry is not an isolated notion, while the continuum as given by Cantor's and Dedekind's construction of the real numbers are hybrid conceptions. The set 2^N of all sequences of 0s and 1s isolates the set-theoretical component of Cantor's construction, while the set S(N) of all subsets of N isolates that of Dedekind's construction, but both of these lose entirely the basic geometric intuition of the continuum. On the other hand, it does not count against Cantor's and Dedekind's conceptions of the continuum in the form of the real number system R that they are hybrids of geometrical, arithmetical and set-theoretical notions. On the contrary, by a kind of miracle of synergy, R has proved to serve together with the natural numbers N as one of the two core structures of mathematics; together they are the *sine qua non* of our subject, both pure and applied. Moreover, Cantor's construction of R is readily reconstrued in foundational programs such as those of intuitionism and predicativism that are alternatives to set-theoretical platonism. These alternative programs more fully meet the view of the nature of mathematics advanced by conceptual structuralism in sec. 2. Finally, such questions as Cantor's continuum hypothesis should not take the core role of R in pure and applied mathematics for support as a definite mathematical problem, but have to be considered on their own merits and in my view only make sense within the foundational framework of set-theoretical platonism.[26]

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